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Approximation by Marcinkiewicz Means of Walsh-Kaczmarz-Fourier Series in the Hardy Space $H_{2/3}$

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Gát, Goginava and Nagy proved that the maximal operator $\sigma^{\kappa,*}$ of Marcinkiewicz-Fejér means of Walsh-Kaczmarz-Fourier series, is bounded from the dyadic Hardy space H_p into the space L_p for $p > 2/3$ [4]. Moreover, Goginava and Nagy showed that $\sigma^{\kappa,*}$ is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$ [8]. The main aim of this paper is to investigate the endpoint case $p = 2/3$. In the present work we give necessary and sufficient conditions for the convergence of Walsh-Kaczmarz-Marcinkiewicz means in terms of modulus of continuity on the Hardy space $H_{2/3}$.

Keywords: Walsh-Kaczmarz system, Marcinkiewicz means, maximal operator, two-dimensional system, modulus of continuity, Hardy space, convergence, Fourier series.

AMS Subject Classification: 42C10.

1. Introduction

In 1948, Šneider [17] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^\kappa(x)}{\log n} \geq C > 0$$

holds a.e. In 1974, Schipp [14] and Young [21] proved that the Walsh-Kaczmarz system is a convergence system. In 1981, Skvortsov [16] showed the uniform convergence of the Fejér means with respect to the Walsh-Kaczmarz system for any continuous functions f. Gát [2] proved, that the Walsh-Kaczmarz-Fejér means of an integrable function converge almost everywhere to the function. He showed that the maximal operator $\sigma^{\kappa,*}$ of Walsh-Kaczmarz-Fejér means is weak type $(1,1)$ and of type (p,p) for all $1 < p \leq \infty$. Gát's result was generalized by Simon [15], he showed that the maximal operator $\sigma^{\kappa,*}$ is of type (H_p, L_p) for $p > 1/2$. In the endpoint case $p = 1/2$ Goginava [5] proved that the maximal operator $\sigma^{\kappa,*}$ is not

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bounded from the dyadic Hardy space $H_{1/2}$ to the space $L_{1/2}$. Moreover, Weisz [24] showed that the maximal operator is of weak type $(H_{1/2}, L_{1/2})$.

In [5, 18] it was proved that the maximal operators $\widetilde{\sigma}_p^{\kappa,*}$ defined by

$$\widetilde{\sigma}_p^{\kappa,*} := \sup_{n\in\mathbb{N}} \frac{|\sigma_n^\kappa|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)},$$

(where $0 < p \leq 1/2$ and $[x]$ denotes the integer part of x) are bounded from the Hardy space H_p to the space L_p. Moreover, it was proved that the sequence $\{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)\}_{n=1}^{\infty}$ can not be weakened.

The second author [19] (see also [20]) found a necessary and sufficient condition for the convergence $\sigma_n^\kappa f \to f$ in H_p norm, in terms of modulus of continuity of the martingale $f \in H_p$.

In particular, it was proved that if

$$\omega_{H_p}\left(1/2^k, f\right) = o\left(2^{k(1/p-2)}\log^{2[1/2+p]}k\right), \text{ as } k \to \infty,$$

then

$$\|\sigma_n^\kappa f - f\|_{H_p} \to 0, \text{ as } n \to \infty, \quad (0 < p \leq 1/2)\,. \tag{1}$$

Moreover, there exists a martingale $f \in H_p$, $(0 < p \leq 1/2)$, for which

$$\omega\left(1/2^k, f\right)_{H_p} = O\left(2^{k(1/p-2)}\log^{2[1/2+p]}k\right), \qquad \text{as } k \to \infty$$

and

$$\|\sigma_n^\kappa f - f\|_p \nrightarrow 0, \qquad \text{as } n \to \infty.$$

In 1939, for the two-dimensional trigonometric Fourier partial sums $S_{j,j}f$ Marcinkiewicz [9] has proved that the means

$$\sigma_n f = \frac{1}{n}\sum_{j=1}^{n} S_{j,j}f$$

converge a.e. to f as $n \to \infty$ for any $f \in L\log L([0,2\pi]^2)$. Zhizhiashvili [25] improved this result for $f \in L_1([0,2\pi]^2)$.

In 2006, the a.e. convergence of Walsh-Kaczmarz-Marcinkiewicz means was proved by the first author [10]. He also proved that the maximal operator

$$\sigma^{\kappa,*}f := \sup_{n\in\mathbb{P}}|\sigma_n^\kappa f| = \sup_{n\in\mathbb{P}}\frac{1}{n}\left|\sum_{j=0}^{n} S_{j,j}^\kappa f\right|$$

is of weak type $(1,1)$ and of type (p,p) for all $1 < p \leq \infty$. In [4] it was proved that the maximal operator $\sigma^{\kappa,*}$ is bounded from the Hardy space H_p to the space L_p for $p > 2/3$. In the paper [8] Goginava and the first author showed that, $\sigma^{\kappa,*}$ is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$. This means that, it is interesting to discuss what does happen at the endpoint $p = 2/3$. Recently, it was showed in [11] that the maximal operator $\tilde{\sigma}^{\kappa,*} f := \sup_{n\in\mathbb{P}} \frac{|\sigma_n^\kappa f|}{\log^{3/2}(n+1)}$, is bounded from the Hardy space $H_{2/3}$ into the space $L_{2/3}$. Moreover, it was proved that the sequence $\{\log^{3/2}(n+1)\}_{n=1}^{\infty}$ can not be weakened. That is, the order of deviant behaviour of the n-th Walsh-Kaczmarz-Marcinkiewicz mean is exactly $\log^{3/2}(n+1)$ in the endpoint case $p = 2/3$. As a corollary we get that

$$\|\sigma_n^\kappa f\|_{2/3} \le c \log^{3/2}(n+1)\|f\|_{H_{2/3}} \tag{2}$$

for all $f \in H_{2/3}$.

The main aim of this paper is to continue the investigations at the endpoint $p = 2/3$. We give necessary and sufficient conditions for the convergence of Walsh-Kaczmarz-Marcinkiewicz means in terms of modulus of continuity on the Hardy space $H_{2/3}$. That is, we give the two-dimensional version of the result of the second author [19] (see equation (1)).

2. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis [1, 13].

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is the elements of $\mathbb{Z}_2$ are 0,1, and the group operation is the modulo 2 addition. Let every subset be open. The Haar measure on $\mathbb{Z}_2$ is given such that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are sequences of the form $x = (x_0, x_1, ..., x_k, ...)$ with coordinates $x_k \in \{0,1\}\,(k \in \mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) is the product measure and the topology is the product topology. The compact Abelian group G is called the Walsh group.

The dyadic intervals are defined by

$$I_0(x) := G,\ I_n(x) := I_n(x_0, ..., x_{n-1}) := \{y \in G :\ y = (x_0, ..., x_{n-1}, y_n, y_{n+1}, ...)\},$$

where $(x \in G, n \in \mathbb{N})$. They form a base for the neighbourhoods of G. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of G, and $I_n := I_n(0)\ (n \in \mathbb{N})$. Set $e_n := (0, ..., 0, 1, 0, ...) \in G$, the n-th coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$.

For $k \in \mathbb{N}$ the k-th Rademacher function is defined by

$$r_k(x) := (-1)^{x_k}, \quad (x = (x_0, x_1, ..., x_k, ...) \in G)$$

If $n \in \mathbb{N}$, then n can be expressed in the number system of base 2. That is, $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0,1\}$ $(i \in \mathbb{N})$. Denote the order of n by $|n| := \max\{j \in \mathbb{N} : n_j \ne 0\}$. We immediately have that $2^{|n|} \le n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the product system of Rademacher functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum\limits_{k=0}^{|n|-1} n_k x_k} \qquad (x \in G, n \in \mathbb{P}).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

It is well-known that the set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions are equal in dyadic blocks. Namely,

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{w_n : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbb{P}$ and $\kappa_0 = w_0$.

Define the transformation $\tau_A : G \to G$ by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...)$$

for $A \in \mathbb{N}$. Skvortsov [16] gave a relation between the function κ_n and w_n. Namely,

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in G). \tag{3}$$

The Dirichlet kernels are defined by

$$D_n^{\alpha} := \sum_{k=0}^{n-1} \alpha_k,$$

where $\alpha_n = w_n$ (for all $n \in \mathbb{P}$) or κ_n (for all $n \in \mathbb{P}$), $D_0^{\alpha} := 0$. The 2^n-th Dirichlet kernels have a closed form (see e.g. [13])

$$D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n, \\ 2^n, & \text{if } x \in I_n. \end{cases} \tag{4}$$

The norm (or quasi-norm) of the space $L_p(G^2)$ (for the simplicity we write L_p) is defined by

$$\|f\|_p := \left(\int_{G^2} |f(x_1, x_2)|^p \, d\mu(x_1, x_2) \right)^{1/p} \qquad (0 < p < \infty).$$

The space $weak - L_p$ consists of all measurable functions f for which

$$\|f\|_{weak-L_p} := \sup_{\lambda > 0} \lambda \mu (f > \lambda)^{1/p} < +\infty.$$

The σ-algebra generated by the 2-dimensional cube of measure 2^{-2k} will be denoted by F_k $(k \in \mathbb{N})$. Denote by $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ the one-parameter martingale with respect to $(F_n, n \in \mathbb{N})$ (for details see, e. g. [22, 23]). The maximal function of a martingale f is defined by $f^* = \sup_{n\in\mathbb{N}} \left|f^{(n)}\right|$. For $0 < p < \infty$ the Hardy martingale space $H_p(G^2)$ consists of all martingales for which $\|f\|_{H_p} := \|f^*\|_p < \infty$ (for simplicity we use the notation H_p).

The Kronecker product $(\alpha_{n,m} : n, m \in \mathbb{N})$ of two Walsh-(Kaczmarz) system is said to be the two-dimensional Walsh-(Kaczmarz) system. That is, $\alpha_{n,m}\left(x^1, x^2\right) = \alpha_n\left(x^1\right)\alpha_m\left(x^2\right)$.

If $f \in L_1$, then the number $\widehat{f}^\alpha(n,m) := \int_{G^2} f\alpha_{n,m}$ $\quad (n, m \in \mathbb{N})$ is said to be the (n,m)-th Walsh-(Kaczmarz)-Fourier coefficient of f. We can extend this definition for martingales in the usual way (see Weisz [22, 23]). Denote the (n,m)-th rectangular partial sum of the Walsh-(Kaczmarz)-Fourier series of a martingale f by $S^\alpha_{n,m}$. Namely,

$$S^\alpha_{n,m} f(x^1, x^2) := \sum_{k=0}^{n-1}\sum_{i=0}^{m-1} \widehat{f}^\alpha(k,i)\alpha_{k,i}(x^1, x^2).$$

The Marcinkiewicz-Fejér means of a martingale f are defined by

$$\sigma^\alpha_n f\left(x^1, x^2\right) := \frac{1}{n}\sum_{k=0}^{n} S^\alpha_{k,k} f(x^1, x^2).$$

The two-dimensional Dirichlet kernels and Marcinkiewicz-Fejér kernels are defined by

$$D^\alpha_{k,l}(x^1, x^2) := D^\alpha_k(x^1)D^\alpha_l(x^2), \quad K^\alpha_n(x^1, x^2) := \frac{1}{n}\sum_{k=0}^{n} D^\alpha_{k,k}(x^1, x^2).$$

During the proof of our main theorem we will use the following bellow estimation of Marcinkiewicz-Fejér kernels on the special indices, which was proved in [7]:

Lemma 2.1 (Goginava [7]) **:** *Let*

$$x^1 \in I_{4A}\left(0, ..., 0, x^1_{4m} = 1, 0, ..., 0, x^1_{4l} = 1, x^1_{4l+1}, ..., x^1_{4A-1}\right)$$

and

$$x^2 \in I_{4A}\left(0, ..., 0, x^2_{4l} = 1, x^1_{4l+1} ..., x^1_{4q-1}, 1 - x^1_{4q}, x^2_{4q+1}, ..., x^2_{4A-1}\right).$$

Then

$$n_{A-1}\left|K_{n_{A-1}}\left(x^1, x^2\right)\right| \geq 2^{4q+4l+4m-3},$$

where

$$n_A = 2^{4A} + 2^{4A-4} + ... + 2^4 + 2^0.$$

The concept of modulus of continuity in H_p $(0 < p \le 1)$ is given by

$$\omega_{H_p}\left(1/2^n, f\right) := \left\| f - S_{2^n,2^n} f \right\|_{H_p}.$$

Let the maximal operators $\sigma^{\kappa,*}$ and $\sigma^{\kappa,\#}$ be defined by

$$\sigma^{\kappa,*} f = \sup_{n\ge 1} \left|\sigma_n^{\kappa} f\right|, \quad \sigma^{\kappa,\#} f = \sup_{n\in\mathbb{N}} \left|\sigma_{2^n}^{\kappa} f\right|.$$

For the maximal operator $\sigma^{\kappa,\#}$ Gát, Goginava and Nagy [3] proved that the following is true:

Theorem 2.2 (Gát, Goginava and Nagy [3])**:** *The maximal operator $\sigma^{\kappa,\#}$ is bounded from the Hardy space H_p to the space L_p when $p > 1/2$.*

Later, it was shown that the maximal operator $\sigma^{\kappa,\#}$ is not bounded from the Hardy space H_p to the space L_p for $0 < p \le 1/2$. Moreover, the maximal operator $\sigma^{\kappa,\#}$ is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$ (see [6]).

For the martingale

$$f = \sum_{n=0}^{\infty} \left(f^{(n)} - f^{(n-1)} \right)$$

the conjugate transforms are defined as

$$\widetilde{f^{(t)}} = \sum_{n=0}^{\infty} r_n(t) \left(f^{(n)} - f^{(n-1)} \right),$$

where $t \in G$ is fixed. Note that $\widetilde{f^{(0)}} = f$. It is well-known (see [22]) that

$$\left\| \widetilde{f^{(t)}} \right\|_{H_p} = \|f\|_{H_p}, \quad \|f\|_{H_p}^p \sim \int_{[0,1)} \left\| \widetilde{f^{(t)}} \right\|_p^p dt. \tag{5}$$

As a consequence, we have that the conjugate transform of the n-th Marcinkievicz means of a function f coincides with the n-th Marcinkievicz means of the conjugate transform of f.

3. Formulation of main results

Our main result reads as follows:

Theorem 3.1 : *a) Let*

$$\omega\left(\frac{1}{2^k}, f\right)_{H_{2/3}} = o\left(\frac{1}{k^{3/2}}\right), \ \ as \ \ k \to \infty. \tag{6}$$

Then

$$\|\sigma_n^\kappa f - f\|_{H_{2/3}} \to 0, \text{ when } n \to \infty.$$

b) There exists a martingale $f \in H_{2/3}$, for which

$$\omega\left(\frac{1}{2^{2^k}}, f\right)_{H_{2/3}} = O\left(\frac{1}{2^{3k/2}}\right), \text{ as } k \to \infty$$

and

$$\|\sigma_n^\kappa f - f\|_{2/3} \nrightarrow 0 \text{ as } n \to \infty.$$

Proof: During the proof we follow the method of the second author in [19] (for dimension 1), but for the completeness we give the details. Moreover, the proof is based on the result of the first author [11] (see inequality (2)) and the method of Weisz [22] improved for conjugate transforms (see equality (5)). Combining (2) and (5) we have

$$\begin{aligned}
\|\sigma_n^\kappa f\|_{H_{2/3}}^{2/3} &= \int_{[0,1)} \left\|\widetilde{(\sigma_n^\kappa f)}^{(t)}\right\|_{2/3}^{2/3} dt = \int_{[0,1)} \left\|\sigma_n^\kappa \widetilde{f^{(t)}}\right\|_{2/3}^{2/3} dt \qquad (7)\\
&\le c\log(n+1)\int_{[0,1)} \left\|\widetilde{f^{(t)}}\right\|_{H_{2/3}}^{2/3} dt\\
&= c\log(n+1)\int_{[0,1)} \|f\|_{H_{2/3}}^{2/3} dt\\
&= c\log(n+1)\|f\|_{H_{2/3}}^{2/3}.
\end{aligned}$$

Let $2^N < n \le 2^{N+1}$.

$$\begin{aligned}
\|\sigma_n^\kappa f - f\|_{H_{2/3}}^{2/3} \le \left\|\sigma_n^\kappa f - \sigma_n^\kappa S_{2^N,2^N} f\right\|_{H_{2/3}}^{2/3} + \left\|\sigma_n^\kappa S_{2^N,2^N} f - S_{2^N,2^N} f\right\|_{H_{2/3}}^{2/3}\\
+ \left\|S_{2^N,2^N} f - f\right\|_{H_{2/3}}^{2/3}
\end{aligned}$$

The inequality (7) gives immediately

$$\begin{aligned}
\left\|\sigma_n^\kappa f - \sigma_n^\kappa S_{2^N,2^N} f\right\|_{H_{2/3}}^{2/3} + \left\|S_{2^N,2^N} f - f\right\|_{H_{2/3}}^{2/3} &\le \left\|\sigma_n^\kappa\left(S_{2^N,2^N} f - f\right)\right\|_{H_{2/3}}^{2/3}\\
&\quad + \left\|S_{2^N,2^N} f - f\right\|_{H_{2/3}}^{2/3}\\
&\le c\left(\log(n+1)+1\right)\omega^{2/3}\left(\frac{1}{2^N}, f\right)_{H_{2/3}}
\end{aligned}$$

For $\left\|\sigma_n^\kappa S_{2^N,2^N} f - S_{2^N,2^N} f\right\|_{H_{2/3}}^{2/3}$ we will show that

$$\left\|\sigma_n^\kappa S_{2^N,2^N} f - S_{2^N,2^N} f\right\|_{H_{2/3}}^{2/3} \leq \left(\frac{2^N}{n}\right)^{2/3} \left\|S_{2^N,2^N}\left(\sigma_{2^N}^\kappa f - f\right)\right\|_{H_{2/3}}^{2/3}$$
$$\leq \left\|\sigma_{2^N}^\kappa f - f\right\|_{H_{2/3}}^{2/3} \to 0, \text{ while } n \to \infty. \tag{8}$$

That is, we get that if

$$\omega\left(\frac{1}{2^n}, f\right)_{H_{2/3}} = o\left(\frac{1}{n^{3/2}}\right), \text{ as } n \to \infty,$$

then

$$\|\sigma_n^\kappa f - f\|_{H_{2/3}} \to 0, \text{ while } n \to \infty.$$

At last, we have to show inequality (8). We write

$$\begin{aligned}
\sigma_n^\kappa S_{2^N,2^N} f - S_{2^N,2^N} f &= \frac{1}{n}\sum_{k=0}^{2^N} S_{k,k}^\kappa S_{2^N,2^N} f + \frac{1}{n}\sum_{k=2^N+1}^{n} S_{k,k}^\kappa S_{2^N,2^N} f - S_{2^N,2^N} f \\
&= \frac{2^N}{n}\left(\sigma_{2^N}^\kappa f - S_{2^N,2^N} f\right) \\
&= \frac{2^N}{n}\left(S_{2^N,2^N}\sigma_{2^N}^\kappa f - S_{2^N,2^N} f\right) \\
&= \frac{2^N}{n} S_{2^N,2^N}\left(\sigma_{2^N}^\kappa f - f\right).
\end{aligned}$$

Combining (5), Theorem 2.2 and following the steps of estimation (7) we get inequality (8). It completes the proof of the first part of our theorem.

Now, we prove the second part of Theorem 3.1. We use the martingale constructed in [12]. We set

$$a_i(x^1, x^2) = 2^{2^i}\left(D_{2^{2^i+1}}(x^1) - D_{2^{2^i}}(x^1)\right)\left(D_{2^{2^i+1}}(x^2) - D_{2^{2^i}}(x^2)\right)$$

and

$$f^{(n)}(x^1, x^2) = \sum_{i=1}^{n} \frac{a_i(x^1, x^2)}{2^{3i/2}}.$$

In the paper [12], it is shown that the martingale f satisfies the conditions of the second part of Theorem 3.1. That is, $f \in H_{2/3}$ and $\omega\left(\frac{1}{2^{2^k}}, f\right)_{H_{2/3}} = O\left(\frac{1}{2^{3k/2}}\right),$ as $k \to \infty$.

Now, we show that

$$\|\sigma_n^\kappa f - f\|_{2/3} \nrightarrow 0 \text{ as } n \to \infty.$$

It is easy to calculate the Fourier coefficients

$$\widehat{f}^{\kappa}(i,j)=\begin{cases}\frac{2^{2^k}}{2^{3k/2}}, & \text{if } (i,j)\in\left\{2^{2^k},...,2^{2^k+1}-1\right\}^2,\ k=0,1,...\\ 0, & \text{if } (i,j)\notin\bigcup\limits_{k=0}^{\infty}\left\{2^{2^k},...,2^{2^k+1}-1\right\}^2.\end{cases} \tag{9}$$

Set $n_{2^{A-2}}=2^{4\cdot 2^{A-2}}+2^{4\cdot 2^{A-2}-4}+...+2^4+2^0=2^{2^A}+2^{2^A-4}+...+2^4+2^0$ as in Lemma 1.

$$\sigma^{\kappa}_{n_{2^{k-2}}}f-f=\frac{2^{2^k}\sigma^{\kappa}_{2^{2^k}}f}{n_{2^{k-2}}}+\frac{1}{n_{2^{k-2}}}\sum_{j=2^{2^k}+1}^{n_{2^{k-2}}}S^{\kappa}_{j,j}f-\frac{2^{2^k}f}{n_{2^{k-2}}}-\frac{n_{2^{k-2}-1}f}{n_{2^{k-2}}} \tag{10}$$

Let $2^{2^k}<j\leq n_{2^{k-1}}$. Using equations (3) and (9) we have

$$S^{\kappa}_{j,j}f\left(x^1,x^2\right)=S_{2^{2^k},2^{2^k}}f\left(x^1,x^2\right)+\sum_{v=2^{2^k}}^{j-1}\sum_{s=2^{2^k}}^{j-1}\widehat{f}(v,s)\kappa_{v,s}\left(x^1,x^2\right)$$

$$=S_{2^{2^k},2^{2^k}}f\left(x^1,x^2\right)+\frac{2^{2^k}}{2^{3k/2}}\sum_{v=0}^{j-2^{2^k}-1}\sum_{s=0}^{j-2^{2^k}-1}\kappa_{v+2^{2^k}}\left(x^1\right)\kappa_{s+2^{2^k}}\left(x^2\right)$$

$$=S_{2^{2^k},2^{2^k}}f\left(x^1,x^2\right)+\frac{2^{2^k}r_{2^k}\left(x^1+x^2\right)}{2^{3k/2}}\sum_{v=0}^{j-2^{2^k}-1}\sum_{s=0}^{j-2^{2^k}-1}w_v\left(\tau_{2^k}\left(x^1\right)\right)w_s\left(\tau_{2^k}\left(x^2\right)\right)$$

$$=S_{2^{2^k},2^{2^k}}f\left(x^1,x^2\right)+\frac{2^{2^k}r_{2^k}\left(x^1+x^2\right)D^w_{j-2^{2^k}}\left(\tau_{2^k}\left(x^1\right)\right)D^w_{j-2^{2^k}}\left(\tau_{2^k}\left(x^2\right)\right)}{2^{3k/2}}.$$

Hence, we write the following

$$\frac{1}{n_{2^{k-2}}}\sum_{j=2^{2^k}+1}^{n_{2^{k-2}}}S^{\kappa}_{j,j}f\left(x^1,x^2\right)=\frac{n_{2^{k-2}-1}S_{2^{2^k},2^{2^k}}f\left(x^1,x^2\right)}{n_{2^{k-2}}}$$

$$+\frac{2^{2^k}r_{2^k}\left(x^1+x^2\right)}{n_{2^{k-2}}2^{3k/2}}\sum_{j=1}^{n_{2^{k-2}-1}}D^w_j\left(\tau_{2^k}\left(x^1\right)\right)D^w_j\left(\tau_{2^k}\left(x^2\right)\right)$$

$$=\frac{n_{2^{k-2}-1}S_{2^{2^k},2^{2^k}}f\left(x^1,x^2\right)}{n_{2^{k-2}}}$$

$$+\frac{2^{2^k}r_{2^k}\left(x^1+x^2\right)n_{2^{k-2}-1}K^w_{n_{2^{k-2}-1}}\left(\tau_{2^k}\left(x^1\right),\tau_{2^k}\left(x^2\right)\right)}{n_{2^{k-2}}2^{3k/2}}.$$

Equality (10) yields

$$\begin{aligned}\|\sigma^{\kappa}_{n_{2^{k-2}}}f-f\|^{2/3}_{2/3}\geq \frac{c}{2^k}\|n_{2^{k-2}-1}K^w_{n_{2^{k-2}-1}}\circ(\tau_{2^k}\times\tau_{2^k})\|^{2/3}_{2/3}\\ -\left(\frac{2^{2^k}}{n_{2^{k-2}}}\right)^{2/3}\|\sigma^{\kappa}_{2^{2^k}}f-f\|^{2/3}_{2/3}\\ -\left(\frac{n_{2^{k-2}-1}}{n_{2^{k-2}}}\right)^{2/3}\|S_{2^{2^k},2^{2^k}}f-f\|^{2/3}_{2/3}.\end{aligned}$$

For a fixed 2^k we give a subset of G^2 as the following disjoint union:

$$G\times G\supseteq\bigcup_{m=1}^{2^k-3}\bigcup_{l=m+1}^{2^k-2}\bigcup_{q=l+1}^{2^k-1}\left(J^{m,l}_{2^k}\times L^{l,q}_{2^k}\right),$$

where

$$\begin{aligned}J^{m,l}_{2^k}:=\big\{&x^1\in G: x^1_{2^k-1}=...=x^1_{2^k-4m}=0,\\ &x^1_{2^k-4m-1}=1,x^1_{2^k-4m-2}=...=x^1_{2^k-4m-4l}=0,x^1_{2^k-4l-1}=1\big\}\end{aligned}$$

and

$$\begin{aligned}L^{l,q}_{2^k}:=\big\{&x^2\in G: x^2_{2^k-1}=...=x^2_{2^k-4l}=0,\\ &x^2_{2^k-4l-1}=1,x^1_{2^k-4l-2},...,x^1_{2^k-4q},x^2_{2^k-4q-1}=1-x^1_{2^k-4q-1}\big\}.\end{aligned}$$

Notice that, for any $\left(x^1,x^2\right)\in J^{m,l}_{2^k}\times L^{l,q}_{2^k}$, by the definition of τ_{2^k} and Lemma 1 we have

$$n_{_{2^{k-2}-1}}\left|K^w_{n_{2^{k-2}-1}}\left(\tau_{2^k}\left(x^1\right),\tau_{2^k}\left(x^2\right)\right)\right|\geq 2^{4q+4l+4m-3}.$$

This immediately yields

$$\begin{aligned}&\int_{G^2}(n_{_{2^{k-2}-1}}|K^w_{n_{2^{k-2}-1}}(\tau_{2^k}(x^1),\tau_{2^k}(x^2))|)^{2/3}d\mu(x^1,x^2)\\ &\geq c\sum_{m=1}^{2^{k-2}-3}\sum_{l=m+1}^{2^{k-2}-2}\sum_{q=l+1}^{2^{k-2}-1}\int_{J^{m,l}_{2^k}\times L^{l,q}_{2^k}}(n_{2^{k-2}-1}|K^w_{n_{2^{k-2}-1}}(\tau_{2^k}(x^1),\tau_{2^k}(x^2))|)^{2/3}d\mu(x^1,x^2)\\ &\geq c\sum_{m=1}^{2^{k-2}-3}\sum_{l=m+1}^{2^{k-2}-2}\sum_{q=l+1}^{2^{k-2}-1}\mu\left(J^{m,l}_{2^k}\times L^{l,q}_{2^k}\right)2^{(8q+8l+8m)/3}\end{aligned}$$

$$\geq c \sum_{m=1}^{2^{k-2}-3} \sum_{l=m+1}^{2^{k-2}-2} \sum_{q=l+1}^{2^{k-2}-1} 2^{(8q+8l+8m)/3} 2^{-4l} 2^{-4q}$$

$$\geq c \sum_{m=1}^{2^{k-2}-3} 2^{8m/3} \sum_{l=m+1}^{2^{k-2}-2} 2^{-4l/3} \sum_{q=l+1}^{2^{k-2}-1} 2^{-4q/3}$$

$$\geq c \sum_{m=1}^{2^{k-2}-3} 1 \geq c2^k.$$

By inequality (11) we have

$$\limsup_{k\to\infty} \|\sigma^{\kappa}_{n_{2^{k-2}}} f - f\|_{2/3} \geq c > 0.$$

The proof of Theorem 3.1 is complete. □

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