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Uniform and Pointwise Polynomial Inequalities in Regions without Cusps in the Weighted Lebesgue Space

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In this present work, we study the estimation of the modulus of algebraic polynomials in the bounded and unbounded quasidisks in the weighted Lebesgue space.

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1. Introduction and main results

Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; let $G \subset \mathbb{C}$ be a bounded Jordan region, with $0 \in G$ and the boundary $L := \partial G$ being a simple closed Jordan curve, $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$; $\Delta := \Delta(0,1) := \{w : |w| > 1\}$. Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$. For $R > 1$, let us set $L_R := \{z : \ |\Phi(z)| = R\}$, $G_R := intL_R$, $\Omega_R := extL_R$.

Let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $\{z_j\}_{j=1}^m$ be a fixed system of distinct points on the curve L located in the positive direction. For $1 < R < \infty$ and $z \in \overline{G}_R$, consider the so-called generalized Jacobi weight function $h(z)$ defined as follows:

$$h(z) := \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \tag{1}$$

where $\gamma_j > -1$ for all $j = \overline{1,m}$ (i.e. $j = 1, 2, ..., m$).

For any $p > 0$ we introduce:

$$\|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) \, |P_n(z)|^p \, d\sigma_z \right)^{1/p} < \infty, \tag{2}$$

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where σ_z is the two-dimensional Lebesgue measure,

$$\|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int\limits_L h(z) \left|P_n(z)\right|^p |dz| \right)^{1/p} < \infty, \tag{3}$$

when L is rectifiable and

$$\|P_n\|_{C(\overline{G})} := \max_{z\in\overline{G}} |P_n(z)| \,. \tag{4}$$

The well-known Bernstein-Walsh Lemma [20] says:

$$|P_n(z)| \leq |\Phi(z)|^n \|P_n\|_{C(\overline{G})}\,, \quad z \in \Omega. \tag{5}$$

Then, according to the above notations, (5) can be written as follows:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n \|P_n\|_{C(\overline{G})} \,. \tag{6}$$

Hence, setting $R = 1 + \frac{const.}{n}$, according to (6), we see that the $C-$norm of polynomials $P_n(z)$ in $\overline{G}_R$ and $\overline{G}$ are equivalent, i.e. the norm $\|P_n\|_{C(\overline{G})}$ increases with no more than a constant.

The same effect is observed for the $\|P_n\|_{\mathcal{L}_p(h,L)}$ from (3) according to the following estimate [7]:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq R^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}\,, \quad p > 0. \tag{7}$$

To give similar to (6) and (7) inequalities for the $\|P_n\|_{A_p(h,G)}$ from (2), we need to give some definitions and notations.

Let $B := B(0,1) := \{w : |w| < 1\}$, $z = \psi(w)$ be the univalent conformal mapping of B onto the G normalized by $\psi(0) = 0,\ \psi'(0) > 0$.

Definition 1.1: A bounded Jordan region G is called a κ-quasidisk, $0 \leq \kappa < 1$, if any conformal mapping ψ can be extended to a K-quasiconformal, $K = \frac{1+\kappa}{1-\kappa}$, homeomorphism of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$. In that case the curve $L := \partial G$ is called a κ-quasicircle. The region G (curve L) is called a quasidisk (quasicircle), if it is a κ-quasidisk (κ-quasicircle) with some $0 \leq \kappa < 1$.

We denote this class as $Q(\kappa)$, $0 \leq \kappa < 1$, and say that $L = \partial G \in Q(\kappa)$, if $G \in Q(\kappa)$, $0 \leq \kappa < 1$. Further, we denote $G \in Q$, if $G \in Q(\kappa)$ for some $0 \leq \kappa < 1$, and $L \in Q$, if $L \in Q(\kappa)$ for some $0 \leq \kappa < 1$.

Definition 1.2: We say that $G \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$, if $G \in Q(\kappa)$ and $L := \partial G$ is rectifiable.

We know well that a quasicircle may not even be locally rectifiable [9, p.104].

The Bernstein-Walsh-type estimate for $\|P_n\|_{A_p(h,G)}$, $p > 0$, for the quasidiscs G

with weight function $h(z)$, which is defined in (1) with $\gamma_j > -2$, is as follows ([2]):

$$\|P_n\|_{A_p(h,G_R)} \le c_2 R^{*^{n+\frac{1}{p}}} \|P_n\|_{A_p(h,G)}, \quad p > 0, \tag{8}$$

where $R^* := 1 + c_3(R-1)$ and $c_2 = c_2(G, c_3) > 0$, $c_3 = c_3(G) > 0$. Therefore, if we choose $R = 1 + \frac{1}{n}$, we see from (8) that the value $\|P_n\|_{A_p(h,\,\cdot)}$ of polynomials $P_n(z)$ in G_R and G are equivalent.

N. Stylianopoulos in [15] replaced the norm $\|P_n\|_{C(\overline{G})}$ with $\|P_n\|_{A_2(G)}$ on the right-hand side of (5) and so have a new version of the Bernstein-Walsh lemma as following:

Lemma A (N. Stylianopoulos). *Assume that L is a quasicircle and rectifiable. Then there exists a constant $c = c(L) > 0$, such that for any $n \in \mathbb{N}$ and $P_n \in \wp_n$*

$$|P_n(z)| \le c \frac{\sqrt{n}}{d(z,L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega, \tag{9}$$

where $d(z,L) := \inf\{|\zeta - z| : \ \zeta \in L\}$.

Analogous results to (9) with $\|P_n\|_{A_p(h,G)}$ on the right-hand side, for some different unbounded regions and weight function $h(z)$ (1) with $\gamma_j > -2$ were obtained in our works [3]-[4].

In this work, firstly, we study a problem similar to (9) for quasidisks and for generalized Jacobi weight function $h(z)$, defined as in (1) with $\gamma_j > -1$ for all $j = \overline{1,m}$, where $\|P_n\|_{\mathcal{L}_p(h,L)}$ was used instead of $\|P_n\|_{A_2(G)}$. Secondly, we obtain an estimate for $|P_n(z)|$ on the $\overline{G}$ as follows:

$$\|P_n\|_{C(\overline{G})} \le c\mu_n(G,h,p) \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{10}$$

where $c = c(G,p) > 0$ is a constant, independent of n and P_n. Note that, analogously to (10) estimates for different bounded regions exist in literature. Here we want to list some of them. The first result in this direction, in case $h(z) \equiv 1$ for $L = \{z : |z| = 1\}$ and $0 < p < \infty$ was found by Jackson [8]. Suetin [16], [17] investigated this problem with a sufficiently smooth Jordan curve. The estimate of (10) type for $0 < p < \infty$ and $h(z) \equiv 1$ when L is a rectifiable Jordan curve was investigated by Mamedhanov [11]. More general results of the (10)-type were obtained by Nikolskii [12, pp.122-133] and others for the polynomials of several variables.

Finally, we obtain the evaluation for $|P_n(z)|$ in the whole complex plane, depending on the geometrical properties of the region G and weigth function $h(z)$.

Now, we give the main results. We give some notations which we will use.

Throughout this paper, we denote by $c, c_0, c_1, c_2, ...$ positive constants and by $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ sufficiently small positive constants (in general, different in different relations) that depend on G in general and on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

Let the points $\{z_j\}_{j=1}^m$ be on the curve L in the positive direction and weight function $h(z)$ defined as in (1) for $\gamma_j > -1$ for all $j = \overline{1,m}$. For $k \le m$, $\alpha \in (\frac{1}{2}, 1]$, $p > 1$ and $\kappa \in [0,1)$, we set: $\Gamma_k := \{\gamma_j, \ j = \overline{1,k}\}$, $\Gamma_k^{(1)} := \{\gamma_k \in \Gamma_m : \gamma_k > \alpha(p-1)\}$ (with possible skips), $\Gamma_k^{(2)} := \{\gamma_k \in \Gamma_m : \gamma_k = \alpha(p-1)\}$, $\Gamma_k^{(3)} := \{\gamma_k \in \Gamma_m : -1 < \gamma_k < \alpha(p-1)\}$, $\widetilde{\Gamma}_k^{(1)} :=$

$\left\{\gamma_k \in \Gamma_m : \gamma_k > \frac{p-1}{1+\kappa}\right\}$ (with possible skips), $\widetilde{\Gamma}_k^{(2)} := \left\{\gamma_k \in \Gamma_m : \gamma_k = \frac{p-1}{1+\kappa}\right\}$, $\widetilde{\Gamma}_k^{(3)} := \left\{\gamma_k \in \Gamma_m : -1 < \gamma_k < \frac{p-1}{1+\kappa}\right\}$, $\gamma_k^* = \max\left\{\gamma_k : \gamma_k \in \Gamma_k^{(1)}\right\}$, $\gamma^* := \gamma_m^*$, $\widetilde{\gamma}_k^* = \max\left\{0; \gamma_k : \gamma_k \in \Gamma_k^{(1)}\right\}$, $\widetilde{\gamma}^* := \widetilde{\gamma}_m^*$.

1.1. *General case*

Here we provide the main results for quasidisks with some additional conditions.

Definition 1.3: We say that $G \in Q_\alpha^\beta$, if $G \in Q$ and $\Phi \in H^\alpha(\overline{G})$ and $\Psi \in H^\beta(\overline{\Delta})$ for some $\frac{1}{2} < \alpha \leq 1$ and $0 < \beta \leq 1$.

Definition 1.4: We say that $G \in \widetilde{Q}_\alpha^\beta$, if $G \in Q_\alpha^\beta$ and L is rectifiable.

We note, that the class Q_α^β and $\widetilde{Q}_\alpha^\beta$ is sufficiently large. This can be seen from the following:

Remark 1: a) If L is a smooth curve having continuous tangent line, then $G \in \widetilde{Q}_\alpha^\beta$ for all $\alpha, \beta < 1$.

b) If G is "L-shaped" region, then $G \in \widetilde{Q}_\alpha^\beta$ for $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{2}$.

c) If L is quasi-smooth, that is, for every pair z_1, $z_2 \in L$, if $s(z_1, z_2)$ represents the smaller of the lengths of the arcs joining z_1 to z_2 on L, there exists a constant $c > 1$, such that $s(z_1, z_2) \leq c\,|z_1 - z_2|$, then $G \in \widetilde{Q}_\alpha^\beta$ for $\alpha = \frac{1}{2}(1 - \frac{1}{\pi}\arcsin\frac{1}{c})^{-1}$ and $\beta = \frac{2}{(1+c)^2}$ [18], [19].

d) If L is "c-quasiconformal" (see, for example, [10]), then $G \in Q_\alpha^\beta$ for $\alpha = \frac{\pi}{2(\pi - \arcsin\frac{1}{c})}$ and $\beta = \frac{2(\arcsin\frac{1}{c})^2}{\pi(\pi - \arcsin\frac{1}{c})}$. Also, if L is an asymptotic conformal curve, then $G \in \widetilde{Q}_\alpha^\beta$ for all $\alpha, \beta < 1$ [10].

Now, we begin to give new results for the regions $G \in \widetilde{Q}_\alpha^\beta$.

Theorem 1.5: *Assume that $G \in \widetilde{Q}_\alpha^\beta$ for some $\frac{1}{2} < \alpha \leq 1$ and $0 < \beta \leq 1$; $P_n \in \wp_n$, $n \in \mathbb{N}$; $R_1 = 1 + \frac{\varepsilon_1}{n}$ and $h(z)$ defined as in (1). Then, for any $p > 1$*

$$|P_n(z)| \leq c_1 \frac{D_{n;1}}{d(z, L)} \|P_n\|_{\mathcal{L}_p(h,L)} |\Phi(z)|^{n+1}, \ z \in \Omega_{R_1}, \tag{11}$$

where $c_1 = c_1(G, \alpha, \beta, p) > 0$ and

$$D_{n;1} := \begin{cases} \sum\limits_{i=1}^{m} n^{\frac{\gamma_i}{\alpha p} - (1-\frac{1}{p})\beta}, \, if \ \gamma_i \in \Gamma_m^{(1)} \ for \ \ all \ i = \overline{1,m}, \\ (n^{1-\beta}\ln n)^{1-\frac{1}{p}}, \ if \ \gamma_i \in \Gamma_m^{(2)} \ for \ \ all \ i = \overline{1,m}, \\ n^{(1-\beta)(1-\frac{1}{p})}, \qquad if \gamma_i \in \Gamma_m^{(3)} \ for \ \ all \ i = \overline{1,m}. \end{cases}$$

Theorem 1.5 is local, that is, each term in the sum on the right side of $D_{n;1}$ shows the growth of $|P_n(z)|$, depending on the behavior of the weight function $h(z)$ and the boundary L in the neighborhood of a single point $z_j \in L$ for any $j = \overline{1,m}$.

Comparing the terms in the sum for each point $\{z_j\}$, $j = \overline{1,m}$, and using the above notations, we can obtain the following result of global character:

Theorem 1.6: *Assume that* $G \in \widetilde{Q}_{\alpha}^{\beta}$ *for some* $\frac{1}{2} < \alpha \leq 1$ *and* $0 < \beta \leq 1$; $P_n \in \wp_n$, $n \in \mathbb{N}$; $R_1 = 1 + \frac{\varepsilon_1}{n}$ *and* $h(z)$ *defined as in (1). Then, for any* $p > 1$

$$|P_n(z)| \leq c_2 \frac{D_{n;2}}{d(z,L)} \|P_n\|_{\mathcal{L}_p(h,L)} \; |\Phi(z)|^{n+1}, \quad z \in \Omega_{R_1}, \tag{12}$$

where $c_2 = c_2(G,p,m) > 0$ *and*

$$D_{n;2} := \begin{cases} n^{\frac{\gamma_k^*}{\alpha p} - (1-\frac{1}{p})\beta}, & \text{if there is at least one } \gamma_i \in \Gamma_k^{(1)} \\ & \text{for some} i = \overline{1,k}, \; k \leq m; \\ (n^{1-\beta} \ln n)^{1-\frac{1}{p}}, & \text{if there is at least one } \gamma_i \in \Gamma_k^{(2)} \setminus \Gamma_m^{(1)} \\ & \text{for some } i = \overline{1,k}, \; k \leq m; \\ n^{(1-\beta)(1-\frac{1}{p})}, & \text{if } \gamma_i \in \Gamma_m^{(3)} \text{ for all } i = \overline{1,m}. \end{cases}$$

In particular, in the case of having a single singular point on $z_1 \in L$ (i.e. $m = 1$), we obtain the following :

Corollary 1.7: *Assume that* $G \in \widetilde{Q}_{\alpha}^{\beta}$ *for some* $\frac{1}{2} < \alpha \leq 1$ *and* $0 < \beta \leq 1$; $P_n \in \wp_n$, $n \in \mathbb{N}$; $R_1 = 1 + \frac{\varepsilon_1}{n}$ *and* $h(z)$ *defined as in (1) for* $m = 1$. *Then, for any* $p > 1$, *we have*

$$|P_n(z)| \leq c_3 \frac{D_{n;3}}{d(z,L)} \|P_n\|_{\mathcal{L}_p(h,L)} |\Phi(z)|^{n+1}, \; z \in \Omega_{R_1}, \tag{13}$$

where $c_3 = c_3(G, \; p) > 0$ *and*

$$D_{n;3} := \begin{cases} n^{\frac{\gamma_1}{\alpha p} - (1-\frac{1}{p})\beta}, & \text{if } \gamma_1 \in \Gamma_1^{(1)}, \\ (n^{1-\beta} \ln n)^{1-\frac{1}{p}}, & \text{if } \gamma_1 \in \Gamma_1^{(2)}, \\ n^{(1-\beta)(1-\frac{1}{p})}, & \text{if } \gamma_1 \in \Gamma_1^{(3)}. \end{cases}$$

Now let's give a similar result for the finite region of $G \in \widetilde{Q}_{\alpha}^{\beta}$ for all $z \in \overline{G}$.

Theorem 1.8: *Assume that* $G \in \widetilde{Q}_{\alpha}^{\beta}$ *for some* $\frac{1}{2} < \alpha \leq 1$ *and* $0 < \beta \leq 1$; $P_n \in \wp_n$, $n \in \mathbb{N}$ *and* $h(z)$ *defined as in (1). Then, for any* $p > 1$ *there exists a constant* $c_4 = c_4(G) > 0$ *such that the following is fulfilled:*

$$\|P_n\|_{C(\overline{G})} \leq c_4 n^{\frac{\eta+p}{\alpha p} - (1-\frac{1}{p})\beta} \|P_n\|_{\mathcal{L}_p(h, \; L)}, \tag{14}$$

where $\eta := \max\{0; \gamma_k, \; k = \overline{1,m}\}$.

In particular, for $m = 1$ *the following is true:*

$$|P_n(z_1)| \leq c_4 n^{\frac{\gamma_1+p}{\alpha p} - (1-\frac{1}{p})\beta} \|P_n\|_{\mathcal{L}_p(h,L)}. \tag{15}$$

Now, combining Theorems 1.6 and 1.8, we obtain the growth of $|P_n(z)|$ on the whole complex plane:

Corollary 1.9: *Assume that* $G \in \widetilde{Q}_{\alpha}^{\beta}$ *for some* $\frac{1}{2} < \alpha \leq 1$ *and* $0 < \beta \leq 1$;

$P_n \in \wp_n,\ n \in \mathbb{N}$; $R_1 = 1 + \frac{\varepsilon_1}{n}$ *and* $h(z)$ *defined as in (1). Then, for any* $p > 1$

$$|P_n(z)| \leq c_5 \left\| P_n \right\|_{\mathcal{L}_p(h,L)} \begin{cases} n^{\frac{\eta+p}{\alpha p} - (1-\frac{1}{p})\beta}, & \text{for all } \mathrm{z} \in \overline{\mathrm{G}}, \\ \frac{|\Phi(z)|^{n+1}}{d(z,L)} D_{n;2}\ , & \text{for all } \mathrm{z} \in \Omega_{\mathrm{R}_1}, \end{cases} \tag{16}$$

where $c_5 = c_5(G,p,m) > 0$*,* $\eta := \max\{0; \gamma_k,\ k = \overline{1,m}\}$ *and* $D_{n;2}$ *is as in Theorem 1.6.*

We note that, according to Remark **1**, we can calculate α and β in the right hand side of estimates (11)-(16) for each region defined in Remark **1**, respectively.

1.2. *Special case*

In this part, we provide the main results for κ-quasidisks.

Theorem 1.10: *Assume that* $G \in \widetilde{Q}(\kappa)$ *for some* $0 \leq \kappa < 1$*;* $P_n \in \wp_n,\ n \in \mathbb{N}$; $R_1 = 1 + \frac{\varepsilon_1}{n}$ *and* $h(z)$ *defined as in (1). Then, for any* $p > 1$ *the following is true:*

$$|P_n(z)| \leq c_7 \frac{D_{n;4}}{d(z,L)} \left| \Phi(z) \right|^{n+1} \left\| P_n \right\|_{\mathcal{L}_p(h,L)}, \quad z \in \Omega_{R_1}, \tag{17}$$

where $c_7 = c_7(G,p) > 0$ *and*

$$D_{n;4} := \begin{cases} \sum_{i=1}^{m} n^{\frac{\gamma_i(1+\kappa)}{p} - (1-\frac{1}{p})(1-\kappa)}, & \text{if } \gamma_\mathrm{i} \in \widetilde{\Gamma}_\mathrm{m}^{(1)} \text{ for all } \mathrm{i} = \overline{1,\mathrm{m}}, \\ (n^\kappa \ln n)^{1-\frac{1}{p}}, & \text{if } \gamma_\mathrm{i} \in \widetilde{\Gamma}_\mathrm{m}^{(2)}, \text{ for all } \mathrm{i} = \overline{1,\mathrm{m}}, \\ n^{\kappa(1-\frac{1}{p})}, & \text{if } \gamma_\mathrm{i} \in \widetilde{\Gamma}_\mathrm{m}^{(3)}, \text{ for all } \mathrm{i} = \overline{1,\mathrm{m}}. \end{cases}$$

Theorem 1.11: *Assume that* $G \in \widetilde{Q}(\kappa)$ *for some* $0 \leq \kappa < 1$*.* $P_n \in \wp_n,\ n \in \mathbb{N}$; $R_1 = 1 + \frac{\varepsilon_1}{n}$ *and* $h(z)$ *defined as in (1). Then, for any* $p > 1$ *the following is true:*

$$|P_n(z)| \leq c_8 \frac{D_{n;5}}{d(z,L)} \left| \Phi(z) \right|^{n+1} \left\| P_n \right\|_{\mathcal{L}_p(h,L)}, \quad z \in \Omega_{R_1}, \tag{18}$$

where $c_8 = c_8(G,p) > 0$ *and*

$$D_{n;5} := \begin{cases} n^{\frac{\gamma_k^*(1+\kappa)}{p} - (1-\frac{1}{p})(1-\kappa)}, & \text{if at least one } \gamma_\mathrm{i} \in \widetilde{\Gamma}_\mathrm{m}^{(1)} \\ & \text{for some } \mathrm{i} = \overline{1,\mathrm{k}}, \mathrm{k} \leq \mathrm{m}, \\ (n^\kappa \ln n)^{1-\frac{1}{p}}, & \text{if at least one } \gamma_\mathrm{i} \in \widetilde{\Gamma}_\mathrm{k}^{(2)} \setminus \widetilde{\Gamma}_\mathrm{m}^{(1)} \\ & \text{for some } \mathrm{i} = \overline{1,\mathrm{k}},\ \mathrm{k} \leq \mathrm{m}; \\ n^{\kappa(1-\frac{1}{p})}, & \text{if } \gamma_\mathrm{i} \in \widetilde{\Gamma}_\mathrm{m}^{(3)}, \\ & \text{for all } \mathrm{i} = \overline{1,\mathrm{m}}. \end{cases}$$

Theorem 1.12: *Assume that* $G \in \widetilde{Q}(\kappa)$ *for some* $0 \leq \kappa < 1$; $h(z)$ *defined as in (1). Then, for any* $p > 1$ *there exists a constant* $c_9 = c_9(G,p) > 0$*, independent from* z *and* n *such that, for any* $n \in \mathbb{N}$ *and* $P_n \in \wp_n$ *the following is true:*

$$\left\| P_n \right\|_{C(\overline{G})} \leq c_9 n^{\frac{(\eta+1)(1+\kappa)}{p} + 2\kappa\left(1-\frac{1}{p}\right)} \cdot \left\| P_n \right\|_{\mathcal{L}_p(h,\ L)}, \tag{19}$$

where $\eta := \max\{0; \gamma_k,\ k = \overline{1,m}\}$.

Therefore, combining the estimates (17) with (19), we obtain an estimate on the growth of $|P_n(z)|$ in the whole complex plane:

Corollary 1.13: *Assume that* $G \in \widetilde{Q}(\kappa)$ *for some* $0 \le \kappa < 1$; $h(z)$ *defined as in (1). Then there exists a constant* $c_{10} = c_{10}(G,p) > 0$ *such that, for any* $n \in \mathbb{N}$ *and* $P_n \in \wp_n$ *we have*

$$|P_n(z)| \le c_{10} \|P_n\|_{\mathcal{L}_p(h,L)} \begin{cases} n^{\frac{(\eta+1)(1+\kappa)}{p}+2\kappa\left(1-\frac{1}{p}\right)}, & z \in \overline{G}, \\ \frac{|\Phi(z)|^{n+1}}{d(z,L)} D_{n;5}\ , & z \in \Omega_{R_1}, \end{cases} \tag{20}$$

where $D_{n;5}$ *is as in Theorem 1.11.*

Remark 2: As we see from (6), the estimates (14) and (19) are also true in $\overline{G}_R$ with another constant for some $R = 1 + \frac{c_{11}}{n}$. Therefore, if we choose

$$\widetilde{R} := \sup\left\{R > 1 : \|P_n\|_{C(\overline{G}_R)} \le c_{12} \|P_n\|_{C(\overline{G})}\right\}, \tag{21}$$

the estimate of (11)- (13), (17) and (18) will be significant for $z \in \Omega_{\widetilde{R}}$.

Thus, in order to estimate the $|P_n(z)|$ in $\overline{G}_{1+\frac{c}{n}}$, for some constant $c > 0$, we need to know a similar estimate for the $|P_n(z)|$ in $\overline{G}$, and then use (6) to achieve the desired estimate. Therefore, the estimates in (11)-(13), (17) and (18) are meaningful for points $z \in \Omega$, that are not close to the boundary (i.e. for the points $z \in \Omega_{\widetilde{R}}$, where $\widetilde{R}$ is defined as in (21)). Let's note that, for the estimate of $|P_n(z)|$ on the whole $\mathbb{C}$, it was possible to estimate $|P_n(z)|$ on $\overline{G}$ and Ω separately, after $\mathbb{C}$ was divided as $\mathbb{C} = \overline{G} \cup \Omega$. However, according to the above mentioned Remark **2**, we have divided $\mathbb{C}$ as $\mathbb{C} = \overline{G}_{\widetilde{R}} \cup \Omega_{\widetilde{R}}$ and estimated $|P_n(z)|$ for each $\overline{G}_{\widetilde{R}}$ and $\Omega_{\widetilde{R}}$. Finally, we have found the estimate of $|P_n(z)|$ in whole $\mathbb{C}$. That's why we can replace R_1, which was used in (11)-(13), (17) and (18), with $\widetilde{R}$ in (21).

Remark 3: Theorems 1.5, 1.6, 1.10, 1.11 and their corollaries can be formulated for $p > 1$. For simplicity of presentation of the facts, we give the result for the $p > 0$, similarly to Corollary1.7. Analogous results for the others cases can be given.

Corollary 1.14: *Assume that* $G \in \widetilde{Q}^{\beta}_{\alpha}$ *for some* $\frac{1}{2} < \alpha \le 1$ *and* $0 < \beta \le 1$; $P_n \in \wp_n$, $n \in \mathbb{N}$; $R_1 = 1 + \frac{\varepsilon_1}{n}$ *and* $h(z)$ *defined as in (1) for* $m = 1$. *Then, for any* $p > 0$ *we have*

$$|P_n(z)| \le c_{13} \frac{D_{n;6}}{d^{\frac{2}{p}}(z,L)} \|P_n\|_{\mathcal{L}_p(h,L)} |\Phi(z)|^{n+1},\ z \in \Omega_{R_1},$$

where $c_{13} = c_{13}(G,\ p) > 0$ *and*

$$D_{n;6} := \begin{cases} n^{\left(\frac{\gamma_1}{\alpha}-\beta\right)\frac{1}{p}}, & \text{if } \gamma_1 > \alpha, \\ (n^{1-\beta}\ln n)^{\frac{1}{p}}, & \text{if } \gamma_1 = \alpha, \\ n^{\frac{1-\beta}{p}}, & \text{if } \gamma_1 < \alpha. \end{cases}$$

1.3. *Sharpness of estimates*

The sharpness of estimates (11)-(20) for some special cases can be discussed by comparing them with the following results:

Remark 4: For any $n \in \mathbb{N}$ and $i = 1, 2$ there exist polynomials $P_n^{(i)} \in \wp_n$ and regions $G^i \subset \mathbb{C}$, such that

$$\left\| P_n^{(1)} \right\|_{C(\overline{G^1})} \geq c_{14} n^{\frac{1}{p}} \left\| P_n^{(1)} \right\|_{\mathcal{L}_p(L^1)} \tag{22}$$

and

$$\left| P_n^{(2)}(z) \right| \geq c_{15} \left| \Phi(z) \right|^{n+1} \left\| P_n^{(2)} \right\|_{\mathcal{L}_2(L^2)}, \ \forall z \in F \Subset C\overline{G^2}, \tag{23}$$

where $c_{14} = c_{14}(G^1) > 0$, $c_{15} = c_{15}(G^2) > 0$ and $L^i := \partial G^i, \ i = 1, 2$.

2. Some auxiliary results

For the $a > 0$ and $b > 0$, we use the expression "$a \preceq b$" (order inequality), if $a \leq cb$ and the expression "$a \asymp b$" means that $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

The following definitions of the K-quasiconformal curves are well known (see, for example, [5], [9, p.97] and [14]):

Definition 2.1: The Jordan arc (or curve) L is called K-quasiconformal ($K \geq 1$), if there is a K-quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).

We denote by $F(L)$ the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and let

$$K_L := \inf \left\{ K(f) : f \in F(L) \right\},$$

where $K(f)$ is the maximal dilatation of such a mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a K-quasiconformal curve, if $K_L \leq K$.

Remark 1: It is well known that when we are not interested in the coefficients of quasiconformality of the curve, then the definitions of "quasicircle" and "quasiconformal curve" are equivalent. But, when we are also interested in the coefficients of quasiconformality of the given curve, we will consider that if the curve L is K-quasiconformal, then it is κ-quasicircle with $\kappa = \frac{K^2-1}{K^2+1}$.

Following Remark **1**, for simplicity, we will use both terms, depending on the situation.

We note that, the region D in Definition 2.1 may be taken as $D \subset \mathbb{C}$ [14] or $D = \mathbb{C}$ [5],[9, p.97]. Case $D = \mathbb{C}$ gives the global definition of a K-quasiconformal arc or curve consequently. At the same time, we can consider the bounded domain $D \supset L$ as the neighborhood of the curve L. In this case, Definition 2.1 will be called local definition. This local definition has an advantage in determining the coefficients of quasiconformality for some simple arcs or curves.

Lemma 2.2: *[1] Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.
b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2},$$

where $0 < r_0 < 1$ is a constant, depending on G and $L_{r_0} := \{z = \psi(w) : |w| = r_0\}$.

Lemma 2.3: *Let $G \in Q(\kappa)$ for some $0 \leq \kappa < 1$. Then*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{1+\kappa},$$

for all $w_1, w_2 \in \overline{\Delta}$.

This fact follows from appropriate results for the estimate $|\Psi'(\tau)|$ (see, [13, p.287, Lemma 9.9] and [6, Th.2.8]):

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{24}$$

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of the points on L and the weight function $h(z)$ defined as (1):

Lemma 2.4: *Let L be a rectifiable Jordan curve; $h(z)$ as defined in (1). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$, we have*

$$\|P_n\|_{\mathcal{L}_p(h, L_R)} \leq R^{n+\frac{1+\gamma_*}{p}} \|P_n\|_{\mathcal{L}_p(h, L)}, \ p > 0. \tag{25}$$

Proof: Let us set:

$$f_n(t) := \prod_{j=1}^{m} \left[t(\Psi(\frac{1}{t}) - \Psi(w_j)\right]^{\frac{\gamma_j}{p}} t^n P_n\left(\Psi\left(\frac{1}{t}\right)\right) \cdot (\Psi'(\frac{1}{t}))^{\frac{1}{p}}.$$

Since f_n is analytic in B and L is rectifiable, f_n belongs to Hardy class H_p, and then, by Hardy's convexity theorem, we have

$$\int_{|t|=\frac{1}{R}} |f_n(t)|^p \frac{|dt|}{|t|} \leq \int_{|t|=1} |f_n(t)|^p |dt|,$$

which implies that

$$\int_{L_R} h(z)\left|P_n(z)\right|^p |dz|$$
$$=\int_{|w|=R}\prod_{j=1}^{m}\left|(\Psi(w)-\Psi(w_j)\right|^{\gamma_j}\left|P_n\left(\Psi(w)\right)\left(\Psi^{'}(w)\right)^{\frac{1}{p}}\right|^p |dw|$$
$$=\int_{|t|=\frac{1}{R}}\prod_{j=1}^{m}\left|(\Psi(\frac{1}{t})-\Psi(w_j)\right|^{\gamma_j}\left|P_n\left(\Psi\left(\frac{1}{t}\right)\right)(\Psi^{'}(\frac{1}{t}))^{\frac{1}{p}}\right|^p |dt|$$
$$\leq R^{np+1+\gamma_j}\int_{|t|=1}\prod_{j=1}^{m}\left|(\Psi(\frac{1}{t})-\Psi(w_j)\right|^{\gamma_j}\left|P_n\left(\Psi\left(\frac{1}{t}\right)\right)(\Psi^{'}(\frac{1}{t}))^{\frac{1}{p}}\right|^p |dt|$$
$$=R^{np+1+\gamma_j}\int_{L} h(z)\left|P_n(z)\right|^p |dz|.$$

We completed the proof of (25). □

Remark 2: In case of $h(z)\equiv 1$, the estimate (25) has been proved in [7].

3. Proof of Theorems

3.0.1. Proof of Theorem 1.5

Proof: For $0<\delta<\delta_0:=\frac{1}{4}\min\left\{|z_i-z_j|:i,j=\overline{1,m},\ i\neq j\right\}$, let $\Omega(z_j,\ \delta):=\Omega\cap\{z:|z-z_j|\leq\delta\}$, $\Omega(\delta):=\bigcup_{j=1}^{m}\Omega(z_j,\ \delta)$; $\Delta_j:=\Phi(\Omega(z_j,\ \delta))$. Let $w_j:=\Phi(z_j)$ and for $\varphi_j:=\arg w_j$, $j=\overline{1,m}$, we put $\Delta_j^{'}:=\left\{t=Re^{i\theta}:R>1,\ \frac{\varphi_{j-1}+\varphi_j}{2}\leq\theta<\frac{\varphi_j+\varphi_{j+1}}{2}\right\}$, where $\varphi_0\equiv\varphi_m$, $\varphi_1\equiv\varphi_{m+1}$; $\Omega^j:=\Psi(\Delta_j^{'})$, $L_R^j:=L_R\cap\overline{\Omega}^j$. Clearly, $\Omega=\bigcup_{j=1}^{m}\Omega^j$. $F^i:=\Phi(L^i)=\overline{\Delta^{'}}_i\cap\{\tau:|\tau|=1\}$, $F_R^i:=\Phi(L_R^i)=\overline{\Delta^{'}}_i\cap\{\tau:|\tau|=R\}$, $i=\overline{1,m}$.

Let $R_1:=1+\frac{\varepsilon_1}{n}$ for the sufficiently small $\varepsilon_1>0$. For any $z\in\Omega_{R_1}$, let us set:

$$G_n(z):=\frac{P_n(z)}{\Phi^{n+1}(z)}. \tag{26}$$

Cauchy's integral representation gives:

$$G_n(z)=-\frac{1}{2\pi i}\int_{L_{R_1}}G_n(\zeta)\frac{d\zeta}{\zeta-z},\ z\in\Omega_{R_1}.$$

Then,

$$|G_n(z)| \le \frac{1}{2\pi} \int\limits_{L_{R_1}} |G_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|} \le \frac{1}{2\pi d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)| \, |d\zeta| \, .$$

Replacing the variable $\tau = \Phi(\zeta)$, multiplying the numerator and determinator of the integral by $\prod\limits_{i=1}^{m} |\Psi(\tau) - \Psi(w_i)|^{\frac{\gamma_i}{p}} \; |\Psi'(\tau)|^{\frac{1}{p}}$, according to the Hölder inequality, we get:

$$|G_n(z)| \le \frac{1}{2\pi d(z, L_{R_1})} \int\limits_{|\tau|=R_1} |P_n(\Psi(\tau))| \left|\Psi'(\tau)\right| |d\tau| \tag{27}$$

$$\le \frac{1}{2\pi d(z, L_{R_1})} \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau) - \Psi(w_j)|^{\gamma_j} |P_n(\Psi(\tau))|^p \left|\Psi'(\tau)\right| |d\tau| \right)^{\frac{1}{p}}$$

$$\times \left(\int\limits_{|\tau|=R_1} \frac{|\Psi'(\tau)|}{\prod\limits_{j=1}^{m} |\Psi(\tau) - \Psi(w_j)|^{\frac{q\gamma_j}{p}}} |d\tau| \right)^{\frac{1}{q}}$$

$$=: \frac{1}{2\pi d(z, L_{R_1})} A_{n,p} \times B_{n,q}, \; \frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1,$$

since $(1 - \frac{1}{p})q = 1$. First of all, for the integral $A_{n,p}$ we find the following from Lemma 2.4

$$A_{n,p} \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \, . \tag{28}$$

In order to estimate the integral $B_{n,q}$, taking into account the estimate for the $|\Psi'|$ (24), we get

$$B_{n,q} = \left(\sum_{i=1}^{m} \int\limits_{F_{R_1}^i} \frac{|\Psi'(\tau)|}{\prod\limits_{i=1}^{m} |\Psi(\tau) - \Psi(w_i)|^{\gamma_j(q-1)}} |d\tau| \right)^{\frac{1}{q}} \tag{29}$$

$$=: \left(\sum_{i=1}^{m} B_{n,q}^i \right)^{\frac{1}{q}} \le \sum_{i=1}^{m} \left(B_{n,q}^i\right)^{\frac{1}{q}},$$

where

$$B_{n,q}^i := \int\limits_{F_{R_1}^i} \frac{|\Psi'(\tau)|}{\prod\limits_{i=1}^{m} |\Psi(\tau) - \Psi(w_i)|^{\gamma_j(q-1)}} |d\tau| \, , \; i = \overline{1, m}.$$

Since the points $w_i := \Phi(z_i)$ are distinct, for the integral $B_{n,q}^i$, we have:

$$B_{n,q}^i \asymp \int\limits_{F_{R_1}^i} \frac{|\Psi'(\tau)|}{|\Psi(\tau)-\Psi(w_i)|^{\gamma_i(q-1)}}\,|d\tau| := J(F_{R_1}^i). \tag{30}$$

For simplicity, we set $i=1$ and $F_{R_1}^1 = F_{R_1}^{11} \cup F_{R_1}^{12}$, where

$$\begin{aligned} F_{R_1}^{11} &: = \left\{\tau : \tau \in F_{R_1}^1,\ |\tau - w_1| < c_1 n^{-1}\right\}, \\ F_{R_1}^{12} &: = \left\{\tau : \tau \in F_{R_1}^1,\ c_1 n^{-1} \le |\tau - w_1| < c_2\right\}. \end{aligned}$$

Taking into consideration these designations, (30) can be written as:

$$\begin{aligned} J(F_{R_1}^1) &= \int\limits_{F_{R_1}^1} \frac{|\Psi'(\tau)|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}}\,|d\tau| \\ &= \int\limits_{F_{R_1}^{11}} \frac{|\Psi'(\tau)|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}}\,|d\tau| \\ &\quad + \int\limits_{F_{R_1}^{12}} \frac{|\Psi'(\tau)|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}}\,|d\tau| \\ &= : J(F_{R_1}^{11}) + J(F_{R_1}^{12}). \end{aligned} \tag{31}$$

We will consider the estimates of the following integral separately, according to the given possible values of $\gamma_1 :\ -1 < \gamma_1 \le 0$ and $\gamma_1 > 0$.

Let $\gamma_1 > 0$. For $\tau : |\tau| = R_1 > 1$ denote by $\tau^* := \frac{\tau}{R_1}$. Since $G \in \widetilde{Q}_\alpha^\beta$, then

$$d(\Psi(\tau), L) \asymp |\Psi(\tau) - \Psi(\tau^*)| \preceq |\tau - \tau^*|^\beta \preceq n^{-\beta}.$$

Then, according to Lemma 2.2, from (29) we obtain:

$$J(F^{1k}_{R_1}) = \int\limits_{F^{1k}_{R_1}} \frac{|\Psi'(\tau)|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}}\,|d\tau| \tag{32}$$

$$\asymp \int\limits_{F^{1k}_{R_1}} \frac{d(\Psi(\tau),L)}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}\,(|\tau|-1)}\,|d\tau|$$

$$\preceq \int\limits_{|\tau|=R_1} \frac{|\tau-\tau^*|^{\beta}}{|\tau-w_1|^{\frac{\gamma_1(q-1)}{\alpha}}\,(|\tau|-1)}\,|d\tau|$$

$$\preceq n^{1-\beta} \int\limits_{|\tau|=R_1} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma_1(q-1)}{\alpha}}}$$

$$\preceq \begin{cases} n^{\frac{\gamma_1(q-1)}{\alpha}-\beta}, & \gamma_1(q-1)>\alpha,\\ n^{1-\beta}\ln n, & \gamma_1(q-1)=\alpha,\\ n^{1-\beta}, & \gamma_1(q-1)<\alpha, \end{cases} \quad k=1,2.$$

Since $R_1 = 1+\frac{\varepsilon_1}{n}$ for the sufficiently small $\varepsilon_1>0$, then from Lemma 2.4 we have:

$$(B_{n,q})^q \preceq \begin{cases} n^{\frac{\gamma_1(q-1)}{\alpha}-\beta}, & \gamma_1(q-1)>\alpha,\\ n^{1-\beta}\ln n, & \gamma_1(q-1)=\alpha,\\ n^{1-\beta}, & \gamma_1(q-1)<\alpha. \end{cases} \tag{33}$$

Let $-1<\gamma_1\le 0$. Let us put $d(\Psi(\tau),L) := |\widetilde{z}-z^*|$, $\widetilde{z}\in L_{R_1}$, $z^*\in L$ and $\widetilde{w} := \Phi(\widetilde{z})$, $w^* := \Phi(z^*)$. Since $G\in \widetilde{Q}^{\beta}_{\alpha}$, we get $|\widetilde{z}-z^*| \preceq |\widetilde{w}-w^*|^{\beta}$, and in this case, from (29) we give:

$$J(F^{11}_{R_1}) = \int\limits_{F^{11}_{R_1}} \frac{|\Psi'(\tau)|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}}\,|d\tau| \tag{34}$$

$$\asymp \int\limits_{F^{11}_{R_1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)(q-1)}}{(|\tau|-1)^{1-\beta}}\,|d\tau|$$

$$\preceq n^{1-\beta} \int\limits_{F^{11}_{R_1}} |\tau-w_1|^{(-\gamma_1)(q-1)\beta}\,|d\tau|$$

$$\preceq n^{1-\beta+\gamma_1(q-1)\beta}\cdot mesF^{11}_{R_1} \preceq n^{\gamma_1(q-1)\beta-\beta} \preceq 1.$$

$$J(F_{R_1}^{12}) = \int\limits_{F_{R_1}^{12}} \frac{|\Psi'(\tau)|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}}\,|d\tau| \tag{35}$$

$$\asymp \int\limits_{F_{R_1}^{12}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)(q-1)}}{(|\tau|-1)^{1-\beta}}\,|d\tau|$$

$$\preceq n^{1-\beta}\cdot mesF_{R_1}^{12} \preceq n^{1-\beta}.$$

Therefore, in this case we get:

$$(B_{n,q})^q \preceq n^{1-\beta}.$$

Combining (26)-(35) we obtain:

$$|P_n(z)| \preceq \frac{|\Phi(z)|^{n+1}}{d(z,L_{R_1})}\,\|P_n\|_{\mathcal{L}_p(h,L)}$$

$$\times \begin{cases} \sum_{i=1}^m n^{\frac{\gamma_i}{\alpha p}-(1-\frac{1}{p})\beta}, & \text{if } \gamma_i > \alpha(p-1), \text{ for all } i=\overline{1,m}, \\ \left(n^{1-\beta}\ln n\right)^{1-\frac{1}{p}}, & \text{if } \gamma_i = \alpha(p-1), \text{ for all } i=\overline{1,m}, \\ n^{(1-\beta)(1-\frac{1}{p})}, & \text{if } -1<\gamma_i<\alpha(p-1), \text{ for all } i=\overline{1,m}, \end{cases} \quad z\in\Omega_{R_1}.$$

On the other hand, according to Lemma 2.2, we obtain that

$$d(z,L_{R_1}) \succeq d(z,L) \text{ for all } z\in\Omega_{\widetilde{R}}, \tag{36}$$

$R_1 := 1+\frac{\varepsilon_1}{n}$ for the sufficiently small $\varepsilon_1>0$ and $\widetilde{R}$ defined as in (21), and the proof is completed. □

3.0.2. Proof of Theorem 1.8

Proof: Let $R=1+\frac{c_1}{n}$. Let $\|P_n\|_{C(\overline{G})} = |P_n(z')|$, $z'\in L$ and let's put $w' := \Phi(z')$. Then, there exists $z_{i_0}\in\{z_i\}_{i=1}^m$ such that $|z'-z_{i_0}|\le\delta\le|z'-z_i|$, for some $\delta>0$ and $i=1,2,..i_0-1,i_0+1,...,m$. For simplicity, let us take $i_0=1$, $w_1=\Phi(z_1)$. Cauchy's integral representation for the bounded region G_R gives:

$$P_n(z') = \frac{1}{2\pi i}\int\limits_{L_R} P_n(\zeta)\frac{d\zeta}{\zeta-z'}\ ,\ z\in G_R.$$

Replacing the variable $\tau=\Phi(\zeta)$, multiplying the numerator and determinator of the integrant by $\prod_{i=1}^m |\Psi(\tau)-\Psi(w_i)|^{\frac{\gamma_i}{p}}\ |\Psi'(\tau)|^{\frac{1}{p}}$, according to the Hölder inequality, we get:

$$
\begin{aligned}
\left|P_n\left(z'\right)\right| &\leq \frac{1}{2\pi}\int\limits_{L_R}\left|P_n\left(\zeta\right)\right|\frac{|d\zeta|}{|\zeta-z'|}\\
&=\frac{1}{2\pi}\int\limits_{|\tau|=R}\frac{\left|P_n\left(\Psi(\tau)\right)\right|\left|\Psi'(\tau)\right|}{\left|\Psi(\tau)-\Psi(w')\right|}\left|d\tau\right|\\
&\leq\frac{1}{2\pi}\left(\int\limits_{|\tau|=R}\prod_{j=1}^{m}\left|\Psi(\tau)-\Psi(w_j)\right|^{\gamma_j}\left|P_n\left(\Psi(\tau)\right)\right|^p\left|\Psi'(\tau)\right|\left|d\tau\right|\right)^{\frac{1}{p}}\\
&\quad\times\left(\int\limits_{|\tau|=R}\frac{\left|\Psi'(\tau)\right|^{\left(1-\frac{1}{p}\right)q}}{\prod\limits_{j=1}^{m}\left|\Psi(\tau)-\Psi(w_j)\right|^{\frac{q\gamma_j}{p}}\left|\Psi(\tau)-\Psi(w')\right|^q}\left|d\tau\right|\right)^{\frac{1}{q}}\\
&=\frac{1}{2\pi}\left(\int\limits_{L_R}h(\zeta)\left|P_n\left(\zeta\right)\right|^p\left|d\zeta\right|\right)^{\frac{1}{p}} \qquad (37)\\
&\quad\times\left(\int\limits_{|\tau|=R}\frac{\left|\Psi'(\tau)\right|}{\prod\limits_{j=1}^{m}\left|\Psi(\tau)-\Psi(w_j)\right|^{\gamma_j(q-1)}\left|\Psi(\tau)-\Psi(w')\right|^q}\left|d\tau\right|\right)^{\frac{1}{q}}\\
&=\frac{1}{2\pi}J_{n,1}\times J_{n,2},
\end{aligned}
$$

where

$$
\begin{aligned}
J_{n,1} &:= \left\|P_n\right\|_{\mathcal{L}_p(h,L_R)},\\
(J_{n,2})^q &:= \int\limits_{|\tau|=R}\frac{\left|\Psi'(\tau)\right|}{\prod\limits_{j=1}^{m}\left|\Psi(\tau)-\Psi(w_i)\right|^{\gamma_i(q-1)}\left|\Psi(\tau)-\Psi(w')\right|^q}\left|d\tau\right|.
\end{aligned}
$$

Then, from Lemma 2.4 we have:

$$
\left|P_n\left(z'\right)\right| \preceq J_{n,1}\cdot J_{n,2} \preceq \left\|P_n\right\|_{\mathcal{L}_p(h,\ L)}\cdot J_{n,2}. \qquad (38)
$$

We begin to estimate the integral $J_{n,2}$. From (24) and

$$
d(\Psi\left(\tau\right),L) \preceq |\tau|-1^{\beta},\ \ \left|\Psi(\tau)-\Psi(\mathrm{w}')\right| \succeq \left|\tau-\mathrm{w}'\right|^{\frac{1}{\alpha}}, \qquad (39)
$$

since $G\in\widetilde{Q}_{\alpha}^{\beta}$, and taking into consideration that the points $w_i := \Phi(z_i)$ are distinct, we have:

$$\begin{aligned}(J_{n,2})^q &\preceq \int\limits_{|\tau|=R} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}\,|\tau-w'|^{\frac{q}{\alpha}}\,(|\tau|-1)^{1-\beta}} \\ &= \int\limits_{F^{11}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}\,|\tau-w'|^{\frac{q}{\alpha}}\,(|\tau|-1)^{1-\beta}} \\ &\quad+ \int\limits_{F^{12}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}\,|\tau-w'|^{\frac{q}{\alpha}}\,(|\tau|-1)^{1-\beta}} \\ &=: \left(J^1_{n,2}\right)^q + \left(J^2_{n,2}\right)^q,\end{aligned} \tag{40}$$

where $F^{11} := \{\tau : |\tau| = R, |\tau - w_1| \le \delta\ \}$; $F^{12} := \{\tau : |\tau| = R, |\tau - w_1| > \delta\ \}$. So, we will consider the estimates of the integral (40) according to the possible values of γ_1.

Case 1. Let $\gamma_1 > 0$.

From (39) and (40) we obtain:

$$\begin{aligned}\left(J^1_{n,2}\right)^q &:= \int\limits_{F^{11}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}\,|\tau-w'|^{\frac{q}{\alpha}}\,(|\tau|-1)^{1-\beta}} \\ &\preceq n^{1-\beta} \int\limits_{F^{11}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma_1(q-1)}{\alpha}}\,|\tau-w'|^{\frac{q}{\alpha}}}.\end{aligned}$$

Denote by

$$\mathcal{F}^1_1 := \left\{\tau \in F^{11} : |\tau - w_1| < \left|\tau - w'\right|\right\}$$

and

$$\mathcal{F}^1_2 := \left\{\tau \in F^{11} : |\tau - w_1| \ge \left|\tau - w'\right|\right\}.$$

Then

$$\left(J^1_{n,2}\right)^q \preceq n^{1-\beta} \int\limits_{\mathcal{F}^1_1} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma_1(q-1)}{\alpha}+\frac{q}{\alpha}}} + n^{1-\beta} \int\limits_{\mathcal{F}^1_2} \frac{|d\tau|}{|\tau-w'|^{\frac{\gamma_1(q-1)}{\alpha}+\frac{q}{\alpha}}} \preceq n^{\frac{\gamma_1(q-1)}{\alpha}+\frac{q}{\alpha}-\beta}, \tag{41}$$

for any $q > 1,\ \frac{1}{2} < \alpha \le 1,\ 0 < \beta \le 1$, and

$$\begin{aligned}\left(J^2_{n,2}\right)^q &:= \int\limits_{F^{12}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}\,|\tau-w'|^{\frac{q}{\alpha}}\,(|\tau|-1)^{1-\beta}} \\ &\preceq n^{1-\beta} \int\limits_{F^{12}} \frac{|d\tau|}{|\tau-w'|^{\frac{q}{\alpha}}} \preceq n^{\frac{q}{\alpha}-\beta}.\end{aligned} \tag{42}$$

since $|\Psi(\tau) - \Psi(w_1)| \succeq 1$ according to Lemma 2.3.

Therefore, in case $\gamma_1 > 0$, from (40), (41) and (42), for any $q > 1$, $\frac{1}{2} < \alpha \leq 1$, $0 < \beta \leq 1$ we get:

$$J_{n,2} \preceq n^{\frac{\gamma_1(q-1)+q}{q\alpha} - \frac{\beta}{q}}. \tag{43}$$

Case 2. Let $-1 < \gamma_1 \leq 0$. In this case, repeating the discussion in all integrals of Case 1, we obtain estimates for integrals $J_{n,2}^k$, $k = 1, 2$, analogous to (41)-(42):

$$\left(J_{n,2}^1\right)^q = \int_{F^{11}} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma_1)(q-1)}\,|d\tau|}{|\tau - w'|^{\frac{q}{\alpha}}\,(|\tau| - 1)^{1-\beta}} \tag{44}$$
$$\preceq n^{1-\beta} \int_{F^{11}} \frac{|d\tau|}{|\tau - w'|^{\frac{q}{\alpha}}} \preceq n^{\frac{q}{\alpha}-\beta},$$

and

$$\left(J_{n,2}^2\right)^q \preceq \int_{F^{12}} \frac{|d\tau|}{(|\tau| - 1)^{1-\beta}} \preceq n^{\frac{q}{\alpha}-\beta}, \tag{45}$$

for any $q > 1$, $\frac{1}{2} < \alpha \leq 1$, $0 < \beta \leq 1$. Therefore, in this case, from (40), (44) and (45), for any $q > 1$, $\frac{1}{2} < \alpha \leq 1$, $0 < \beta \leq 1$, we obtain:

$$J_{n,2} \preceq n^{\frac{1}{\alpha} - \frac{\beta}{q}}. \tag{46}$$

From (43), (46) and (38), we find the following:

$$\left|P_n\left(z'\right)\right| \preceq n^{\frac{\eta+1}{\alpha p} + (1-\frac{1}{p})(\frac{1}{\alpha}-\beta)} \cdot \|P_n\|_{\mathcal{L}_p(h,\ L)},$$

where $\eta := \max\{0; \gamma_k,\ k = \overline{1, m}\}$, and the proof is completed. □

3.0.3. Proof of Theorem 1.10

Proof: We are going to track the proof of Theorem 1.5. However, instead of (39) we are going to use the following estimates (47), that are obtained from Lemma 2.3 and (24) for the region $G \in Q(\kappa)$:

$$\left|\Psi'(\tau)\right| \preceq \frac{1}{(|\tau| - 1)^{\kappa}}, \tag{47}$$
$$d(\Psi(\tau), L) \preceq (|\tau| - 1)^{1-\kappa},\ \left|\Psi(\tau) - \Psi(w')\right| \succeq \left|\tau - w'\right|^{1+\kappa}.$$

Let $\gamma_1 > 0$. Since $G \in Q(\kappa)$, for arbitrary $z^* \in L$ such that $d(\Psi(\tau), L) = |\Psi(\tau) - z^*|$ and $w^* := \Phi(z^*)$, we have:

$$d(\Psi(\tau), L) = |\Psi(\tau) - z^*| \preceq |\tau - w^*|^{1-\kappa}.$$

According to (47), from (32), we obtain:

$$
\begin{aligned}
J(F_{R_1}^{1k}) &= \int\limits_{F_{R_1}^{1k}} \frac{|\Psi'(\tau)|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}}\,|d\tau| \\
&\preceq \int\limits_{F_{R_1}^{1k}} \frac{|d\tau|}{|\tau-w_1|^{\gamma_1(q-1)(1+\kappa)}\,(|\tau|-1)^{\kappa}} \\
&\preceq n^{\kappa} \int\limits_{|\tau|=R_1} \frac{|d\tau|}{|\tau-w_1|^{\gamma_1(q-1)(1+\kappa)}} \\
&\preceq \begin{cases} n^{\gamma_1(q-1)(1+\kappa)+\kappa-1}, & \gamma_1(q-1)(1+\kappa)>1, \\ n^{\kappa}\ln n, & \gamma_1(q-1)(1+\kappa)=1, \quad k=1,2. \\ n^{\kappa}, & \gamma_1(q-1)(1+\kappa)<1, \end{cases}
\end{aligned}
$$

Since $R_1 := 1+\frac{\varepsilon_1}{n}$ for the sufficiently small $\varepsilon_1>0$, in this case, from Lemma 2.4, we have:

$$
(B_{n,q}^{1})^{q} \preceq \begin{cases} n^{\gamma_1(q-1)(1+\kappa)+\kappa-1}, & \gamma_1(q-1)(1+\kappa)>1, \\ n^{\kappa}\ln n, & \gamma_1(q-1)(1+\kappa)=1, \\ n^{\kappa}, & \gamma_1(q-1)(1+\kappa)<1. \end{cases} \tag{48}
$$

Let $-1<\gamma_1\leq 0$. Let us put $d(\Psi_R(\tau),L) := |\widetilde{z}-z^{*}|,\ \widetilde{z}\in L_{R_1}$ and $\widetilde{w} := \Phi(\widetilde{z}),\ w^{*} := \Phi(z^{*})$. Since $G\in Q(\kappa)$, according to (47), we get $|\widetilde{z}-z^{*}| \preceq |\widetilde{w}-w^{*}|^{1-\kappa}$, and in this case (32) gives:

$$
\begin{aligned}
J(F_{R_1}^{11}) &= \int\limits_{F_{R_1}^{11}} \frac{|\Psi'(\tau)|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}}\,|d\tau| \\
&\asymp \int\limits_{F_{R_1}^{11}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)(q-1)}}{(|\tau|-1)^{\kappa}}\,|d\tau| \\
&\preceq n^{\kappa} \int\limits_{F_{R_1}^{11}} |\tau-w_1|^{(-\gamma_1)(q-1)(1-\kappa)}\,|d\tau| \\
&\preceq n^{\gamma_1(q-1)(1-\kappa)+\kappa}\cdot mesF_{R_1}^{11} \\
&\preceq n^{\gamma_1(q-1)(1-\kappa)+\kappa-1} \preceq 1.
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
J(F_{R_1}^{12}) &\asymp \int\limits_{F_{R_1}^{12}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)(q-1)}}{(|\tau|-1)^{\kappa}}\,|d\tau| \\
&\preceq n^{\kappa}\cdot mesF_{R_1}^{12} \preceq n^{\kappa}.
\end{aligned} \tag{50}
$$

Therefore,

$$(B^1_{n,q})^q \preceq n^{\kappa}. \tag{51}$$

Combining (26)-(31) with (47)-(51), we obtain:

$$|P_n(z)| \preceq \frac{|\Phi(z)|^{n+1}}{d(z, L_{R_1})} \|P_n\|_{\mathcal{L}_p(h,L)}$$

$$\times \begin{cases} \sum_{i=1}^m n^{\frac{\gamma_i(1+\kappa)}{p} - (1-\frac{1}{p})(1-\kappa)}, & \text{if } \gamma_\mathrm{i} > \frac{1}{1+\kappa}(\mathrm{p}-1) \ \text{ for all } \ \mathrm{i} = \overline{1,\mathrm{m}}; \\ (n^\kappa \ln n)^{1-\frac{1}{p}}, & \text{if } \gamma_\mathrm{i} = \frac{1}{1+\kappa}(\mathrm{p}-1), \ \text{ for all } \ \mathrm{i} = \overline{1,\mathrm{m}}; \\ n^{\kappa(1-\frac{1}{p})}, & \text{if } -1 < \gamma_\mathrm{i} < \frac{1}{1+\kappa}(\mathrm{p}-1), \ \text{ for all } \ \mathrm{i} = \overline{1,\mathrm{m}}, \end{cases} \quad z \in \Omega_{R_1},$$

and we complete the proof. □

3.0.4. Proof of Theorem 1.12

Proof: This time we will track the proof of Theorem 1.8. However, we are going to use the estimates in (47) instead of (39). Then, analogously to (40)-(46), we obtain:

a) for $\gamma_1 > 0$;

$$\begin{aligned}(J_{n,2})^q &\preceq \int\limits_{|\tau|=R} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)} |\tau - w'|^{q(1+\kappa)} (|\tau|-1)^\kappa} \\ &\preceq \int\limits_{F^{11}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)} |\tau - w'|^{q(1+\kappa)} (|\tau|-1)^\kappa} \\ &\quad + \int\limits_{F^{12}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)} |\tau - w'|^{q(1+\kappa)} (|\tau|-1)^\kappa}\end{aligned} \tag{52}$$

$$\begin{aligned}&\preceq n^\kappa \int\limits_{F^{11}} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1(q-1)(1+\kappa)} |\tau - w'|^{q(1+\kappa)}} + n^\kappa \int\limits_{F^{12}} \frac{|d\tau|}{|\tau - w'|^{q(1+\kappa)}} \\ &\le n^\kappa \left\{ \int\limits_{\mathcal{F}^1_1} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1(q-1)(1+\kappa)+q(1+\kappa)}} + \int\limits_{\mathcal{F}^1_2} \frac{|d\tau|}{|\tau - w'|^{\gamma_1(q-1)(1+\kappa)+q(1+\kappa)}} \right\} \\ &+ n^\kappa \int\limits_{F^{12}} \frac{|d\tau|}{|\tau - w'|^{q(1+\kappa)}} \\ &\preceq n^{\gamma_1(q-1)(1+\kappa)+q(1+\kappa)+\kappa-1} + n^{q(1+\kappa)+\kappa-1} = n^{(\gamma_1+1)(q-1)(1+\kappa)+2\kappa};\end{aligned}$$

$$J_{n,2} \preceq n^{\frac{(\gamma_1+1)(q-1)(1+\kappa)}{q} + \frac{2\kappa}{q}} = n^{\frac{(\gamma_1+1)(1+\kappa)}{p} + 2\kappa\left(1-\frac{1}{p}\right)}. \tag{53}$$

b) for $\gamma_1 < 0$;

$$(J_{n,2})^q = \int\limits_{|\tau|=R} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma_1)(q-1)} |d\tau|}{|\tau - w'|^{q(1+\kappa)} (|\tau| - 1)^\kappa} \tag{54}$$

$$= \left(\int\limits_{F^{11}} + \int\limits_{F^{12}} \right) \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma_1)(q-1)} |d\tau|}{|\tau - w'|^{q(1+\kappa)} (|\tau| - 1)^\kappa}$$

$$=: \left(J^1_{n,2}\right)^q + \left(J^2_{n,2}\right)^q .$$

Further

$$\left(J^1_{n,2}\right)^q = \int\limits_{F^{11}} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma_1)(q-1)} |d\tau|}{|\tau - w'|^{q(1+\kappa)} (|\tau| - 1)^\kappa} \tag{55}$$

$$\preceq n^\kappa \int\limits_{F^{11}} \frac{|\tau - w_1|^{(-\gamma_1)(q-1)(1-\kappa)} |d\tau|}{|\tau - w'|^{q(1+\kappa)}}$$

$$= n^\kappa \left(\int\limits_{F^{11}_1} + \int\limits_{F^{11}_2} \right) \frac{|\tau - w_1|^{(-\gamma_1)(q-1)(1-\kappa)} |d\tau|}{|\tau - w'|^{q(1+\kappa)}},$$

where

$$F^{11}_1 := \left\{ \tau : \tau \in F^{11},\ |\tau - w_1| < c_1 n^{-1} \right\},$$
$$F^{11}_2 := \left\{ \tau : \tau \in F^{11},\ c_1 n^{-1} \leq |\tau - w_1| < c_2 \right\}.$$

Furthermore,

$$\int\limits_{F^{11}_1} \frac{|\tau - w_1|^{(-\gamma_1)(q-1)(1-\kappa)} |d\tau|}{|\tau - w'|^{q(1+\kappa)}} \tag{56}$$

$$\preceq n^{\gamma_1(q-1)(1-\kappa)} \int\limits_{F^{11}_1} \frac{|d\tau|}{|\tau - w'|^{q(1+\kappa)}} \preceq n^{\gamma_1(q-1)(1-\kappa)+q(1+\kappa)-1},$$

$$\int\limits_{F^{11}_2} \frac{|\tau - w_1|^{(-\gamma_1)(q-1)(1-\kappa)} |d\tau|}{|\tau - w'|^{q(1+\kappa)}} \preceq \int\limits_{F^{11}_2} \frac{|d\tau|}{|\tau - w'|^{q(1+\kappa)}} \preceq n^{q(1+\kappa)-1}, \tag{57}$$

for any $q > 1,\ 0 \leq \kappa < 1$, and similarly we get:

$$\left(J^2_{n,2}\right)^q \preceq n^\kappa \int\limits_{F^{12}} \frac{|d\tau|}{|\tau - w'|^{q(1+\kappa)}} \preceq n^{q(1+\kappa)+\kappa-1}. \tag{58}$$

Therefore, in this case, for any $q > 1$, $0 \leq \kappa < 1$, we obtain:

$$J_{n,2} \preceq n^{\frac{q(1+\kappa)+\kappa-1}{q}}. \tag{59}$$

From (52) and (59), we find the following:

$$\left|P_n\left(z'\right)\right| \preceq \|P_n\|_{\mathcal{L}_p(h,\ L)} \begin{cases} n^{\frac{(\gamma_1+1)(1+\kappa)}{p}+2\kappa\left(1-\frac{1}{p}\right)}, & \gamma_1 \geq 0, \\ n^{\frac{(1+\kappa)}{p}+2\kappa\left(1-\frac{1}{p}\right)}, & \gamma_1 < 0. \end{cases} \tag{60}$$

From (60), we get:

$$\left|P_n\left(z'\right)\right| \preceq n^{\frac{(\eta+1)(1+\kappa)}{p}+2\kappa\left(1-\frac{1}{p}\right)} \cdot \|P_n\|_{\mathcal{L}_p(h,\ L)},$$

where $\eta := \max\{0; \gamma_k,\ k = \overline{1,m}\}$ and the proof is completed. □

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