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On some Properties of Conjugate Fourier-Stieltjes Series

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A theorem of Ferenc Lukács [9] states that the partial sums of conjugate Fourier series of a periodic Lebesgue integrable function f diverge at the logarithmic rate at the points of first kind discontinuity of f.

The aim of this paper is to investigate analogous problems in terms of Fourier-Stieltjes series and Abel-Poisson means of the Fourier-Stieltjes series.

Keywords: Fourier-Stieltjes series, Bounded variation.

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1. Introduction

Let f be a 2π periodic Lebesgue integrable function. The Fourier trigonometric series of the function f is defined by

$$\frac{a_0}{2} + \sum_{i=1}^{\infty}(a_i \cos ix + b_i \sin ix), \tag{1}$$

where

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos ix dx \text{ and } b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin ix dx,$$

are the Fourier coefficients of f. The conjugate series of (1) is defined by

$$\sum_{i=1}^{\infty}(a_i \sin ix - b_i \cos ix). \tag{2}$$

Let $\tilde{S}_k(f;x)$ be the k-th partial sum of series (2). Lukács [9] proved the following theorem.

Theorem 1.1 : *If $f \in L(-\pi,\pi]$ and the finite limit*

$$\lim_{t\to 0+}[f(x+t) - f(x-t)] = d_x(f),$$

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exists at some point $x \in (-\pi, \pi]$ *then*

$$\lim_{k\to+\infty} \frac{\tilde{S}_k(f;x)}{\ln k} = -\frac{d_x(f)}{\pi}.$$

B. Golubov obtained a formula for the jump of a function of bounded p-variation at a given point in terms of derivatives of partial sums of its Fourier series.

R. Riad [13] proved an analogous theorem of the Lukács theorem in terms of the conjugate Walsh series.

G. Kvernadze, T. Hagstrom, H. Shapiro ([5]-[8]) investigate how to determined jumps for class of function generalized variation in terms of Jacobi polynomials, also they utilize the truncated Fourier series as a tool for the approximation of the points of discontinuities and the magnitudes of jumps of a 2π-periodic bounded function in terms of derivative of the partial sums, they also use integrals.

F. Móricz ([10], [11]) generalized Lukács's theorem in terms of the Abel-Poisson means and proved estimate of the partial derivative of the Abel-Poisson mean of an integrable function at those points where functions are smooth.

Pinsky [12] generalized Fourier partial sums by using a family of convolution operators with some classes of kernels.

Q. Shi and X. Shi [14] discuss the concentration factor methods for determination of jumps in terms of spectral data.

P. Zhou and S. Zhou [19] generalize Lukács theorem in terms of the linear operators which satisfy some certain conditions.

D. Yu, P. Zhou and S. Zhou [17] show how jumps can be determined by the higher order partial derivatives of the of its Abel-Poisson means.

The authors ([20], [21]) examine the analogous theorems for the generalized Cesáro means, introduced by Akhobadze ([1]-[3]), as well as positive regular linear means, and consider ([22], [23]) Lukás theorem for the functions and series introduced by Taberski ([15], [16]) as well as generalized Cesáro, positive regular linear and Abel-Poisson means.

2. Formulation of the results

Our interest is to study same tasks for Fourier-Stiltjes series. Let the function f be 2π periodic and have bounded variation on the $[-\pi;\pi]$, the Fourier coefficients are defined as

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos kx df(x) \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin kx df(x). \tag{3}$$

By $\tilde{S}_n(df;x)$ we define the n-th partial sum of Fourier-Stieltjes series of the f. Also define

$$\varphi_x(t) = f(x+t) + f(x-t) - 2f(x) - t \cdot (f'(x+) - f'(x-)).$$

Theorem 2.1 : *For any point x where there exist numbers $f'(x+)$, $f'(x-)$ and*

$$\overset{t}{\underset{0}{V}}(\varphi_x) = o(t), \qquad t \to 0+, \tag{4}$$

we have

$$\lim_{n\to+\infty} \frac{\tilde{S}_n(df;x)}{\ln n} = -\frac{f'(x+) - f'(x-)}{\pi}. \tag{5}$$

where $\overset{t}{\underset{0}{V}}(\varphi_x)$ is a variation of the function $\varphi_x(\cdot)$ on the set $[0;t]$.

It is natural to ask: is the analogue of the last statement valid for a Abel-Poisson summability method?

Abel-Poisson means of the conjugate Fourier-Stieltjes series is defined as

$$d\tilde{f}(r,x) = \sum_{k\in\mathbb{N}} r^k(a_k \sin kx - b_k \cos kx) = -\frac{1}{\pi}\int_0^\pi Q(r,x)d(f(x+t) + f(x-t)), \tag{6}$$

where

$$Q(r,t) = \sum_{k\in\mathbb{N}} r^k \sin kt = \frac{r\sin t}{(1-r)^2 + 4r\sin^2(t/2)}. \tag{7}$$

Now we examine the analogous of the theorem 2.1 for Abel-Poisson mean.

Theorem 2.2 : *For any point x where there exist numbers $f'(x+)$, $f'(x-)$ we have*

$$\lim_{r\to 1-} \frac{d\tilde{f}(r,x)}{\ln(1-r)} = \frac{f'(x+) - f'(x-)}{\pi}. \tag{8}$$

Note that in Theorem 2.2 we omit condition (4). It is natural to ask about condition (4). If the function f is absolutely continuous then the mentioned condition follows automatically, but in this case we provide the known results in the introduction. Arises a question: is condition (4) equivalent to absolutely continuity of function f?

Proposition 2.3 : *There exists function f of bounded variation which is not absolutely continuous but for which (4) holds.*

It is interesting if there is a possibility to replace (4) with a weaker condition. Our hypothesis is that (4) is the best option to guarantee (5).

3. Proofs

Proof (of Theorem 2.1):

$$\tilde{S}_n(df;x) = -\frac{1}{\pi}\int_0^{\pi} \tilde{D}_k(t)d(f(x+t)+f(x-t))$$

$$= -\frac{1}{\pi}\int_0^{\pi} \tilde{D}_k(t)d(f(x+t)+f(x-t)-2f(x)-t\cdot(f'(x+)-f'(x-)))$$

$$-\frac{f'(x+)-f'(x-)}{\pi}\int_0^{\pi} \tilde{D}_k(t)dt = -A_1(n)-A_2(n). \quad (9)$$

Let estimate $A_1(n)$. By (4) for every $\varepsilon > 0$ we can choose $\delta \equiv \delta(\varepsilon) > 0$ such that

$$\overset{t}{\underset{0}{V}}(\varphi_x) < \varepsilon \cdot t, \qquad t \in (0;t). \quad (10)$$

Let us choose n such that $1/n < \delta$. We get

$$A_1(n) = \frac{1}{\pi}\left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}\right)\tilde{D}_k(t)d\varphi_x(t)$$

$$= B_1(n)+B_2(n)+B_3(n). \quad (11)$$

Since $|\tilde{D}_n(t)| \le n$ for all t, (see [18, Ch. II, (5.11)]), by (10) we have

$$B_1(n) \le \frac{n}{\pi}\int_0^{1/n} d\overset{t}{\underset{0}{V}}(\varphi_x) < \frac{\varepsilon}{\pi}. \quad (12)$$

By estimation $|\tilde{D}_n(t)| \le 2/t$, $t \in (0;\pi]$, (see [18, Ch. II, (5.11)]) and integration by parts with respect to t and (10) we have

$$B_2(n) \le \frac{2}{\pi}\int_{1/n}^{\delta}\frac{1}{t}d\overset{t}{\underset{0}{V}}(\varphi_x) < \varepsilon + \frac{2}{\pi}\int_{1/n}^{\delta}\frac{1}{t^2}\overset{t}{\underset{0}{V}}(\varphi_x)dt < \varepsilon + \frac{2\varepsilon}{\pi}\int_{n}^{\delta}\frac{1}{t}dt = o(\ln n). \quad (13)$$

$$B_3(n) \leq \frac{2}{\pi}\int_{\delta}^{\pi}\frac{1}{t}d\bigvee_0^t(\varphi_x) \leq \frac{2}{\pi\delta}\int_{\delta}^{\pi}d\bigvee_0^t(\varphi_x) \leq \frac{2}{\pi\delta}\bigvee_0^{\pi}(\varphi_x) = O(1). \tag{14}$$

By (11)-(14) we get

$$\lim_{n\to+\infty} A_1(n)/\ln n = 0. \tag{15}$$

It is well known that

$$\int_0^{\pi}\tilde{D}_n(t)dt \simeq \ln n,$$

therefore we get

$$\lim_{n\to+\infty}\frac{\pi\cdot A_2(n)}{(f'(x+)-f'(x-))\cdot\ln n} = 1.$$

Combining (9), (15) and the last estimate we prove (5). □

Proof (of Theorem 2.2):

$$d\tilde{f}(r,x) = -\frac{1}{\pi}\int_0^{\pi}Q(r,x)d\varphi_x(t) - \frac{f'(x+)-f'(x-)}{\pi}\int_0^{\pi}Q(r,x)dt$$

$$= -D_1(r) - D_2(r). \tag{16}$$

Let us estimate $D_2(r)$. Consider

$$(2^{-1}\ln((1-r)^2+4r\sin^2(t/2)))' = \frac{r\sin t}{(1-r)^2+4r\sin^2(t/2)}.$$

Therefore we have

$$D_2(r) = \frac{f'(x+)-f'(x-)}{\pi}(2^{-1}\ln((1-r)^2+4r\sin^2(t/2)))|_0^{\pi}$$

$$= \frac{f'(x+)-f'(x-)}{\pi}(\ln(1+r)-\ln(1-r)) \simeq -\frac{f'(x+)-f'(x-)}{\pi}\ln(1-r). \tag{17}$$

By definition of the numbers $f'(x+)$ and $f'(x-)$, for any $\varepsilon > 0$ we choose $\delta = \delta(\varepsilon) > 0$ such that when $t \in [0;\delta]$ we have

$$|f(x+t)-f(x)-f'(x+)\cdot t|/t < \varepsilon/2, \quad |f(x-t)-f(x)+f'(x-)\cdot t|/t < \varepsilon/2.$$

Therefore

$$|f(x+t)-f(x)-f'(x+)\cdot t|<\varepsilon\cdot t/2,\quad |f(x-t)-f(x)+f'(x-)\cdot t|<\varepsilon\cdot t/2.$$

By definition of $\varphi_x(t)$ we have

$$|\varphi_x(t)|=|f(x+t)+f(x-t)-2f(x)-t\cdot(f'(x+)-f'(x-)|$$

$$\leq |f(x+t)-f(x)-t\cdot f'(x+)|+|f(x-t)-f(x)+t\cdot f'(x-)|<\varepsilon\cdot t. \tag{18}$$

Consider $D_1(r)$.

$$D_1(r)=\frac{1}{\pi}\int\limits_0^{\delta} Q(r,x)d\varphi_x(t)+\frac{1}{\pi}\int\limits_{\delta}^{\pi} Q(r,x)d\varphi_x(t)=E_1+E_2. \tag{19}$$

Integration by parts with respect to t we have

$$E_1=\frac{1}{\pi}Q(r,\delta)\varphi_x(\delta)-\frac{1}{\pi}\int_0^{\delta} Q'(r,t)\varphi_x(t)dt=F_1+F_2.$$

We have

$$F_1=O(1). \tag{20}$$

If β is a point such that $Q'(r,t)$ changes the sign on it, then by (18) and integrating by parts with respect to t we have

$$|F_2|\leq \varepsilon\int\limits_0^{\delta} t\cdot|Q'(r,t)|dt=\varepsilon\int\limits_0^{\beta} t\cdot Q'(r,t)dt-\varepsilon\int\limits_{\beta}^{\delta} t\cdot Q'(r,t)dt$$

$$=\varepsilon\beta Q(r,\beta)-\varepsilon\int\limits_0^{\beta} Q(r,t)dt-\varepsilon\delta Q(r,\delta)+\varepsilon\beta Q(r,\beta)+\varepsilon\int\limits_{\beta}^{\delta} Q(r,t)dt$$

$$\leq 2\varepsilon\beta Q(r,\beta)+2\varepsilon\int\limits_0^{\pi} Q(r,t)dt=o(1)+o(\ln(1-r))=o(\ln(1-r)). \tag{21}$$

On the other hand, by the representation of $Q(r,t)$ (see [18, Ch. III, (6.3)]) we have

$$\frac{r\sin t}{(1-r)^2+4r\sin^2(t/2)}\leq\frac{\sin t}{4\sin^2(t/2)}=\frac{1}{2}\cot(t/2).$$

Then

$$|E_2| \leq \frac{1}{2\pi} \int\limits_{\delta}^{\pi} \cot(t/2) d \overset{t}{\underset{0}{V}}(\varphi_x) \leq \frac{1}{\pi} \int\limits_{\delta}^{\pi} \frac{1}{t} d \overset{t}{\underset{0}{V}}(\varphi_x) \leq \frac{1}{\delta\pi} \int\limits_{\delta}^{\pi} d \overset{t}{\underset{0}{V}}(\varphi_x) = O(1). \quad (22)$$

Finally, by (16)-(22) we prove (8). Theorem 2.2 is proved. □

Proof (of Proposition 2.3): Let us introduce the set

$$A = \{1,\ 1/2,\ 1/3, ...\}\,. \quad (23)$$

Now choose some number $\gamma > 1$ and define the function f

$$f(x) = \begin{cases} n^{-\gamma} - (n+1)^{-\gamma} & \text{if } x = \frac{1}{n}, \\ 0 & \text{if } x \in [-\pi; \pi] \backslash A. \end{cases}$$

It is easy to see that f is not absolutely continuous, f has bounded variation and (4) is valid at the point 0. Let as consider $\varphi_0(t)$ because $f'(0) = 0$ when $t > 0$ we have $\varphi_0(t) = f(0+t) + f(0-t) - 2f(0) - t \cdot (f'(0+) - f'(0-)) = f(t)$, then

$$\overset{t}{\underset{0}{V}}(\varphi_0) = 2 \sum_{n \geq 1/t}^{\infty} \left(\frac{1}{n^\gamma} - \frac{1}{(n+1)^\gamma} \right) = 2t^\gamma = o(t), \quad t \to 0+\,.$$

Proposition 2.3 is proved. □

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References

[1] T. Akhobadze, *On generalized Cesáro summability of trigonometric Fourier series*, Bull. Georgian Acad. Sci., **170** (2004), 23-24

[2] T. Akhobadze, *On the convergence of generalized Cesáro means of trigonometric Fourier series I*, Acta Math. Hungar., **115** (2007), 59-78

[3] T. Akhobadze, *On the convergence of generalized Cesáro means of trigonometric Fourier series II*, Acta Math. Hungar., **115** (2007), 79-100

[4] B. I. Golubov, *Determination of the jumps of a function of bounded p-variation by its Fourier series*, Math. Notes, **12** (1972), 444-449

[5] G. Kvernadze, *Determination of the jumps of a bounded function by its Fourier series*, J. Approx. Theory, **92** (1998), 167-190

[6] G. Kvernadze, *Approximation of the singularities of a bounded function by the partial sums of its differentiated Fourier series*, Appl. Comput. Harmon. Anal., **11** (2001), 439-454

[7] G. Kvernadze, T. Hagstrom, H. Shapiro, *Locating discontinuities of a bounded function by the partial sums of its Fourier series*, J. Sci. Comput., **14** (1999), 301-327

[8] G. Kvernadze, T. Hagstrom, H. Shapiro, *Detecting the singularities of a function of Vp class by its integrated Fourier series*, Comput. Math. Appl., **39** (2000), 25-43

[9] F. Lukács, *Über die bestimmung des sprunges einer funktion aus ihrer Fourierreihe*, J, Reine Angew. Math., **150** (1920), 107-112

[10] F. Móricz, *Determination of jumps in terms of Abel-Poisson means*, Acta Math. Hungar., **98** (2003), 259-262

[11] F. Móricz, *Ferenc Lukács type theorems in terms of the Abel-Poisson mean of conjugate series*, Proc. Amer. Math. Soc., **131** (2003), 1234-1250

[12] M.A. Pinsky, *Inverse problems in Fourier analysis and summability theory*, Methods Appl. Anal., **11** (2004), 317-330

[13] R. Riad, *On the conjugate of the Walsh series*, Ann. Univ. Sci. Budapest. Rolando Eotvös, Sect. Math., **30** (1987), 69-76

[14] Q. Shi, X. Shi, *Determination of jumps in terms of spectral data*, Acta Math. Hungar., **110** (2006), 193-206

[15] R. Taberski, *Convergence of some trigonometric sums*, Demonstratio Mathematica, **5** (1973), 101-117

[16] R. Taberski, *On general Dirichlet's integrals*, Anales soc. Math Polonae, Series I: Prace mathematyczne **XVII** (1974), 499-512

[17] D.S. Yu, P. Zhou, S. Zhou, *On determinatioin of jumps in terms of Abel-Poisson mean of Fourier series*, J. Math. Anal. Appl., **341** (2008), 12-23

[18] A. Zigmund, *Trigonometric Series*, Cambridge University Press, **1** (1959)

[19] P. Zhou, S.P. Zhou, *More on determination of jumps*, Acta Math. Hungar., **118** (2008), 41-52

[20] Sh. Zviadadze, *On the Statement of F. Lukács*, Bull. Georgian Acad. Sci., **171** (2005), 411-412

[21] Sh. Zviadadze, *On a Lukács theorem for regular linear means of conjugate trigonometric Fourier series*, Bull. Georgian Acad. Sci., **173** (2006), 435-438

[22] Sh. Zviadadze, *On generalizations of a theorem of Ferenc Lukás*, Acta Math. Hungar., **122** (2009), 105-120

[23] Sh. Zviadadze, *On some properties conjugate trigonometric Fourier series*, Georgian Math. Journal, 20 (2013), 397-413