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# Solutions of Nonlinear Fractional Integrodifferential Equations with Boundary Conditions

Zhenyu Guo[1*], Min Liu[2], Donglei Wang[3]

[1,2] *School of Sciences, Liaoning Shihua University*
*Fushun, Liaoning* 113001*, China*
[3] *WangHua High School, Fushun, Liaoning, China*
[1]*E-mail: guozy@163.com,* [2]*E-mail: min_liu@yeah.net*

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A nonlinear fractional integrodifferential equation with boundary conditions is studied in this article, and some sufficient conditions for existence and uniqueness of a solution for the equation is established by Krasnoselskii fixed point theorem and Banach contraction principle, respectively.

**Keywords:** Fractional integrodifferential equations, Krasnoselskii fixed point theorem, Banach contraction principle, Boundary conditions.

**AMS Subject Classification**: 34A12, 34G20, 34K05.

## 1. Introduction

This article is concerned with the following nonlinear fractional integrodifferential equation with boundary conditions:

$$\begin{aligned} D^q x(t) &= f\big(t, x(t), (\phi x)(t), (\psi x)(t)\big), \quad 0 < t < 1,\ 2 < q < 3, \\ x(0) &= x'(0) = x'(1) = 0, \end{aligned} \tag{1.1}$$

where $D^q$ denotes the standard Riemann-Liouville fractional derivative, $f : [0,1] \times X \times X \times X \to X$ is a given continuous function, $(X, \|\cdot\|)$ is a Banach space and $C = C([0,1], X)$ is the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$, for $\gamma, \delta : [0,1] \times [0,1] \to [0, +\infty)$,

$$(\phi x)(t) = \int_0^t \gamma(t,s)x(s)ds, \quad (\psi x)(t) = \int_0^t \delta(t,s)x(s)ds. \tag{1.2}$$

Recently, fractional order differential equations and systems have been of great interest. For detailed discussion on this topic, refer to the monographs of Kilbas et al.[10], Podlubny [17], and the papers by Anguraj et al. [1], Ahmad and Alsaedi [2,3], Cui [4], Guo and Liu [5-9], Kosmatov [11], Lakshmikantham and Vatsala [12], Li and Deng [13], Li [14], Li and Guérékata [15], Mao et al.[16] and the references therein.

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Applying Krasnoselskii fixed point theorem and Banach contraction principle, we obtain theorems of existence and uniqueness of solutions for equation (1.1).

## 2. Preliminaries

Let us recall some basic definitions on fractional calculus, which can be found in the literature.

**Definition 2.1:** The Riemann-Liouville fractional integral of order $q$ is defined by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \ q > 0, \tag{2.1}$$

provided the integral exists.

**Definition 2.2:** The Riemann-Liouville fractional derivaltive of order $q$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-q-1} f(s)ds, \ n-1 < q \leq n, \ q > 0, \tag{2.2}$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

**Lemma 2.3:** (*see* [17]) *For $q > 0$, let $x, D^q x \in C(0,1) \cap L(0,1)$. Then*

$$I^q D^q x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n}, \tag{2.3}$$

*where $c_i \in \mathbb{R}, i = 1, 2, \cdots, n$($n$ is the smallest integer such that $n \geq q$).*

**Lemma 2.4:** (*see* [17]) *Let $x \in L(0,1)$. Then*
*(i) $D^p I^q x(t) = I^{q-p}, \ q > p > 0$;*
*(ii) $D^q t^{a-1} = (\Gamma(a)/\Gamma(a-q)) t^{a-q-1}, \ q > 0, a > 0$.*

**Theorem 2.5:** (*Krasnoselskii fixed point theorem, see* [18]) *Let $D$ be a closed convex and nonempty subset of a Banach space $X$, and $A, B$ be two operators such that*
*(i) $Ax + By \in D$ whenever $x, y \in D$;*
*(ii) $A$ is compact and continuous;*
*(iii) $B$ is a contraction mapping.*
*Then there exists $z \in D$ such that $z = Az + Bz$.*

**Lemma 2.6:** *Given $f \in C(0,1) \cap L(0,1)$, the unique solution of*

$$\begin{aligned} D^q x(t) &= f(t), \quad 0 < t < 1, \ 2 < q < 3, \\ x(0) &= x'(0) = x'(1) = 0 \end{aligned} \tag{2.4}$$

*is*

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \int_0^1 \frac{t^{q-1}(1-s)^{q-2}}{\Gamma(q)} f(s)ds. \tag{2.5}$$

**Proof:** It follows from Lemma 2.3 that the fractional differential equation in (2.4) is equivalent to the integral equation

$$x(t) = I^q f(t) + c_1 t^{q-1} + c_2 t^{q-2} + c_3 t^{q-3}$$

$$= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + c_1 t^{q-1} + c_2 t^{q-2} + c_3 t^{q-3}, \tag{2.6}$$

where $c_1, c_2, c_3 \in \mathbb{R}$. From the boundary conditions for (2.4), we have $c_2 = c_3 = 0$ and

$$c_1 = -\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q)} f(s)ds, \tag{2.7}$$

which completes the proof. □

Now list the following hypotheses for convenience:

(H1) There exist positive functions $L_1(t), L_2(t), L_3(t)$ such that

$$\begin{gathered} \|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \\ \leq L_1(t)\|x_1 - x_2\| + L_2(t)\|y_1 - y_2\| + L_3(t)\|z_1 - z_2\|, \\ \forall t \in [0,1], x_1, x_2, y_1, y_2, z_1, z_2 \in X. \end{gathered} \tag{2.8}$$

Further,

$$\begin{gathered} \gamma_0 = \sup_{t\in[0,1]} \Big| \int_0^t \gamma(t,s)ds \Big|, \qquad \delta_0 = \sup_{t\in[0,1]} \Big| \int_0^t \delta(t,s)ds \Big|, \\ I_L^q = \sup_{t\in[0,1]} \{|I^q L_1(t)|, |I^q L_2(t)|, |I^q L_3(t)|\}, \\ I^{q-1}L(1) = \max\{|I^{q-1}L_1(1)|, |I^{q-1}L_2(1)|, |I^{q-1}L_3(1)|\}. \end{gathered} \tag{2.9}$$

(H2) $\|f(t,x,y,z)\| \leq \mu(t)$, for all $(t,x,y,z) \in [0,1]\times X\times X\times X, \mu \in L^1([0,1], \mathbb{R}^+)$.

## 3. Main results

In this section, the theorems of uniqueness and existence of a solution for equation (1.1) will be given.

**Theorem 3.1 :** *Assume that $f : [0,1] \times X \times X \times X \to X$ is jointly continuous and satisfies (H1) and (H2). If*

$$(1 + \gamma_0 + \delta_0)I^{q-1}L(1) < q - 1, \tag{3.1}$$

*then the fractional integrodifferential equation (1.1) has at least one solution.*

**Proof:** Consider $B_r = \{x \in C : \|x\| \leq r\}$, where

$$r \geq \frac{\|\mu\|_{L^1}(2q-1)}{(q-1)\Gamma(q+1)}. \tag{3.2}$$

Define two mappings $A$ and $B$ on $B_r$ by

$$
\begin{aligned}
(Ax)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big)ds,\\
(Bx)(t) &= -\int_0^1 \frac{t^{q-1}(1-s)^{q-2}}{\Gamma(q)} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big)ds.
\end{aligned}
\tag{3.3}
$$

For $x, y \in B_r$, by (H2), we obtain

$$
\begin{aligned}
&\|(Ax)(t) + (By)(t)\|\\
&\quad\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big\|f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big)\big\|ds\\
&\qquad + \int_0^1 \frac{t^{q-1}(1-s)^{q-2}}{\Gamma(q)} \big\|f\big(s, y(s), (\phi y)(s), (\psi y)(s)\big)\big\|ds\\
&\quad\le \|\mu\|_{L^1} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \|\mu\|_{L^1} \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q)} ds\\
&\quad= \|\mu\|_{L^1} \Big(\frac{t^q}{\Gamma(q+1)} + \frac{1}{(q-1)\Gamma(q)}\Big)\\
&\quad\le \frac{\|\mu\|_{L^1}(2q-1)}{(q-1)\Gamma(q+1)} \le r,
\end{aligned}
\tag{3.4}
$$

which means $Ax + By \in B_r$.

It is claimed that $A$ is compact and continuous. Continuity of $f$ implies that $(Ax)(t)$ is continuous. $(Ax)(t)$ is uniformly bounded on $B_r$ as

$$
\|Ax\| \le \frac{\|\mu\|_{L^1}}{\Gamma(q+1)}. \tag{3.5}
$$

Since $f$ is bounded on the compact set $[0,1] \times B_r \times B_r \times B_r$, let $\sup_{(t,x,\phi x,\psi x)\in[0,1]\times B_r\times B_r\times B_r} \|f(t, x, \phi x, \psi x)\| = f_{\max}$. Then, for $t_1, t_2 \in [0,1]$, we get

$$
\begin{aligned}
&\|(Ax)(t_2) - (Ax)(t_1)\|\\
&\quad= \frac{1}{\Gamma(q)} \Big\| \int_0^{t_1} \big((t_2-s)^{q-1} - (t_1-s)^{q-1}\big) f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big)ds\\
&\qquad + \int_{t_1}^{t_2} (t_2-s)^{q-1} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big)ds \Big\|\\
&\quad\le \frac{f_{\max}}{\Gamma(q)} \Big| -\frac{(t_2-t_1)^q}{q} + \frac{t_2^q}{q} - \frac{t_1^q}{q} + \frac{(t_2-t_1)^q}{q} \Big|\\
&\quad\le \frac{f_{\max}}{\Gamma(q+1)} |t_2^q - t_1^q|,
\end{aligned}
\tag{3.6}
$$

which is independent of $x$. Therefore, $A$ is relatively compact on $B_r$. By Arzela-Ascoli's Theorem, $A$ is compact on $B_r$.

For $x, y \in B_r$ and $t \in [0,1]$, by (H1), we have

$$
\begin{aligned}
&\|(Bx)(t) - (By)(t)\| \\
&\le \int_0^1 \frac{t^{q-1}(1-s)^{q-2}}{\Gamma(q)} \big\| f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \\
&- f\big(s, y(s), (\phi y)(s), (\psi y)(s)\big)\big\| ds \\
&\le \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q)} \Big(L_1(s)\|x-y\| + L_2(s)\|\phi x - \phi y\| + L_3(s)\|\psi x - \psi y\|\Big) ds \\
&\le \frac{1}{q-1} \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \Big(L_1(s) + \gamma_0 L_2(s) + \delta_0 L_3(s)\Big)\|x-y\| ds \\
&\le \frac{1}{q-1}\big(I^{q-1}L_1(1) + \gamma_0 I^{q-1}L_2(1) + \delta_0 I^{q-1}L_3(1)\big)\|x-y\| \\
&\le \frac{(1+\gamma_0+\delta_0)I^{q-1}L(1)}{q-1}\|x-y\|.
\end{aligned}
\tag{3.7}
$$

It follows from (3.1) that $B$ is a contraction mapping. Thus, by Krasnoselskii fixed point theorem, (1.1) has at least one solution. □

**Theorem 3.2:** *Assume that $f : [0,1] \times X \times X \times X \to X$ is jointly continuous and satisfies (H1). If*

$$
\lambda = (1+\gamma_0+\delta_0)\Big(I_L^q + \frac{I^{q-1}L(1)}{q-1}\Big) < 1, \tag{3.8}
$$

*then the fractional integrodifferential equation (1.1) has a unique solution.*

**Proof:** Define a mapping $F : C \to C$ by

$$
\begin{aligned}
(Fx)(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) ds \\
&- \int_0^1 \frac{t^{q-1}(1-s)^{q-2}}{\Gamma(q)} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) ds, \ t \in [0,1].
\end{aligned}
\tag{3.9}
$$

Let $\sup_{t\in[0,1]} |f(t,0,0,0)| = M$, and choose

$$
r \ge \frac{M(2q-1)}{(1-\lambda)(q-1)\Gamma(q+1)}. \tag{3.10}
$$

It is claimed that $FB_r \subset B_r$, where $B_r = \{x \in C : \|x\| \le r\}$. In fact, for $x \in B_r$, by (3.8), (3.10) and (H1), we obtain

$$
\begin{aligned}
\|(Fx)(t)\| \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big\|f\big(s,x(s),(\phi x)(s),(\psi x)(s)\big)\big\|ds \\
& + \int_0^1 \frac{t^{q-1}(1-s)^{q-2}}{\Gamma(q)} \big\|f\big(s,x(s),(\phi x)(s),(\psi x)(s)\big)\big\|ds \\
\leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \Big(\big\|f\big(s,x(s),(\phi x)(s),(\psi x)(s)\big) - f(s,0,0,0)\big\| \\
& + \|f(s,0,0,0)\|\Big)ds \\
& + \int_0^1 \frac{t^{q-1}(1-s)^{q-2}}{\Gamma(q)} \Big(\big\|f\big(s,x(s),(\phi x)(s),(\psi x)(s)\big) \\
& - f(s,0,0,0)\big\| + \|f(s,0,0,0)\|\Big)ds \\
\leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \Big(L_1(s)\|x(s)\| + L_2(s)\|(\phi x)(s)\| \\
& + L_3(s)\|(\psi x)(s)\| + M\Big)ds \\
& + \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q)} \Big(L_1(s)\|x(s)\| + L_2(s)\|(\phi x)(s)\| \\
& + L_3(s)\|(\psi x)(s)\| + M\Big)ds \qquad (3.11) \\
\leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \Big(L_1(s)\|x(s)\| + \gamma_0 L_2(s)\|x(s)\| \\
& + \delta_0 L_3(s)\|x(s)\| + M\Big)ds \\
& + \frac{1}{q-1}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \Big(L_1(s)\|x(s)\| + \gamma_0 L_2(s)\|x(s)\| \\
& + \delta_0 L_3(s)\|x(s)\| + M\Big)ds \\
\leq & \big(I^q L_1(t) + \gamma_0 I^q L_2(t) + \delta_0 I^q L_3(t)\big)r + \frac{Mt^q}{\Gamma(q+1)} \\
& + \frac{1}{q-1}\big(I^{q-1}L_1(1) + \gamma_0 I^{q-1}L_2(1) + \delta_0 I^{q-1}L_3(1)\big)r \\
& + \frac{M}{\Gamma(q)}\int_0^1 (1-s)^{q-2}ds \\
\leq & I_L^q(1+\gamma_0+\delta_0)r + \frac{M}{\Gamma(q+1)} + \frac{I^{q-1}L(1)}{q-1}(1+\gamma_0+\delta_0)r \\
& + \frac{M}{(q-1)\Gamma(q)} = \lambda r + \frac{M(2q-1)}{(q-1)\Gamma(q+1)} \leq \lambda r + (1-\lambda)r = r.
\end{aligned}
$$

It is declared that $F$ is a contraction mapping. For $x, y \in C$ and $t \in [0,1]$, by (3.8) and (H1), we have

$$
\begin{aligned}
&\|(Fx)(t)-(Fy)(t)\| \\
&\quad\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big\| f\big(s,x(s),(\phi x)(s),(\psi x)(s)\big) \\
&\quad\quad - f\big(s,y(s),(\phi y)(s),(\psi y)(s)\big)\big\| ds \\
&\quad\quad + \int_0^1 \frac{t^{q-1}(1-s)^{q-2}}{\Gamma(q)} \big\| f\big(s,x(s),(\phi x)(s),(\psi x)(s)\big) \\
&\quad\quad - f\big(s,y(s),(\phi y)(s),(\psi y)(s)\big)\big\| ds \\
&\quad\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \Big( L_1(s)\|x-y\| + L_2(s)\|\phi x-\phi y\| + L_3(s)\|\psi x-\psi y\|\Big) ds \\
&\quad + \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q)} \Big( L_1(s)\|x-y\| + L_2(s)\|\phi x-\phi y\| \\
&\quad + L_3(s)\|\psi x-\psi y\|\Big) ds \\
&\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \Big( L_1(s)+\gamma_0 L_2(s)+\delta_0 L_3(s)\Big)\|x-y\| ds \\
&\quad + \frac{1}{q-1}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \Big( L_1(s)+\gamma_0 L_2(s)+\delta_0 L_3(s)\Big)\|x-y\| ds \qquad (3.16)\\
&\le \big(I^q L_1(t)+\gamma_0 I^q L_2(t)+\delta_0 I^q L_3(t)\big)\|x-y\| \\
&\quad + \frac{1}{q-1}\big(I^{q-1}L_1(1)+\gamma_0 I^{q-1}L_2(1)+\delta_0 I^{q-1}L_3(1)\big)\|x-y\| \\
&\le (1+\gamma_0+\delta_0)\Big(I_L^q + \frac{I^{q-1}L(1)}{q-1}\Big)\|x-y\| \\
&= \lambda\|x-y\|.
\end{aligned}
$$

$\lambda < 1$ ensures that $F$ is contractive. Therefore, the conclusion of the theorem follows from the contraction mapping principle. □

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