ON SOLVABILITY OF FOURTH ORDER OPERATOR-DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE ON WEIGHT SPACES

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Abstract. Solvability of the operator-differential equations of the fourth order of elliptic type with partial derivatives in the weighted spaces are considered in the paper. Solvability conditions in some weighted spaces, expressed by the coefficients of given operator-differential equations are obtained. Connection between lower bound of the spectrum of operator participating in the main part of the equation and exponent of the weight function is also determined. In this paper are found conditions oh the lower boundary of the spectrum of the main operator for operationally-differential equation depending of two and of the order of the weighted spaces which provide existence and uniqueness of the regular solution of a class of the operationally-differential equation of the fourth order in weighted spaces on all axis.

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Let$H$ – be a separable Hilbert space, A – a positive definite self-adjoin operator in H. Let $\gamma = (\gamma_1, \gamma_2) \in R^2$, $R = (-\infty, \infty)$. By $L_{2,\gamma}(R^2; H)$ we denote a Hilbert space of vector-function $f(x,y)$ with values in H for which

$$\|f\|_{L_{2,\gamma}} = \left(\int\int_{R^2} \|f(x,y)\|^2 e^{-2\gamma_1 x - 2\gamma_2 y} dxdy \right)^{\frac{1}{2}} < \infty.$$

On $D(R^2; H_4)$- set of infinitely differentiable in H vector-function $u(x,y)$, determines in R^2, with values $H_4 = D(A^4)$ with compact supports, we determine the norm

$$\|u\|_{W^4_{2,\gamma}} = \left(\sum_{\substack{k,j=0 \\ k+j \le 4}} \left\| A^{4-(k+j)} \frac{\partial^{k+j} u}{\partial x^k \partial y^j} \right\|_{L_{2,\gamma}} \right)^{\frac{1}{2}}.$$

By $W^4_{2,\gamma}(R^2; H)$ we denote a completion of $D(R^2; H_4)$ by the norm $\|u\|_{W^4_{2,\gamma}}$ Notice that for $\gamma = 0 = (0,0)$ the space $L_{2,0}(R^2; H) = L_2(R^2; H)$ and $W_2, 0^4(R^2; H) = W^4_2(R^2; H)$. In the space$H$consider the following operator-differential equation

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}+A^4 u+\sum_{\substack{k,j=0\\ k+j\le 4}}^{4} A_{k,j}\frac{\partial^{k+j}u}{\partial x^k\partial y^j}=f(x,y),(x,y)\in R^2, \qquad (1)$$

where $f(x,y)$, $u(x,y)$ are vector functions with values in H, and the operators A and $A_{k,j}$ $(k,j=\overline{0,4}, k+j\le 4)$ satisfy the following conditions:

1) A is a positive-definite operator with lower boundary of the spectrum μ_0, i.e., $A\ge \mu_0 E$

2) the operators $B_{k,j}=A_{k,j}\ A^{(k+j)-4}$ $(k,j=\overline{0,4}, k+j\le 4)$ are bounded operators in H.

Definition. If for $f(x,y)\in L_{2,\gamma}(R^2;H)$ there exists a vector-function $u(x,y)\in W^4_{2,\gamma}(R^2;H)$, that satisfies equation (1) almost everywhere in R^2, we'll call it a regular solution of equation (1). If for any $f(x,y)\in L_{2,\gamma}(R^2;H)$ there exists a regular solution of equation (1) that satisfies the estimation

$$\|u\|_{W^4_{2,\gamma}}\le const\,\|f\|_{L_{2,\gamma}},$$

then equation (1) is said to be regularly solvable.

In the present paper, we'll find conditions on coefficients of equation (1) and the vector $\gamma=(\gamma_1,\gamma_2)$, that provide regular solvability of the given equation.

Note that for $\gamma=0$ solvability of equation (1) was investigated in the paper [1] for $A_{k,j}=0$ $(k,j=\overline{0,4}, k+j\le 4)$ in the paper [2]. For one variable, when A is an elliptic operator with discrete spectrum, and $A_{ij}\equiv a_{ij}$ are scalar numbers, equation (1) was investigated in the paper [3], when A is a self-adjoint operator and coefficients are unbounded operators see [4].

The following theorem was proved in [2]

Theorem 1.[2]. *Let A-be a positive-definite self-adjoin operator with lower boundary of the spectrum μ_0 and $|\gamma|=\sqrt[4]{\gamma_1^4+\gamma_2^4}<\frac{1}{\sqrt[4]{8}}\mu_0$, then the equation*

$$P_0 u\equiv P_0\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)u\equiv\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}+A^4 u=f(x,y),(x,y)\in R^2 \qquad (2)$$

is regularly solvable.

Further, denote

$$P_1 u=P_1\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)u=\sum_{\substack{k,j=0\\ k+j\le 4}}^{4} A_{kj}\frac{\partial^{k+j}u}{\partial x^k\partial y^j}, \qquad (3)$$

$$u(x,y)\in W^4_{2,\gamma}\left(R^2;H\right)$$

and

$$Pu = P_0 u + P_1 u. \tag{4}$$

At first we prove the following theorem.

Theorem 2. *Let the operator A and vector γ satisfy the conditions of theorem 1. Then for any $u(x,y) \in W_{2,\gamma}^4(R^2;H)$ the following estimations hold:*

$$\left\| A^{4-(k+j)} \frac{\partial^{k+j} u}{\partial x^k \partial y^j} \right\|_{L_{2,\gamma}(R^2;H)} \le C_{k,j}(\gamma;\mu_0) \left\| P_0 u \right\|_{L_{2,\gamma}(R^2;H)} \quad (k,j=\overline{0,4}, k+j \le 4) \tag{5}$$

where

$$C_{0,0}(\gamma;\mu_0) = \frac{\mu_0^4}{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}, \tag{6}$$

$$C_{4,0}(\gamma;\mu_0) = 1 + \frac{4\gamma_1^2}{\sqrt{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}} + \frac{16\gamma_1^4}{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}, \tag{7}$$

$$C_{0,4}(\gamma;\mu_0) = 1 + \frac{4\gamma_2^2}{\sqrt{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}} + \frac{16\gamma_2^4}{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}, \tag{8}$$

for $j=0$, $k=1,2,3$ and $k=0$, $j=1,2,3$

$$C_{k,0} = \left(\frac{k}{4}\right)^k \left(\frac{4-k}{4}\right)^{\frac{4-k}{4}} \left(1 + \frac{4\gamma_1^2}{\sqrt{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}} + \frac{24\gamma_1^4 + 8\gamma_2^4}{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}\right), \tag{9}$$

$$C_{0,j} = \left(\frac{j}{4}\right) \left(\frac{4-j}{4}\right)^{\frac{4-j}{4}} \left(1 + \frac{4\gamma_2^2}{\sqrt{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}} + \frac{24\gamma_2^4 + 8\gamma_1^4}{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}\right), \tag{10}$$

for $k \ne 0$, $j \ne 0$, $k+j=4$ $(k=1, j=3, k=2, j=2, k=3, j=1)$

$$C_{k,j}(\gamma;\mu_0) = \left(\frac{k}{4}\right)^{\frac{k}{4}} \left(\frac{j}{4}\right)^{\frac{j}{4}} \left(1 + \frac{4(\gamma_1^2+\gamma_2^2)}{\sqrt{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}} + \frac{16(\gamma_1^4+\gamma_2^4)}{\mu_0^4 - 8(\gamma_1^4+\gamma_2^4)}\right). \tag{11}$$

And for $2 \le k+j \le 3$, $k \ne 0$, $j \ne 0$, $k=1$, $j=1$; $k=2$, $j=1$; $k=1$, $j=2)$

$$C_{k,j}(\gamma;\mu_0) = \left(\frac{4-(k+j)}{4}\right)^{\frac{4-(k+j)}{4}} \left(\frac{k}{4}\right)^{\frac{k}{4}} \left(\frac{j}{4}\right)^{\frac{j}{4}}. \tag{12}$$

Proof: Obviously, it suffices to prove these inequalities for a vector-function from $D(R^2, H_4)$. Let $u(x,y) \in D(R^2, H_4)$. After substitution $v(x,y) = u(x,y)e^{-\gamma_1 x - \gamma_2 y}$ we get following equivalents of the inequality

$$\left\| A^{4-(k+j)} \left(\tfrac{\partial}{\partial x} + \gamma_1 \right)^k \left(\tfrac{\partial}{\partial y} + \gamma_2 \right)^j v(x,y) \right\|_{L_2(R^2;H)} \leq \\ \leq C_{k,j}(\gamma;\mu_0) \left\| P_0 \left(\tfrac{\partial}{\partial x} + \gamma_1 \right), \left(\tfrac{\partial}{\partial y} + \gamma_2 \right) v \right\|_{L_2(R^2;H)}, \tag{13}$$

where the numbers $C_{k,j}(\gamma;\mu_0)$ are determined from equalities (6)-(12).

After Fourier transformation we get the following equivalent inequalities:

$$\left\| A^{4-(k+j)} (i\xi + \gamma_1)^k (i\eta + \gamma_2)^j \hat{v}(\xi,\eta) \right\|_{L_2(R^2;H)} \leq \\ \leq C_{k,j}(\gamma;\mu_0) \left\| P_0 (i\xi + v_1), (i\xi + v_2) \hat{v}(\xi,\eta) \right\|_{L_2(R^2;H)}. \tag{14}$$

Let

$$P_0 \left((i\xi + \gamma_1), (i\eta + \gamma_2) \right) \hat{v}(\xi,\eta) = \hat{g}(\xi,\eta), \quad (\xi,\eta) \in R^2.$$

Since the operator pencil

$$P_0 \left((i\xi + \gamma_1), (i\eta + \gamma_2) \right) = (i\xi + \gamma_1)^4 E + (i\eta + \gamma_2)^4 E + A^4, (\xi,\eta) \in R^2,$$

is invertible in H for $|\gamma| = \sqrt[4]{\gamma_1^4 + \gamma_2^4} < \frac{1}{\sqrt[4]{8}} \mu_0$ (see [2]), then

$$\hat{v}(\xi,\eta) = P_0^{-1} \left((i\xi + \gamma_1), (i\eta + \gamma_2) \right) \hat{g}(\xi,\eta), (\xi,\eta) \in R^2.$$

Then inequality (14) takes the following form

$$\left\| A^{4-(k+j)} (i\xi + \gamma_1)^k (i\eta + \gamma_2)^j P_0^{-1} \left((i\xi + \gamma_1), (i\xi + \gamma_2) \right) \hat{g}(\xi,\eta) \right\|_{L_2(R^2;H)} \\ \leq C_{k,j}(\gamma;\mu_0) \left\| \hat{g}(\xi,\eta) \right\|_{L_2(R^2;H)}. \tag{15}$$

Now, consider the case $k = 0, j = 0$. Then

$$\left\| A^4 P_0^{-1} \left((i\xi + \gamma_1), (i\eta + \gamma_2) \right) \hat{g}(\xi,\eta) \right\|_{L_2(R^2;H)} \\ \leq \sup_{(\xi,\eta) \in R^2} \left\| A^4 P_0^{-1} \left((i\xi + \gamma_1), (i\eta + \gamma_2) \right) \right\| \left\| \hat{g}(\xi,\eta) \right\|_{L_2(R^2;H)}. \tag{16}$$

Since for any $(\xi,\eta) \in R^2$

$$\begin{aligned} &\left\| A^4 P_0^{-1} \left((i\xi + \gamma_1), (i\eta + \gamma_2) \right) \right\| \\ &= \left\| A^4 \left((i\xi + \gamma_1)^4 + (i\eta + \gamma_2)^4 + A^4 \right)^{-1} \right\| \\ &\leq \sup_{\mu \in \sigma(A)} \left\| \mu^4 \left((i\xi + \gamma_1)^4 + (i\eta + \gamma_2)^4 + \mu^4 \right)^{-1} \right\| \\ &\leq \sup_{\mu \in \sigma(A)} \left\| \mu^4 [Re \left((i\xi + \gamma_1)^4 + (i\eta + \gamma_2)^4 + \mu^4 \right)]^{-1} \right\| \\ &\leq \sup_{\mu \in \sigma(A)} \left\| \mu^4 \left(\xi^4 + \gamma_1^4 + \eta^4 + \gamma_2^4 - 6\xi^2\gamma_1^2 - 6\eta^2\gamma_2^2 \right)^{-1} \right\| \\ &\leq \sup_{\mu \geq \mu_0} \left\| \mu^4 \left(\xi^2 - 3\gamma_1^2 \right)^2 + \left(\eta^2 - 3\gamma_2^2 \right)^2 + \left(\mu^4 - 8 \left(\gamma_1^4 + \gamma_2^4 \right) \right)^{-1} \right\| \\ &\leq \sup_{\mu \geq \mu_0} \left\| \mu^4 \left(\mu^4 - 8 \left(\gamma_1^4 + \gamma_2^4 \right) \right)^{-1} \right\| \leq \frac{\mu_0^4}{\mu_0^4 - 8\left(\gamma_1^4 + \gamma_2^4 \right)} = C_{0,0}(\gamma;\mu_0), \end{aligned} \tag{17}$$

Then taking into account inequality (12) in inequality (16) we prove the validity of inequality (15) for $k = 0,\ j = 0$.

Now, let's consider the case $j = 0, k = 1, 2, 3$. Then

$$\left\| A^{4-k} (i\xi + \gamma_1)^k P_0^{-1} ((i\xi + \gamma_1), (i\eta + \gamma_2))\, \hat{g}(\xi, \eta) \right\|_{L_2(R^2;H)}$$
$$\sup_{(\xi,\eta)\in R^2} \left\| (i\xi + \gamma_1)^k A^4 P_0^1 ((i\xi + \gamma_1), (i\eta + \gamma_2)) \right\| \cdot \| \hat{g}(\xi, \eta) \|_{L_2(R^2;H)} . \tag{18}$$

On the other hand, for $(\xi, \eta) \in R^2$

$$\begin{aligned} &\left\| (i\xi + \gamma_1)^k A^4 P_0^{-1} ((i\xi + \gamma_1), (i\eta + \gamma_2)) \right\| \\ &\le \sup_{\mu\in\sigma(A)} \frac{\mu^{4-k} (\xi^2 + \gamma_1^2)^{\frac{k}{2}}}{(\xi^2 - 3\gamma_1^2)^2 + (\eta^2 - 3\gamma_2^2)^2 + \mu^4 - 8(\gamma_1^4 + \gamma_2^4)} \\ &\le \sup_{\mu\in\sigma(A)} \frac{\mu^{4-k} (\xi^2 + \gamma_1^2)^{\frac{k}{2}}}{(\xi^2 - 3\gamma_1^2)^2 + \mu^4 - 8(\gamma_1^4 + \gamma_2^4)}. \end{aligned} \tag{19}$$

Let $\delta > 0$. Then having applied Young's inequality, we have:

$$\mu^{4-k} (\xi^2 + \gamma_1^2)^{\frac{k}{2}} = (\delta\mu^4)^{\frac{4-k}{4}} \left(\frac{1}{\delta^{\frac{4-k}{k}}} (\xi^2 + \gamma_1^2)^2 \right)^{\frac{k}{4}} \le \delta \frac{4-k}{4} \mu^4 + \frac{k}{4} \frac{1}{\delta^{\frac{4-k}{k}}} (\xi^2 + \gamma_1^2) .$$

Now choose $\delta > 0$.So, that $\frac{4-k}{4}\delta = \frac{k}{4}\frac{1}{\delta^{\frac{4-k}{4}}}$, i.e., $\delta = \left(\frac{k}{4-k}\right)^{\frac{k}{4}}$.
Then

$$\mu^{4-k} (\xi^2 + \gamma_1^2)^{\frac{k}{2}} \le \frac{4-k}{4} \left(\frac{k}{4-k} \right)^{\frac{k}{4}} \left(\mu^4 + (\xi^2 + \gamma_1^2)^2 \right)$$

$$= \left(\frac{k}{4} \right)^{\frac{k}{4}} \left(\frac{4-k}{4} \right)^{\frac{4-k}{4}} \left(\mu^4 + (\xi^2 + \gamma_1^2)^2 \right) .$$

Considering this inequality in (19) we get

$$\begin{aligned} &\left\| (i\xi + \gamma_1)^k A^4 P_0^{-1} ((i\xi + \gamma_1), (i\eta + \gamma_2)) \right\| \\ &\le \left(\frac{k}{4} \right)^{\frac{k}{4}} \left(\frac{4-k}{4} \right)^{\frac{4-k}{4}} \sup_{\mu\in\sigma(A)} \frac{\mu^4 + (\xi^2 + \gamma_1^2)^2}{(\xi^2 - 3\gamma_1^2)^2 + \mu^4 - 8(\gamma_1^4 + \gamma_2^4)} \\ &\le \left(\frac{k}{4} \right)^{\frac{k}{4}} \left(\frac{4-k}{4} \right)^{\frac{4-k}{4}} \\ &\quad \times \sup_{\mu\in\sigma(A)} \frac{\mu^4 - 8(\gamma_1^4 + \gamma_2^4) + (\xi^2 - 3\gamma_1^2)^2 + 8\gamma_1^2 (\xi^2 - 3\gamma_1^2) + 24\gamma_1^4 + 8\gamma_2^4}{(\xi^2 - 3\gamma_1^2)^2 + \mu^4 - 8(\gamma_1^4 + \gamma_2^4)} \end{aligned}$$

$$\leq \left(\frac{k}{4}\right)^{\frac{k}{4}}\left(\frac{4-k}{4}\right)^{\frac{4-k}{4}}$$

$$\times \sup_{\mu\in\sigma(A)}\left(1+8\gamma_1^2\frac{\xi^2-3\gamma_1^2}{\left(\xi^2-3\gamma_1^2\right)+\mu^4-8\left(\gamma_1^4+\gamma_2^4\right)}+\frac{24\gamma_1^4+8\gamma_2^4}{\mu^4-8\left(\gamma_1^4+\gamma_2^4\right)}\right)$$

$$\leq \left(\frac{k}{4}\right)^{\frac{k}{4}}\left(\frac{4-k}{4}\right)^{\frac{4-k}{4}}\left(1+\frac{4\gamma_1^2}{\sqrt{\mu_0^4-8\left(\gamma_1^4+\gamma_2^4\right)}}+\frac{24\gamma_1^4+8\gamma_2^4}{\mu_0^4-8\left(\gamma_1^4+\gamma_2^4\right)}\right)$$

$$= C_{k,0}\left(\gamma;\mu_0\right).$$

Thus, inequality (15) is true also for $j=0, k=1,2,3$. Inequality (15) is similarly proved for $j=0, k=1,2,3$.

Now, consider the case $k\neq 0,\ j\neq 0,\ k+j=4$ ($k=1,\ j=3$; $k=2,\ j=2$; $k=3,\ j=2$). In this case

$$\left\|\left(i\xi+\gamma_1\right)^k\left(i\eta+\gamma_2\right)^j P_0^{-1}\left(\left(i\xi+\gamma_1\right),\left(i\eta+\gamma_2\right)\right)\hat{g}\left(\xi,\eta\right)\right\|_{L_2(R^2;H)}$$

$$\leq \sup_{(\xi,\eta)\in R^2}\left\|\left(i\xi+\gamma_1\right)^k\left(i\eta+\gamma_2\right)^j P_0^{-1}\left(\left(i\xi+\gamma_1\right),\left(i\eta+\gamma_2\right)\right)\right\| \tag{20}$$

$$\times\left\|\hat{g}\left(\xi,\eta\right)\right\|_{L_2(R^2;H)}.$$

For $(\xi,\eta)\in R^2$ we have:

$$\left\|\left(i\xi+\gamma_1\right)^k\left(i\eta+\gamma_2\right)^j P_0^{-1}\left(\left(i\xi+\gamma_1\right),\left(i\xi+\gamma_2\right)\right)\hat{g}\left(\xi,\eta\right)\right\|_{L_2(R^2;H)}$$

$$\leq \sup_{\mu\in\sigma(A)}\frac{\left(\xi^2+\gamma_1^2\right)^{\frac{k}{2}}\left(\eta^2+\gamma_2^2\right)^{\frac{j}{2}}}{\left(\xi^2-3\gamma_1^2\right)^2+\left(\eta^2-3\gamma_2^2\right)^2+\mu^4-8\left(\gamma_1^4+\gamma_2^4\right)}. \tag{21}$$

Since for any $\delta>0$

$$\begin{aligned}\left(\xi^2+\gamma_1^2\right)^{\frac{k}{2}}\left(\eta^2+\gamma_2^2\right)^{\frac{j}{2}} &= \left(\xi^2+\gamma_1^2\right)^{\frac{k}{2}}\left(\eta^2+\gamma_2^2\right)^{\frac{4-k}{2}}\\ &= \left(\delta\left(\xi^2+\gamma_1^2\right)^2\right)^{\frac{k}{4}}\left(\frac{1}{\delta^{\frac{k}{4-k}}}\left(\eta^2+\gamma_2^2\right)^2\right)^{\frac{4-k}{4}}\\ &\leq \frac{k}{4}\delta\left(\xi^2+\gamma_1^2\right)^2+\frac{4-k}{4}\frac{1}{\delta^{\frac{k}{4-k}}}\left(\xi^2+\gamma_1^2\right).\end{aligned}$$

Now, assuming $\delta=\left(\frac{4-k}{k}\right)^{\frac{4-k}{4}}$ we get that

$$\left(\xi^2+\gamma_1^2\right)^{\frac{k}{2}}\left(\eta^2+\gamma_2^2\right)^{\frac{j}{2}}\leq\frac{k}{4}\left(\frac{4-k}{k}\right)^{\frac{4-k}{4}}\left(\left(\xi^2+\gamma_1^2\right)^2+\left(\eta^2+\gamma_2^2\right)^2\right)$$

$$= \left(\frac{k}{4}\right)^{\frac{k}{4}} \left(\frac{j}{4}\right)^{\frac{j}{4}} \Big(\left(\xi^2 - 3\gamma_1^2\right)^2 + 8\gamma_1^2 \left(\xi^2 - 3\gamma_1^2\right)$$
$$+16\gamma_1^4 + \left(\eta^2 - 3\gamma_2^2\right)^2 + 8\gamma_2^2 \left(\eta^2 - 3\gamma_2^2\right) + 16\gamma_2^4 \Big).$$

Thus (21) implies

$$\begin{aligned}
&\left\| (i\xi + \gamma_1)^k (i\eta + \gamma_2)^j P_0^{-1} ((i\xi + \gamma_1), (i\eta + \gamma_2)) \right\| \leq \\
&\leq \left(\tfrac{k}{4}\right)^{\frac{k}{4}} \left(\tfrac{j}{4}\right)^{\frac{j}{4}} \sup_{\mu \in \sigma(A)} \frac{\left(\xi^2+\gamma_1^2\right)^2 + \left(\eta^2+\gamma_2^2\right)}{\left(\xi^2-3\gamma_1^2\right)^2 + \left(\eta^2-3\gamma_2^2\right)^2 + \mu^4 - 8\left(\gamma_1^4+\gamma_2^4\right)} = \\
&= \left(\tfrac{k}{4}\right)^{\frac{k}{4}} \left(\tfrac{j}{4}\right)^{\frac{j}{4}} \frac{\left(\xi^2-3\gamma_1^2\right)^2 + \left(\eta^2-3\gamma_2^2\right)^2 + 8\gamma_1^2\left(\xi^2-3\gamma_1^2\right) + 8\gamma_2^2\left(\eta^2-3\gamma_2^2\right) + 16\left(\gamma_1^4+\gamma_2^4\right)}{\left(\xi^2-3\gamma_1^2\right)^2 + \left(\eta^2-3\gamma_2^2\right)^2 + \mu_0^4 - 8\left(\gamma_1^4+\gamma_2^4\right)} = \\
&= \left(\tfrac{k}{4}\right)^{\frac{k}{4}} \left(\tfrac{j}{4}\right)^{\frac{j}{4}} \Bigg(1 + 8\gamma_1^2 \frac{\xi^2-3\gamma_1^2}{\left(\xi^2-3\gamma_1^2\right) + \mu_0^4 - 8\left(\gamma_1^4+\gamma_2^4\right)} + \\
&+ 8\gamma_2^2 \frac{\eta^2-3\gamma_2^2}{\left(\eta^2-3\gamma_2^2\right)^2 + \mu_0^4 - 8\left(\gamma_1^4+\gamma_2^4\right)} + \frac{16\left(\gamma_1^4+\gamma_2^4\right)}{\mu_0^4 - 8\left(\gamma_1^4+\gamma_2^4\right)} \Bigg) \leq \\
&\leq \left(\tfrac{k}{4}\right)^{\frac{k}{4}} \left(\tfrac{j}{4}\right)^{\frac{j}{4}} \left(1 + \frac{4\left(\gamma_1^2+\gamma_2^2\right)}{\sqrt{\mu_0^4 - 8\left(\gamma_1^4+\gamma_2^4\right)}} + \frac{16\left(\gamma_1^4+\gamma_2^4\right)}{\mu_0^4 - 8\left(\gamma_1^4+\gamma_2^4\right)} \right) = C_{k,j}(\gamma; \mu_0).
\end{aligned} \tag{22}$$

Now, consider the case $2 \leq k + j \leq 3$, $k \neq 0$, $j \neq 0$. In this case

$$\begin{aligned}
&\left\| A^{4-(k+j)} (i\xi + \gamma_1)^k (i\eta + \gamma_2)^j P_0^{-1} ((i\xi + \gamma_1), (i\eta + \gamma_2)) \hat{g}(\xi, \eta) \right\|_{L_2(R^2;H)} \\
&\leq \sup_{(\xi,\eta) \in R^2} \left\| A^{4-(k+j)} (i\xi + \gamma_1)^k (i\eta + \gamma_2)^j P_0^1 ((i\xi + \gamma_1), (i\eta + \gamma_2)) \right\| \\
&\quad \times \left\| \hat{g}(\xi, \eta) \right\|_{L_2(R^2;H)}.
\end{aligned} \tag{23}$$

Since for $(\xi, \eta) \in R^2$

$$\begin{aligned}
&\left\| A^{4-(k+j)} (i\xi + \gamma_1)^k (i\eta + \gamma_2)^j P_0^{-1} ((i\xi + \gamma_1), (i\eta + \gamma_2)) \right\|_{L_2(R^2;H)} \\
&\leq \sup_{\mu \in \sigma(A)} \left\| \mu^{4-(k+j)} (i\xi + \gamma_1)^k (i\eta + \gamma_2)^j P_0^1 ((i\xi + \gamma_1), (i\eta + \gamma_2)) \right\| \\
&\leq \sup_{\mu \in \sigma(A)} \frac{\mu^{4-(k+j)} \left(\xi^2 + \gamma_1^2\right)^{\frac{k}{2}} \left(\eta^2 + \gamma_2^2\right)^{\frac{j}{2}}}{\left(\xi^2 - 3\gamma_1^2\right)^2 + \left(\eta^2 - 3\gamma_2^2\right)^2 + \mu^4 - 8\left(\gamma_1^4 + \gamma_2^4\right)}.
\end{aligned} \tag{24}$$

Let $\delta_1, \delta_2, \delta_3 > 0$. $\delta_1^{\frac{4-(k+j)}{4}} \cdot \delta_2^{\frac{k}{4}} \cdot \delta_3^{\frac{j}{4}} = 1$. Then we have

$$\begin{aligned}
&\mu^{4-(k+j)} \left(\xi^2 + \gamma_1^2\right)^{\frac{k}{2}} \left(\eta^2 + \gamma_2^2\right)^{\frac{j}{2}} \\
&= \left(\delta_1 \mu^4\right)^{\frac{4-(k+j)}{4}} \left(\delta_2 \left(\xi^2 + \gamma_1^2\right)^2\right)^{\frac{k}{4}} \left(\delta_3 \left(\eta^2 + \gamma_2^2\right)^2\right)^{\frac{j}{4}} \\
&\leq \delta_1 \frac{4 - (k+j)}{4} \mu^4 + \delta_2 \frac{k}{4} \left(\xi^2 + \gamma_1^2\right)^2 + \delta_3 \frac{j}{4} \left(\eta^2 + \gamma_2^2\right)^2.
\end{aligned}$$

Additionally we require $\delta_1 \frac{4-(k+j)}{4} = \delta_2 \frac{k}{4} = \delta_3 \frac{j}{4}$. Then we get

$$\delta_1 = \left(\frac{k}{4-(k+j)} \right)^{\frac{k+j}{4}} \left(\frac{j}{k} \right)^{\frac{j}{4}},$$

i.e.,

$$\mu^{4-(k+j)} \left(\xi^2 + 3\gamma_1^2 \right)^{\frac{k}{2}} \left(\eta^2 + \gamma_2^2 \right)^{\frac{j}{2}} \left(\eta^2 + \gamma_2^2 \right)^{\frac{j}{2}}$$
$$\leq \frac{4-(k+j)}{4} \left(\frac{k}{4-(k+j)} \right)^{\frac{k+j}{4}} \left(\frac{j}{4} \right)^{\frac{j}{4}} \left(\mu^4 + \left(\xi^2 + \gamma_1^2 \right)^2 + \left(\eta^2 + \gamma_2^2 \right)^2 \right)$$

Consequently,

$$\sup_{\mu \in \sigma(A)} \frac{\mu^{4-(k+j)} \left(\xi^2 + \gamma_1^2 \right)^{\frac{k}{2}} \left(\eta^2 + \gamma_2^2 \right)^{\frac{j}{2}}}{\left(\xi^2 - 3\gamma_1^2 \right)^2 + \left(\eta^2 - 3\gamma_1^2 \right)^2 + \mu^4 - 8\left(\gamma_1^4 + \gamma_2^4 \right)}$$
$$\leq \left(\frac{4-(k+j)}{4} \right)^{\frac{4-(k+j)}{4}} \left(\frac{k}{4} \right)^{\frac{k}{4}} \left(\frac{j}{4} \right)^{\frac{j}{4}}$$
$$\times \sup_{\mu \in \sigma(A)} \left\{ \frac{\mu^4 + \left(\xi^2 - 3\gamma_1^2 \right)^2 + \left(\eta^2 - 3\gamma_2^2 \right)^2 + 8\gamma_1^2 \left(\xi^2 - 3\gamma_1^2 \right)^2}{\left(\xi^2 - 3\gamma_1^2 \right)^2 + \left(\eta^2 - 3\gamma_2^2 \right)^2 + \mu^4 - 8\left(\gamma_1^4 + \gamma_2^4 \right)} \right.$$
$$\left. + \frac{8\gamma_2^2 \left(\eta^2 - 3\gamma_2^2 \right) + 24\left(\gamma_1^4 + \gamma_2^4 \right)}{\left(\xi^2 - 3\gamma_1^2 \right)^2 + \left(\eta^2 - 3\gamma_2^2 \right)^2 + \mu^4 - 8\left(\gamma_1^4 + \gamma_2^4 \right)} \right\}$$
$$\leq \left(\frac{4-(k+j)}{4} \right)^{\frac{4-(k+j)}{4}} \left(\frac{k}{4} \right)^{\frac{k}{4}} \left(\frac{j}{4} \right)^{\frac{j}{4}}$$
$$\times \left(1 + \frac{4\left(\gamma_1^2 + \gamma_2^2 \right)}{\mu_0^4 - 8\left(\gamma_1^4 + \gamma_2^4 \right)} + \frac{24\left(\gamma_1^4 + \gamma_2^4 \right)}{\mu_0^4 - \left(\gamma_1^4 + \gamma_2^4 \right)} \right) = C_{k,j}\left(\gamma; \mu_0 \right).$$

The theorem is proved.
Now, prove a theorem on solvability of equation (1).

Theorem 3. *Let the conditions of theorem 1) and condition 2), moreover*

$$\alpha\left(\gamma; \mu_0 \right) = \sum_{\substack{k+j=0 \\ k+j \leq 4}}^{4} C_{k.j}\left(\gamma; \mu_0 \right) \left\| B_{k,j} \right\| < 1,$$

where the numbers $C_{k,j}\left(\gamma; \mu_0 \right)$ are determined from theorem 2 by equalities (6)-(12). Operator $B_{k,j}$ determined from condition 2). Then equation (1) is regularly solvable.

Proof. Write equation (1) in the form $Pu = P_0 u + P_1 u = f$ where $f(x,y) \in L_{2,\gamma}\left(R^2; H \right)$, $u\left(x, y \right) \in W_{2,\gamma}^4\left(R^2; H \right)$. After substitution of $P_0 u = v \in L_{2,\gamma}\left(R^2; H \right)$ we get the equation $\left(+P_1 P_0^{-1} \right) v = f$ in $L_{2,\gamma}\left(R^2; H \right)$. Since

$$\left\|P_1P_0^{-1}u\right\|_{L_{2,\gamma}}=\left\|P_1u\right\|_{L_{2,\gamma}}\leq\sum\nolimits^4_{\substack{k+j=0\\k+j\leq 4}}\left\|A_{k,j}A^{(k+j)-4}\right\|\left\|A^{4-(k+j)}\frac{\partial^{\frac{k+j}{4}}u}{\partial x^k\partial y^j}\right\|_{L_{2,\gamma}}$$

$$\leq\sum_{\substack{k+j=0\\k+j\leq 4}}^{4}C_{k,j}\left(\gamma;\mu_0\right)\left\|B_{k,j}\right\|\left\|P_0u\right\|_{L_{2,\gamma}}=\alpha\left(\gamma;\mu_0\right)\left\|v\right\|_{L_{2,\gamma}}.$$

As $\alpha\left(\gamma;\mu_0\right)<1$, the operator $E+P_1P_0^{-1}$ is invertible in the space $L_{2,\gamma}\left(R^2;H\right)$, then $v=\left(E+P_1P_0^{-1}\right)^{-1}f$ bat $u=P_0^{-1}\left(E+P_1P_0^{-1}\right)^{-1}f$

Hence it follows that

$$\left\|u\right\|_{W^4_{2,\gamma}}\leq const\left\|f\right\|_{L_{2,\gamma}}.$$

The theorem is proved.

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