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# Convergence and Summability of Multiple Fourier Series and Generalized Variation

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In this paper we present results on convergence and Cesàro summability of Multiple Fourier series of functions of bounded generalized variation.

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## 1. Classes of functions of two variables of bounded generalized variation

In 1881 Jordan [20] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. This notion was generalized hereinafter by many authors (quadratic variation, $\Phi$-variation, $\Lambda$-variation ets., see [27]-[4]). In the two dimensional case the class BV of functions of bounded variation was introduced by Hardy [19].

In this section we introduce several classes of bivariate functions of bounded generalized variation and compare them with the class $HBV$ (se Definition 1.1 below), which is important for the applications in Fourier analysis (see Theorem S in Section 2.).

Let $f(x, y)$, $(x, y) \in \mathbb{R}^2$ be a real function of two variables of period $2\pi$ with respect to each variable. Given intervals $I = (a, b)$, $J = (c, d)$ and points $x, y$ from $T := [0, 2\pi]$ we denote

$$f(I, y) := f(b, y) - f(a, y), \qquad f(x, J) = f(x, d) - f(x, c)$$

and

$$f(I, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

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Let $E = \{I_i\}$ be a collection of nonoverlapping intervals from $T$ ordered in arbitrary way and let $\Omega$ be the set of all such collections $E$. Denote by $\Omega_n$ set of all collections of $n$ nonoverlapping intervals $I_k \subset T$.

For the sequence of positive numbers $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we define

$$\Lambda V_1(f) = \sup_y \sup_{E\in\Omega} \sum_n \frac{|f(I_i, y)|}{\lambda_i} \quad (E = \{I_i\}),$$

$$\Lambda V_2(f) = \sup_x \sup_{F\in\Omega} \sum_m \frac{|f(x, J_j)|}{\lambda_j} \qquad (F = \{J_j\}),$$

$$\Lambda V_{1,2}(f) = \sup_{F,\, E\in\Omega} \sum_i \sum_j \frac{|f(I_i, J_j)|}{\lambda_i \lambda_j}.$$

**Definition 1.1:** We say that the function $f$ has bounded $\Lambda$-variation on $T^2 = [0, 2\pi]^2$ and write $f \in \Lambda BV$, if

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda V_{1,2}(f) < \infty.$$

We say that $f$ has bounded partial $\Lambda$-variation and write $f \in P\Lambda BV$ if

$$P\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) < \infty.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty,\ n = 1, 2, \ldots$) the classes $\Lambda BV$ and $P\Lambda BV$ coincide with the Hardy class $BV$ and PBV respectively. Hence it is reasonable to assume that $\lambda_n \to \infty$ and since the intervals in $E = \{I_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,

$$1 < \lambda_1 \le \lambda_2 \le \ldots, \qquad \lim_{n\to\infty} \lambda_n = \infty. \tag{1}$$

In the case when $\lambda_n = n,\ n = 1, 2 \ldots$ we say *Harmonic Variation* instead of $\Lambda$-variation and write $H$ instead of $\Lambda$ ($HBV$, $PHBV$, $HV(f)$, ets).

The notion of $\Lambda$-variation was introduced by D. Waterman [26] in one dimensional case and A. Sahakian [24] in two dimensional case. The class $PBV$ as well as the class $PBV_p$ (see Definition 1.2) was introdused by U. Goginava in [10].

**Definition 1.2:** Let $\Phi$-be a strictly increasing continuous function on $[0, +\infty)$ with $\Phi(0) = 0$. We say that the function $f$ has bounded partial $\Phi$-variation on $T^2$ and write $f \in PBV_\Phi$, if

$$V_\Phi^{(1)}(f) := \sup_y \sup_{\{I_i\}\in\Omega_n} \sum_{i=1}^{n} \Phi\left(\left|f(I_i, y)\right|\right) < \infty, \quad n = 1, 2, ...,$$

$$V_{\Phi}^{(2)}(f) := \sup_{x} \sup_{\{J_j\}\in\Omega_m} \sum_{j=1}^{m} \Phi\left(|f(x, J_j)|\right) < \infty, \quad m = 1, 2, ....$$

In the case when $\Phi(u) = u^p$, $p \geq 1$, we say that $f$ has bounded partial $p$-variation and write $f \in PBV_p$.

In the following theorem the necessary and sufficient conditions are obtained for the inclusion $P\Lambda BV \subset HBV$.

**Theorem 1.3** (U. Goginava, A. Sahakian [11])**:** *Let* $\Lambda = \{\lambda_n\}$ *with* $\lambda_n = n\gamma_n$ *and* $\gamma_n \geq \gamma_{n+1} > 0$, $n = 1, 2, ....$ .
*1) If*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} < \infty, \tag{2}$$

*then* $P\Lambda BV \subset HBV$.
*2) If* $\gamma_n = O(\gamma_{n^{[1+\delta]}})$ *for some* $\delta > 0$ *and*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} = \infty,$$

*then* $P\Lambda BV \not\subset HBV$.

**Corollary 1.4:** $PBV \subset HBV$ *and* $PHBV \not\subset HBV$.

**Corollary 1.5:** *Let* $\Phi$ *and* $\Psi$ *be conjugate functions in the sense of Young* ($ab \leq \Phi(a) + \Psi(b)$) *and let for some* $\{\lambda_n\}$ *satisfying (1),*

$$\sum_{n=1}^{\infty} \Psi\left(\frac{1}{\lambda_n}\right) < \infty. \tag{3}$$

*Then* $PBV_{\Phi} \subset HBV$. *In particular,* $PBV_p \subset HBV$ *for any* $p > 1$.

**Definition 1.6** (U. Goginava [10])**:** The Partial Modulus of Variation of a function $f$ are the functions $v_1(n, f)$ and $v_2(m, f)$ defined by

$$v_1(n, f) := \sup_{y} \sup_{\{I_i\}\in\Omega_n} \sum_{i=1}^{n} |f(I_i, y)|, \quad n = 1, 2, \ldots,$$

$$v_2(m, f) := \sup_{x} \sup_{\{J_k\}\in\Omega_m} \sum_{i=1}^{m} |f(x, J_k)|, \quad m = 1, 2, \ldots.$$

For functions of one variable the concept of the modulus of variation was introduced by Chanturia [4].

**Theorem 1.7** (U. Goginava, A. Sahakian [11])**:** *Let* $f$ *be such that*

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n, f)}}{n^{3/2}} < \infty, \quad j = 1, 2.$$

*Then* $f \in HBV$.

Another class of functions of generalized bounded variation was introduced by M. Dyachenko and D. Waterman in [7]. Denoting by $\Gamma$ the the set of finite collections of nonoverlapping rectangles $A_k := [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset T^2$ they define

$$\Lambda^* V(f) := \sup_{\{A_k\} \in \Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}.$$

**Definition 1.8** (M. Dyachenko, D. Waterman [7])**:** We say that $f \in \Lambda^* BV$ if

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda^* V(f) < \infty.$$

In [14] we introduced a new class of functions of generalized bounded variation and investigate the convergence of Fourier series of function of that class. For the sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we define

$$\Lambda^{\#} V_1(f) = \sup_{\{y_i\} \subset T} \sup_{\{I_i\} \in \Omega} \sum_i \frac{|f(I_i, y_i)|}{\lambda_i},$$

$$\Lambda^{\#} V_2(f) = \sup_{\{x_j\} \subset T} \sup_{\{J_j\} \in \Omega} \sum_j \frac{|f(x_j, J_j|}{\lambda_j}.$$

**Definition 1.9** (U. Goginava, A. Sahakian [11])**:** We say that $f \in \Lambda^{\#} BV$, if

$$\Lambda^{\#} V(f) := \Lambda^{\#} V_1(f) + \Lambda^{\#} V_2(f) < \infty.$$

It is easy to see, that

$$\Lambda^* BV \subset \Lambda^{\#} BV \subset P\Lambda BV. \tag{4}$$

Obviously, the function $f(x, y) = \operatorname{sign}(x - y)$ belongs to $P\Lambda BV \setminus \Lambda^{\#} BV$ for any $\Lambda$. On the other hand, we have proved the following result.

**Theorem 1.10** (U. Goginava, A. Sahakian [14])**:** *If* $\Lambda = \{\lambda_n\}$ *and*

$$\limsup_{n \to \infty} \left( \sum_{k=1}^{n^2} \frac{1}{\lambda_k} \right) \left( \sum_{k=1}^{n} \frac{1}{\lambda_k} \right)^{-1} = +\infty,$$

*then* $\Lambda^{\#} BV \setminus \Lambda^* BV \neq \emptyset$.

In the next theorem we characterize sequences $\Lambda = \{\lambda_n\}$ for which the inclusion $\Lambda^{\#} BV \subset HBV$ holds.

**Theorem 1.11** (U. Goginava, A. Sahakian [14])**:** *Let* $\Lambda = \{\lambda_n\}$.

*a) If*

$$\limsup_{n \to \infty} \frac{\lambda_n \log n}{n} < \infty,$$

*then*

$$\Lambda^{\#}BV \subset HBV.$$

*b) If* $\frac{\lambda_n}{n} \downarrow 0$ *and*

$$\limsup_{n\to\infty} \frac{\lambda_n \log n}{n} = +\infty,$$

*then*

$$\Lambda^{\#}BV \not\subset HBV.$$

**Definition 1.12** (U. Goginava, A. Sahakian [14])**:** Let $\Phi$-be a strictly increasing continuous function on $[0, +\infty)$ with $\Phi(0) = 0$. We say that the function $f \in B^{\#}V_{\Phi}\left(T^2\right)$, if

$$V^{\#}_{\Phi,1}(f) := \sup_{\{y_i\}\subset T} \sup_{\{I_i\}\in\Omega} \sum_i \Phi\left(\left|f\left(I_i, y_i\right)\right|\right) < \infty,$$

and

$$V^{\#}_{\Phi,2}(f) := \sup_{\{x_j\}\subset T} \sup_{\{J_j\}\in\Omega} \sum_j \Phi\left(\left|f\left(x_j, J_j\right)\right|\right) < \infty.$$

Next, we define

$$v_1^{\#}(n, f) := \sup_{\{y_i\}_{i=1}^n} \sup_{\{I_i\}\in\Omega_n} \sum_{i=1}^{n} \left|f\left(I_i, y_i\right)\right|, \quad n = 1, 2, \ldots,$$

$$v_2^{\#}(m, f) := \sup_{\{x_j\}_{j=1}^m} \sup_{\{J_k\}\in\Omega_m} \sum_{j=1}^{m} \left|f\left(x_j, J_j\right)\right|, \quad m = 1, 2, \ldots.$$

**Theorem 1.13** (U. Goginava, A. Sahakian [14])**:** *Let $\Phi$ and $\Psi$ are conjugate functions in the sense of Young ($ab \le \Phi(a) + \Psi(b)$) and let*

$$\sum_{n=1}^{\infty} \Psi\left(\frac{\log n}{n}\right) < \infty.$$

*Then*

$$B^{\#}V_{\Phi} \subset \left\{\frac{n}{\log n}\right\}^{\#} BV.$$

**Theorem 1.14** (U. Goginava, A. Sahakian [14])**:** *Let*

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}(f,n)\log n}{n^2} < \infty, \qquad s = 1, 2.$$

*Then*

$$f \in \left\{ \frac{n}{\log n} \right\}^{\#} BV.$$

Observe that by Theorem 1.11 we have the inclusion $\left\{\frac{n}{\log n}\right\}^{\#} BV \subset HBV$. Now, for a sequence $\Lambda = \{\lambda_n\}$ we denote

$$\Lambda_n := \{\lambda_k\}_{k=n}^{\infty}, \qquad n = 1, 2, \ldots$$

**Definition 1.15** (U Goginava [12]) **:** We say that the function $f \in \Lambda^{\#}BV$ is continuous in $\Lambda^{\#}$-variation and write $f \in C\Lambda^{\#}V$, if

$$\lim_{n\to\infty} \Lambda_n^{\#} V_1(f) = \lim_{n\to\infty} \Lambda_n^{\#} V_2(f) = 0.$$

**Theorem 1.16** (U. Goginava, A. Sahakian [17])**:** *Let the sequence* $\Lambda = \{\lambda_n\}$ *be such that*

$$\liminf_{n\to\infty} \frac{\lambda_{2n}}{\lambda_n} = q > 1.$$

*Then* $\Lambda^{\#}BV = C\Lambda^{\#}V$.

**Theorem 1.17** (U. Goginava [12])**:** *Let* $\alpha + \beta < 1, \alpha, \beta > 0$ *and*

$$\sum_{j=1}^{\infty} \frac{v_s^{\#}(f; 2^j)}{2^{j(1-(\alpha+\beta))}} < \infty, \qquad s = 1, 2.$$

*Then* $f \in C\{n^{1-(\alpha+\beta)}\}^{\#}V$.

## 2. Convergence of double Fourier series

Everywhere in this and in the next section we suppose that the function $f$ is measurable on $\mathbb{R}^2$ and $2\pi$-periodic with respect to each variable. The double Fourier series of a function $f \in L^1(T^2)$ with respect to the trigonometric system is the series

$$S[f] := \sum_{m,n=-\infty}^{+\infty} \widehat{f}(m,n)\, e^{imx} e^{iny},$$

where

$$\widehat{f}(m,n)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}f(x,y)e^{-imx}e^{-iny}dxdy$$

are the Fourier coefficients of $f$. The rectangular partial sums of $S[f]$ are defined as follows:

$$S_{M,N}\left[f,(x,y)\right]:=\sum_{m=-M}^{M}\sum_{n=-N}^{N}\widehat{f}(m,n)\,e^{imx}e^{iny},$$

In this paper we consider only **Pringsheim convergence** of double Fourier series, i.e **convergence of rectangular partial sums** $S_{M,N}\left[f,(x,y)\right]$, as $M,N\to\infty$.

We denote by $C(T^2)$ the space of continuous on $\mathbb{R}^2$ and $2\pi$-periodic with respect to each variable functions with the norm

$$\|f\|_C:=\sup_{x,y\in T^2}|f(x,y)|.$$

For a function $f$ we denote by $f\,(x\pm 0,y\pm 0)$ the open coordinate quadrant limits (if exist) at the point $(x,y)$ and let $f^*(x,y)$ be the arithmetic mean of that quadrant limits:

$$f^*(x,y):=\frac{1}{4}\left\{f\,(x+0,y+0)+f\,(x+0,y-0)\right.$$
$$\left.+f\,(x-0,y+0)+f\,(x-0,y-0)\right\}. \quad (5)$$

**Remark 1:** Observe that for a function $f\in\Lambda BV$ the quadrant limits $f\,(x\pm 0,y\pm 0)$ may not exist. As was shown in [14] for any function $f\in\Lambda^{\#}BV$ the quadrant limits $f\,(x\pm 0,y\pm 0)$ exist at any point $(x,y)\in T^2$.

We say the point $(x,y)\in T^2$ is a **regular point** of a function $f$, if all quadrant limits in (5) exist.

The well known Dirichlet-Jordan theorem (see [29]) states that the Fourier series of a function $g(x),\ x\in T$ of bounded variation converges at every point $x$ to the value $\left[g\,(x+0)+g\,(x-0)\right]/2$. If $g$ is in addition continuous on $T$, the Fourier series converges uniformly on $T$.

Hardy [19] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if the function $f$ has bounded variation in the sense of Hardy $(f\in BV)$, then $S\,[f]$ converges to $f^*(x,y)$ at any regular point $(x,y)$. If $f$ is in addition continuous on $T^2$ then $S\,[f]$ converges uniformly on $T^2$.

**Theorem S** (Sahakian [24]): *The Fourier series of a function $f\in HBV$ converges to $f^*(x,y)$ in any regular point $(x,y)$. The convergence is uniform on any compact $K\subset T^2$, where the function $f$ is continuous.*

Theorem S was proved in [24] under the assumption that the function is continuous on some open set containing $K$, while O. Sargsyan noticed in [23], that the continuity of $f$ on the compact $K$ is sufficient.

**Definition 2.1:** We say that the class of functions $V \subset L^1(T^2)$ is a class of convergence on $T^2$, if for any function $f \in V$

1) the Fourier series of $f$ converges to $f^*(x,y)$ at any regular point $(x,y)$,

2) the convergence is uniform on any compact $K \subset T^2$, where the function $f$ is continuous.

The following results immediately follow from Theorems 1.3, 1.7, Corollary 1.5 and Theorem S.

**Theorem 2.2** (U. Goginava, A. Sahakian [11])**:** *Let $\Lambda = \{\lambda_n\}$ with $\lambda_n = n\gamma_n$ and $\gamma_n \geq \gamma_{n+1} > 0,\ n = 1, 2, \ldots.$ .*

*1) If*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} < \infty,$$

*then the class $P\Lambda BV$ is a class of convergence on $T^2$.*

*2) If $\gamma_n = O(\gamma_{n^{[1+\delta]}})$ for some $\delta > 0$ and*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} = \infty,$$

*then then there exists a continuous function $f \in P\Lambda BV$, the Fourier series of which diverges over cubes at $(0,0)$.*

**Theorem 2.3** (U. Goginava, A. Sahakian [11])**:** *The set of functions $f$ satisfying*

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n,f)}}{n^{3/2}} < \infty, \quad j = 1, 2,$$

*is a class of convergence on $T^2$.*

**Corollary 2.4:** *The set of functions $f$ satisfying $v_1(n,f) = O(n^{\alpha})$, $v_2(n,f) = O\left(n^{\beta}\right)$, $0 < \alpha, \beta < 1$, is a class of convergence on $T^2$.*

**Theorem 2.5** (U. Goginava [10])**:** *The class $PBV_p$, $p \geq 1$, is a class of convergence on $T^2$.*

From Theorem 2.2 it follows that for any $\delta > 0$ the class $f \in P\left\{\frac{n}{\log^{1+\delta} n}\right\} BV$ is a class of convergence. Moreover, one can not take here $\delta = 0$. It is interesting to compare this result with the following one obtained by M. Dyachenko and D. Waterman in [7].

**Theorem DW** (M. Dyachenko and D. Waterman [7]): *If $f \in \left\{\frac{n}{\log n}\right\}^* BV$, then in any point $(x,y) \in T^2$ the quadrant limits (5) exist and the double Fourier series of $f$ converges to $f^*(x,y)$.*
*Moreover, the sequence $\left\{\frac{n}{\log n}\right\}$ can not be replaced with any sequence $\left\{\frac{n\alpha_n}{\log n}\right\}$, where $\alpha_n \to \infty$.*

It is easy to show (see[7]), that $\left\{\frac{n}{\log n}\right\}^* BV \subset HBV$, hence the convergence part of Theorem DW follows from Theorem S. It is essential that the condition $f \in \left\{\frac{n}{\log n}\right\}^{*} BV$ guaranties the existence of quadrant limits.

The following theorem immediately follows from Theorem 1.11 and Theorem S.

**Theorem 2.6** (U. Goginava, A. Sahakian [14])**:** *If $\Lambda = \{\lambda_n\}$ and*

$$\limsup_{n\to\infty} \frac{\lambda_n \log n}{n} < \infty,$$

*then the class $\Lambda^{\#} BV$ is a class of convergence on $T^2$.*

*In particular, the class $\left\{\frac{n}{\log n}\right\}^{\#} BV$ is a class of convergence on $T^2$.*

Theorem DW and (4) imply that the sequence $\left\{\frac{n}{\log n}\right\}$ in Theorem 2.6 can not be replaced with any sequence $\left\{\frac{n\alpha_n}{\log n}\right\}$, where $\alpha_n \to \infty$.

Theorems 1.13, 1.14 and 2.6 imply

**Theorem 2.7** (U. Goginava, A. Sahakian [14])**:** *The class $B^{\#}V_{\Phi}$ is a class of convergence on $T^2$, provided that (2) and (3) hold.*

**Theorem 2.8** (U. Goginava, A. Sahakian [14])**:** *Let*

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}(f,n)\log n}{n^2} < \infty, \quad s = 1, 2.$$

*Then in any point $(x, y) \in T^2$ the quadrant limits (5) exist and the double Fourier series of $f$ converges to $f^*(x, y)$. The convergence is uniform on any compact $K \in T^2$, if $f$ is continuous on $K$.*

## 3. Cesàro summability of double Fourier series

For one-dimensional Fourier series D. Waterman has proved the following theorem.

**Theorem W2** (D. Waterman [25]): *Let $0 < \alpha < 1$. The Fourier series of a function $f \in \{n^{1-\alpha}\}BV$ is everywhere $(C, -\alpha)$ bounded and is uniformly $(C, -\alpha)$ bounded on each closed interval of continuity of $f$.*

*If $f \in C\{n^{1-\alpha}\}BV$, then $S[f]$ is everywhere $(C, -\alpha)$ summable to the value $[f(x+0) + f(x-0)]/2$ and the summability is uniform on each closed interval of continuity.*

Later A. Sablin proved in [22], that for $0 < \alpha < 1$ the classes $\{n^{1-\alpha}\}BV$ and $C\{n^{1-\alpha}\}BV$ coincide.

For double Fourier series the **Cesàro** $(C; \alpha, \beta)$**-means** of a function $f \in L^1(T^2)$ are defined by

$$\sigma_{n,m}^{\alpha,\beta}(f; x, y) := \frac{1}{A_n^{\alpha}} \frac{1}{A_m^{\beta}} \sum_{i=0}^{n} \sum_{j=0}^{m} A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j}\left[f, (x, y)\right],$$

where $\alpha, \beta > -1$ and

$$A_0^\alpha = 1, \ A_k^\alpha = \frac{(\alpha+1)\cdots(\alpha+k)}{k!}, \quad k = 1, 2, ....$$

The double Fourier series of $f$ is said to be $(C;\alpha,\beta)$ **summable to** $s$ **in a point** $(x,y)$, if

$$\lim_{n,m\to\infty} \sigma_{n,m}^{\alpha,\beta}(f;x,y) = s.$$

L. Zhizhiashvili has investigated the convergence of Cesàro means of double Fourier series of functions of bounded variation. In particular, the following theorem was proved.

**Theorem Zh** (L. Zhizhiashvili [28]): *If $f \in BV$, then the double Fourier series of $f$ is $(C;-\alpha,-\beta)$ summable to $f^*(x,y)$ in any regular point $(x,y)$. The convergence is uniform on any compact $K$, where the function $f$ is continuous.*

For functions of partial bounded variation the problem was considered by the first author.

**Theorem G2** (U. Goginava [8]): *Let $\alpha > 0, \ \beta > 0$.*
*1) If $\alpha+\beta < 1$, then for any $f \in C\left(T^2\right) \cap PBV$ the double Fourier series of $f$ is uniformly $(C;-\alpha,-\beta)$ summable to $f$.*
*2) If $\alpha+\beta \geq 1$, then there exists a continuous function $f_0 \in PBV$ such that the sequence $\sigma_{n,n}^{-\alpha,-\beta}(f_0;0,0)$ diverges.*

In [13] we consider the following problem. *Let $\alpha,\beta \in (0,1)$, $\alpha+\beta < 1$. Under what conditions on the sequence $\Lambda = \{\lambda_n\}$ is the double Fourier series of any function $f \in P\Lambda BV$ is $(C;-\alpha,-\beta)$ summable?*

**Theorem 3.1** (U. Goginava, A. Sahakian [13])**:** *Let $\alpha,\beta \in (0,1)$, $\alpha+\beta < 1$ and let the sequence $\Lambda = \{\lambda_k\}$ be such that $\lambda_k k^{(\alpha+\beta)-1} \downarrow 0$.*
*1) If*

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{k^{2-(\alpha+\beta)}} < \infty,$$

*then the double Fourier series of any function $f \in P\Lambda BV$ is $(C;-\alpha,-\beta)$ summable to $f^*(x,y)$ at any regular point $(x,y)$. The summability is uniform on any compact $K$, if $f$ is continuous on the neighborhood of $K$.*
*2) If*

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{k^{2-(\alpha+\beta)}} = \infty,$$

*then there exists a continuous function $f \in P\Lambda BV$ for which the $(C;-\alpha,-\beta)$ means of the double Fourier series diverges over cubes at $(0,0)$.*

**Corollary 3.2** (U. Goginava, A. Sahakian [13])**:** *Let $\alpha,\beta \in (0,1)$, $\alpha+\beta < 1$.*

*1) If* $f \in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log^{1+\varepsilon} n}\right\} BV$ *for some* $\varepsilon > 0$*, then the double Fourier series of the function* $f$ *is* $(C; -\alpha, -\beta)$ *summable to* $f^*(x, y)$ *in any regular point* $(x, y)$*. The summability is uniform on any compact* $K$*, if* $f$ *is continuous on the neighborhood of* $K$*.*

*2) There exists a continuous function* $f \in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log(n+1)}\right\} BV$ *such that* $(C; -\alpha, -\beta)$ *means of two-dimensional Fourier series of* $f$ *diverges over cubes at* $(0, 0)$*.*

**Corollary 3.3** (U. Goginava, A. Sahakian [13])**:** *Let* $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$ *and* $f \in PBV$*. Then the double Fourier series of the function* $f$ *is* $(C; -\alpha, -\beta)$ *summable to* $f^*(x, y)$ *in any regular point* $(x, y)$*. The summability is uniform on any compact* $K$*, if* $f$ *is continuous on the neighborhood of* $K$*.*

In [12] the following problem was considred. *Let* $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$. *Under what conditions on the sequence* $\Lambda = \{\lambda_n\}$ *the double Fourier series of any function* $f \in C\Lambda^{\#}BV$ *is* $(C; -\alpha, -\beta)$ *summable.*

**Theorem 3.4** (U. Goginava [12])**:** *a) Let* $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$ *and* $f \in C\left\{n^{1-(\alpha+\beta)}\right\}^{\#} BV$*. Then the double Fourier series of* $f$ *is* $(C; -\alpha, -\beta)$ *summable to* $f^*(x, y)$ *in any point* $(x, y)$*. The summability is uniform on any compact* $K \subset \mathbb{T}^2$*, if* $f$ *is continuous on the neighborhood of* $K$*.*
*b) Let* $\Lambda := \left\{n^{1-(\alpha+\beta)}\xi_n\right\}$*, where* $\xi_n \uparrow \infty$ *as* $n \to \infty$*. Then there exists a function* $f \in C\left(\mathbb{T}^2\right) \cap C\Lambda^{\#}V$ *for which* $(C; -\alpha, -\beta)$*-means of double Fourier series diverges unboundedly at* $(0, 0)$*.*

Theorems 1.16, 1.17 and 3.4 imply the following results.

**Theorem 3.5:** *Let* $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$ *and* $f \in \left\{n^{1-(\alpha+\beta)}\right\}^{\#} BV$*. Then the double Fourier series of* $f$ *is* $(C; -\alpha, -\beta)$ *summable to* $f^*(x, y)$ *in any point* $(x, y)$*. The summability is uniform on any compact* $K \subset \mathbb{T}^2$*, if* $f$ *is continuous on the neighborhood of* $K$*.*

**Theorem 3.6:** *Let* $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$ *and*

$$\sum_{j=1}^{\infty} \frac{v_s^{\#}\left(f; 2^j\right)}{2^{j(1-(\alpha+\beta))}} < \infty, \quad s = 1, 2.$$

*Then the double Fourier series of* $f$ *is* $(C; -\alpha, -\beta)$ *summable to* $f^*(x, y)$ *in any point* $(x, y)$*. The summability is uniform on any compact* $K \subset \mathbb{T}^2$*, if* $f$ *is continuous on the neighborhood of* $K$*.*

## 4. Classes of functions of $d$ variables of bounded generalized variation

Consider a function $f(x)$ defined on the d-dimensional cube $T^d$ and a collection of intervals

$$J^k = \left(a^k, b^k\right) \subset T, \qquad k = 1, 2, \ldots d.$$

For $d = 1$ we set

$$f\left(J^1\right) := f\left(b^1\right) - f\left(a^1\right).$$

If for any function of $d-1$ variables the expression $f\left(J^1 \times \cdots \times J^{d-1}\right)$ is already defined, then for a function $f$ of $d$ variables the **mixed difference** is defined as follows:

$$f\left(J^1 \times \cdots \times J^d\right) := f\left(J^1 \times \cdots \times J^{d-1}, b^d\right) - f\left(J^1 \times \cdots \times J^{d-1}, a^d\right).$$

For sequences of positive numbers

$$\Lambda^j = \{\lambda_n^j\}_{n=1}^{\infty}, \quad \lim_{n\to\infty} \lambda_n^j = \infty, \quad j = 1, 2, \ldots, d,$$

and for a function $f(x)$, $x = (x_1, \ldots, x_d) \in T^d$ the $\left(\Lambda^1, \ldots, \Lambda^d\right)$**-variation of** $f$ **with respect to the index set** $D := \{1, 2, ..., d\}$ is defined as follows:

$$\left\{\Lambda^1, \ldots, \Lambda^d\right\} V^D\left(f, T^d\right) := \sup_{\{I_{i_j}^j\}\in\Omega} \sum_{i_1,\ldots,i_d} \frac{\left|f\left(I_{i_1}^1 \times \cdots \times I_{i_d}^d\right)\right|}{\lambda_{i_1}^1 \cdots \lambda_{i_d}^d}.$$

For an index set $\alpha = \{j_1, ..., j_p\} \subset D$ and any $x = (x_1, ..., x_d) \in R^d$ we set $\widetilde{\alpha} := D \setminus \alpha$ and denote by $x_\alpha$ the vector of $R^p$ consisting of components $x_j, j \in \alpha$, i.e.

$$x_\alpha = \left(x_{j_1}, ..., x_{j_p}\right) \in R^p.$$

By

$$\left\{\Lambda^{j_1}, ..., \Lambda^{j_p}\right\} V^\alpha\left(f, x_{\widetilde{\alpha}}, T^d\right) \quad \text{and} \quad f\left(I_{i_{j_1}}^1 \times \cdots \times I_{i_{j_p}}^p, x_{\widetilde{\alpha}}\right)$$

we denote respectively the $\left(\Lambda^{j_1}, ..., \Lambda^{j_p}\right)$-variation over the $p$-dimensional cube $T^p$ and mixed difference of $f$ as a function of variables $x_{j_1}, ..., x_{j_p}$ with fixed values $x_{\widetilde{\alpha}}$ of other variables. The $\left(\Lambda^{j_1}, ..., \Lambda^{j_p}\right)$**-variation of** $f$ **with respect to the index set** $\alpha$ is defined as follows:

$$\left\{\Lambda^{j_1}, ..., \Lambda^{j_p}\right\} V^\alpha\left(f, T^p\right) = \sup_{x_{\widetilde{\alpha}}\in T^{d-p}} \left\{\Lambda^{j_1}, ..., \Lambda^{j_p}\right\} V^\alpha\left(f, x_{\widetilde{\alpha}}, T^d\right).$$

**Definition 4.1:** We say that the function $f$ has total bounded $\left(\Lambda^1, ..., \Lambda^d\right)$-variation on $T^d$ and write $f \in \left\{\Lambda^1, ..., \Lambda^d\right\} BV\left(T^d\right)$, if

$$\left\{\Lambda^1, ..., \Lambda^d\right\} V(f, T^d) := \sum_{\alpha\subset D} \left\{\Lambda^1, ..., \Lambda^d\right\} V^\alpha\left(f, T^d\right) < \infty.$$

**Definition 4.2:** We say that the function $f$ is continuous in $\left(\Lambda^1, ..., \Lambda^d\right)$-variation on $T^d$ and write $f \in C\left\{\Lambda^1, ..., \Lambda^d\right\} V\left(T^d\right)$, if

$$\lim_{n\to\infty}\left\{\Lambda^{j_1}, ..., \Lambda^{j_{k-1}}, \Lambda_n^{j_k}, \Lambda^{j_{k+1}}, ..., \Lambda^{j_p}\right\} V^{\alpha}\left(f, T^d\right) = 0, \qquad k = 1, 2, \ldots, p$$

for any $\alpha \subset D,\ \alpha := \{j_1, ..., j_p\}$, where $\Lambda_n^{j_k} := \left\{\lambda_s^{j_k}\right\}_{s=n}^{\infty}$.

The continuity of a function in $\Lambda$-variation was introduced by D. Waterman [25] and was investigated in details by A. Bakhvalov (see [1], [2] and references therein). This property is important for applications in the theory of Fourier series (see Theorem B1 in Section 5).

**Definition 4.3:** We say that the function $f$ has bounded Partial $\left(\Lambda^1, ..., \Lambda^d\right)$-variation and write $f \in P\left\{\Lambda^1, ..., \Lambda^d\right\} BV\left(T^d\right)$ if

$$P\left\{\Lambda^1, ..., \Lambda^d\right\} V(f, T^d) := \sum_{i=1}^{d} \Lambda^i V^{\{i\}}\left(f, T^d\right) < \infty.$$

In the case when $\Lambda^1 = \cdots = \Lambda^d = \Lambda$ we set

$$\begin{aligned} \Lambda BV(T^d) &:= \{\Lambda^1, ..., \Lambda^d\} BV(T^d), \\ C\Lambda V(T^d) &:= C\{\Lambda^1, ..., \Lambda^d\} V(T^d), \\ P\Lambda BV(T^d) &:= P\{\Lambda^1, ..., \Lambda^d\} BV(T^d). \end{aligned}$$

If $\lambda_n = n$ for all $n = 1, 2 \ldots$ we say *Harmonic Variation* instead of $\Lambda$-variation and write $H$ instead of $\Lambda$, *i.e.* $HBV$, $PHBV$, $CHV$, ets.

**Theorem 4.4** (U. Goginava, A. Sahakian [15])**:** *Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $d \geq 2$. If $\lambda_n/n \downarrow 0$ and*

$$\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} < \infty,$$

*then $P\Lambda BV(T^d) \subset CHV(T^d)$.*

For a sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we denote

$$\Lambda^{\#} V_s\left(f, T^d\right) := \sup_{\{x^i\{s\}\}\subset T^{d-1}} \sup_{\{I_i^s\}\in\Omega} \sum_i \frac{\left|f\left(I_i^s, x^i\{s\}\right)\right|}{\lambda_i},$$

where

$$x^i\{s\} := \left(x_1^i, \ldots, x_{s-1}^i, x_{s+1}^i, \ldots, x_d^i\right) \quad \text{for} \quad x^i := \left(x_1^i, \ldots, x_d^i\right).$$

**Definition 4.5:** We say that $f \in \Lambda^{\#} BV\left(T^d\right)$, if

$$\Lambda^{\#} V\left(f, T^d\right) := \sum_{s=1}^{d} \Lambda^{\#} V_s\left(f, T^d\right) < \infty.$$

**Theorem 4.6** (U. Goginava, A. Sahakian [18])**:** *If* $\Lambda = \{\lambda_n\}$ *with*

$$\lambda_n = \frac{n}{\log^{d-1} n}, \quad n = 2, 3, \ldots,$$

*then* $\Lambda^{\#}BV(T^d) \subset HBV(T^d)$.

Now, we denote

$$\Delta := \{\delta = (\delta_1, \ldots, \delta_d) : \delta_i = \pm 1, \ i = 1, 2, \ldots, d\}$$

and

$$\pi_{\varepsilon\delta}(x) := (x_1, \, x_1 + \varepsilon\delta_1) \times \cdots \times (x_d, \, x_d + \varepsilon\delta_d),$$

for $x = (x_1, \ldots, x_d) \in R^d$ and $\varepsilon > 0$. We set $\pi_\delta(x) := \pi_{\varepsilon\delta}(x)$, if $\varepsilon = 1$.

For a function $f$ and $\delta \in \Delta$ we set

$$f_\delta(x) := \lim_{t \in \pi_\delta(x), \ t \to x} f(t), \tag{6}$$

if the last limit exists.

**Theorem 4.7** (U. Goginava, A. Sahakian [18])**:** *Suppose* $\Lambda = \{\lambda_n\}$ *and* $f \in \Lambda^{\#}BV\left(T^d\right)$.

*a) If the limit* $f_\delta(x)$ *exists for some* $x = (x_1, \ldots, x_d) \in T^d$ *and some* $\delta = (\delta_1, \ldots, \delta_d) \in \Delta$, *then*

$$\lim_{\varepsilon \to 0} \Lambda^{\#}V\left(f, \pi_{\varepsilon\delta}(x)\right) = 0.$$

*b) If* $f$ *is continuous on some compact* $K \subset T^d$, *then*

$$\lim_{\varepsilon \to 0} \Lambda^{\#}V\left(f, [x_1 - \varepsilon, x_1 + \varepsilon] \times \cdots \times [x_d - \varepsilon, x_d + \varepsilon]\right) = 0$$

*uniformly with respect to* $x = (x_1, \ldots, x_d) \in K$.

**Theorem 4.8** (U. Goginava, A. Sahakian [18])**:** *If the function* $f(x)$, $x \in T^d$ *satisfies the condition*

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}\left(f, n\right) \log^{d-1} n}{n^2} < \infty, \qquad s = 1, 2, ..., d,$$

*then* $f \in \left\{\frac{n}{\log^{d-1} n}\right\}^{\#} BV\left(T^d\right)$.

## 5. Convergence of multiple Fourier series

The Fourier series of the function $f \in L^1\left(T^d\right)$ with respect to the trigonometric system is the series

$$S[f] := \sum_{n_1,\dots,n_d=-\infty}^{+\infty} \widehat{f}(n_1,....,n_d)\, e^{i(n_1x+\cdots+n_dx_d)},$$

where

$$\widehat{f}(n_1,....,n_d) = \frac{1}{(2\pi)^d}\int_{T^d} f(x_1,...,x_d)e^{-i(n_1x_1+\cdots+n_dx_d)}dx_1\cdots dx_d$$

are the Fourier coefficients of $f$. The rectangular partial sums are defined as follows:

$$S_{N_1,\dots,N_d}[f,(x_1,...,x_d)] = \sum_{n_1=-N_1}^{N_1}\cdots\sum_{n_d=-N_d}^{N_d} \widehat{f}(n_1,....,n_d)\, e^{i(n_1x_1+\cdots+n_dx_d)}$$

We denote by $C(T^d)$ the space of continuous and $2\pi$-periodic with respect to each variable functions with the norm

$$\|f\|_C := \sup_{(x^1,\dots,x^d)\in T^d} |f(x^1,\dots,x^d)|.$$

We say that the point $x := \left(x^1,\dots,x^d\right) \in T^d$ is a **regular point** of a function $f$ if the limits (6) exist for all $\delta \in \Delta$. For a regular point $x \in T^d$ we denote

$$f^*(x) := \frac{1}{2^d}\sum_{\delta\in\Delta} f_\delta(x).$$

**Definition 5.1:** We say that the class of functions $V \subset L^1(T^d)$ is a class of convergence on $T^d$, if for any function $f \in V$

1) the Fourier series of $f$ converges to $f^*(x)$ at any regular point $x \in T^d$,

2) the convergence is uniform on any compact $K \subset T^d$, if $f$ is continuous on the neighborhood of $K$.

In [1] A. Bakhvalov showed that the class $HBV(T^d)$ is not a class of convergence on $T^d$, if $d > 2$. On the other hand, he proved the following

**Theorem B1** (A. Bakhvalov [1]): *The class $CHV(T^d)$ is a class of convergence on $T^d$ for any $d = 1, 2, \dots$*

Convergence of spherical and other partial sums of d-dimensional Fourier series of functions of bounded $\Lambda$-variation was investigated in deatails by M. Dyachenko [5, 6], A. Bakhvalov [1, 3].

The first part of the next theorem is a consequence of Theorem 4.4 and Theorem B1.

**Theorem 5.2** (U. Goginava, A. Sahakian [15])**:** *Let $\Lambda = \{\lambda_n\}$ and $d \geq 2$.*

*a) If* $\lambda_n/n \downarrow 0$ *and*

$$\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} < \infty,$$

*then* $P\Lambda BV$ *is a class of convergence on* $T^d$.

*b) If* $\frac{\lambda_n}{n} = O\left(\frac{\lambda_{[n^\delta]}}{[n^\delta]}\right)$ *for some* $\delta > 1$, *and*

$$\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} = \infty,$$

*then there exists a continuous function* $f \in P\Lambda BV$, *the Fourier series of which diverges at* $(0, \ldots, 0)$.

Theorem 5.2 imply

**Corollary 5.3:** *a) If* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *with*

$$\lambda_n = \frac{n}{\log^{d-1+\varepsilon} n}, \qquad n = 2, 3, \ldots$$

*for some* $\varepsilon > 0$, *then the class* $P\Lambda BV$ *is a class of convergence on* $T^d$.

*b) If* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *with*

$$\lambda_n = \frac{n}{\log^{d-1} n}, \qquad n = 2, 3, \ldots,$$

*then the class* $P\Lambda BV$ *is not a class of convergence on* $T^d$.

**Theorem 5.4** (Goginava, Sahakian [18])**:** *a) If* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *with*

$$\lambda_n = \frac{n}{\log^{d-1} n}, \qquad n = 2, 3, \ldots,$$

*then the class* $\Lambda^{\#}BV\left(T^d\right)$ *is a class of convergence on* $T^d$.

*b)If* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *with*

$$\lambda_n := \left\{\frac{n\xi_n}{\log^{d-1} n}\right\}, \quad n = 2, 3, \ldots,$$

*where* $\xi_n \to \infty$ *as* $n \to \infty$, *then there exists a continuous function* $f \in \Lambda^{\#}BV\left(T^d\right)$ *such that the cubical partial sums of d-dimensional Fourier series of* $f$ *diverge unboundedly at* $(0, ..., 0) \in T^d$.

**Theorem 5.5** (Goginava, Sahakian [18])**:** *For any* $d > 1$ *the class of functions* $f(x)$, $x \in T^d$ *satisfying the following condition*

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}(f, n) \log^{d-1} n}{n^2} < \infty, \quad s = 1, ..., d,$$

*is a class of convergence.*

## 6. Cesàro summability of $d$-dimensional Fourier series

The Cesàro $(C;\alpha_1,...,\alpha_d)$ means of $d$-dimensional Fourier series of function $f\in L^1(T^d)$ is defined by

$$\sigma_{m_1,...,m_d}^{\alpha_1,...,\alpha_d}[f;(x_1,...,x_d)]$$
$$:=\left(\prod_{i=1}^{d}A_{m_i}^{\alpha_i}\right)^{-1}\sum_{p_1=0}^{m_1}\cdots\sum_{p_d=0}^{m_d}\prod_{i=1}^{d}A_{m_i-p_i}^{\alpha_i-1}S_{p_1,...,p_d}[f,(x_1,...,x_d)],$$

where

$$A_0^\alpha=1,\ A_n^\alpha=\frac{(\alpha+1)\cdots(\alpha+n)}{n!},\qquad \alpha>-1.$$

The Fourier series $S[f]$ is said to be $(C;-\alpha_1,...,-\alpha_d)$ summable to $s$ in a point $(x_1,\ldots,x_d)$, if

$$\sigma_{m_1,...,m_d}^{\alpha_1,...,\alpha_d}[f;(x_1,...,x_d)]\to s\quad\text{as}\quad x_1,...,x_d\to\infty.$$

**Definition 6.1:** We say that the class of functions $\Omega\subset L^1(T^d)$ is a class of $(C;-\alpha_1,...,-\alpha_d)$ summability on $T^d$, if the Cesaro $(C;-\alpha_1,...,-\alpha_d)$ means of Fourier series of any function $f\in\Omega$ converges to $f^*(x)$ at any regular point $x\in T^d$. The summability is uniform on any compact $K\subset T^d$, if in addition, $f$ is continuous on the neighborhood of $K$.

The multivariate analog of Theorem W2 from Section 3 was proved by A. Bakhvalov in [2].

**Theorem B2** (A. Bakhvalov [2]): *For any numbers $\alpha_1,...,\alpha_d\in(0,1)$ the class $C\{n^{1-\alpha_1}\},\ldots\{n^{1-\alpha_d}\}V(T^d)$ is a class of $(C;-\alpha_1,...,-\alpha_d)$ summability on $T^d$.*

In the next theorem we consider the problem of $(C;-\alpha_1,...,-\alpha_d)$ summability of the Fourier series of functions of bounded partial $\Lambda$-variation.

**Theorem 6.2** (U. Goginava, A. Sahakian [16])**:** *Suppose $\alpha_1,...,\alpha_d\in(0,1)$, $\alpha_1+\cdots+\alpha_d<1$ and the sequence $\Lambda=\{\lambda_n\}_{n=1}^\infty$ is such that*

$$\frac{\lambda_n}{n^{1-(\alpha_1+\cdots+\alpha_d)}}\downarrow 0\ .$$

*a) If*

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n^{2-(\alpha_1+\cdots+\alpha_d)}}<\infty,$$

*then $P\Lambda BV(T^d)$ is a class of $(C;-\alpha_1,...,-\alpha_d)$ summability on $T^d$.*

*b) If*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n^{2-(\alpha_1+\cdots+\alpha_d)}} = \infty,$$

*then there exists a continuous function $f \in P\Lambda BV(T^d)$ for which the sequence $\sigma_{N,...,N}^{-\alpha_1,...,-\alpha_d}[f,(0,...,0)]$ diverges.*

**Corollary 6.3** (U. Goginava, A. Sahakian [16]**:** *Suppose $\alpha_1,...,\alpha_d \in (0,1)$, $\alpha_1 + \cdots + \alpha_d < 1$ and $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$.*

*a) If*

$$\lambda_n = \frac{n^{1-(\alpha_1+\cdots+\alpha_d)}}{\log^{1+\varepsilon} n}, \qquad n = 2.3.\ldots$$

*for some $\varepsilon > 0$, then the class $P\Lambda BV(T^d)$ is a class of $(C; -\alpha_1,...,-\alpha_d)$ summability on $T^d$.*

*b) If*

$$\lambda_n = \frac{n^{1-(\alpha_1+\cdots+\alpha_d)}}{\log n}, \qquad n = 2.3.\ldots,$$

*then $P\Lambda BV(T^d)$ is not a class of $(C; -\alpha_1,...,-\alpha_d)$ summability on $T^d$.*

**Theorem 6.4** (U. Goginava, A. Sahakian [16])**:** *Let $\alpha_1,...,\alpha_d \in (0,1)$, $\alpha_1 + \cdots + \alpha_d < 1$. Then the set of functions $f$ satisfying the conditions*

$$\sum_{j=0}^{\infty} \frac{\left(v_i\left(2^j, f\right)\right)^{\alpha_i/(\alpha_1+\cdots+\alpha_d)}}{2^{j(\alpha_i/(\alpha_1+\cdots+\alpha_d)-\alpha_i)}} < \infty \quad \text{for} \quad i = 1,...,d,$$

*is a class of $(C; -\alpha_1,...,-\alpha_d)$ summability on $T^d$.*

**Theorem 6.5** (U. Goginava, A. Sahakian [16])**:** *Suppose $\alpha_1,...,\alpha_d \in (0,1)$, $\alpha_1 + \cdots + \alpha_d < 1/p$, $p \geq 1$. Then the class $PBV_p$ is a class of $(C; -\alpha_1,...,-\alpha_d)$ summability on $T^d$.*

In [8] the first author has proved that the class $PBV_p$ is not a class of $(C; -\alpha_1,...,-\alpha_d)$ summability on $T^d$, if $\alpha_1,...,\alpha_d \in (0,1)$, and $\alpha_1 + \cdots + \alpha_d \geq 1/p$.

**Corollary 6.6** (U. Goginava, A. Sahakian [16]) **:** *Suppose $\alpha_1,...,\alpha_d \in (0,1)$, $\alpha_1 + \cdots + \alpha_d < 1$. Then the set of functions $f$ satisfying*

$$v_i\left(2^j, f\right) = O\left(2^{j\gamma}\right) \quad \text{for} \quad i = 1,...,d,,$$

*is a class of $(C; -\alpha_1,...,-\alpha_d)$ summability on $T^d$.*

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## References


[1] A.N. Bakhvalov, *Continuity in $\Lambda$-variation of functions of several variables and the convergence of multiple Fourier series* (Russian), Mat. Sb. **193**, 12 (2002), 3-20; English transl.in *Sb. Math.* **193**, 11-12 (2002), 1731-1748

[2] A.N. Bakhvalov, *Continuity in $\Lambda$-variation and the summation of multiple Fourier series by Cesàro methods* (Russian), Mat. Zametki **90**, 4 (2011), 483-500; translation in Math. Notes **90**, 3-4 (2011), 469-484

[3] A. N. Bakhvalov, *On the localization of Cesàro means of Fourier series of functions of bounded $\Lambda$-variation* (Russian), Izv. Vyssh. Uchebn. Zaved. Mat., **55**, 8 (2011), 9-13; translation in Russian Math., (Iz. VUZ), **55**, 8 (2011), 7-10

[4] Z.A. Chanturia, *The modulus of variation of a function and its application in the theory of Fourier series*, Soviet. Math. Dokl., **15** (1974), 67-71

[5] M.I. Dyachenko, *Waterman classes and spherical partial sums of double Fourier series*, Anal. Math., **21** (1995), 3-21

[6] M.I. Dyachenko, *Two-dimensional Waterman classes and u-convergence of Fourier series* (Russian), Mat. Sb., **190**, 7 (1999), 23-40; English transl.in Sb. Math., **190**, 7-8 (1999), 955-972

[7] M.I. Dyachenko, D. Waterman, *Convergence of double Fourier series and W-classes*, Trans. Amer. Math. Soc., **357** (2005), 397-407

[8] U. Goginava, *On the uniform summability of two-dimensional trigonometric Fourier series*, Proc. A. Razmadze Math. Inst., **124** (2000), 55-72

[9] U. Goginava, *Uniform convergence of Cesáro means of negative order of double Walsh-Fourier series*, J. Approx. Theory, **124** (2003), 96-108

[10] U. Goginava, *On the uniform convergence of multiple trigonometric Fourier series*, East J. Approx., **5**, 3 (1999), 253-266

[11] U. Goginava, A. Sahakian, *On the convergence of Fourier series of functions of bounded partial generalized variation*, East J. Approx., **16**, 2 (2010), 153-165

[12] U. Goginava, *Cesaro means of negative order of double Fourier series and generalized bounded variation*, Siberian Mathematical Journal, **54**, 6 (2013), 1005-1013

[13] U. Goginava, A. Sahakian, O*n the convergence of Cesáro means of negative order of double trigonometric Fourier series of functions of bounded partial generalized variation*, Acta. Sci. Math. (Szeged), **77** (2011), 451-471

[14] U. Goginava, A. Sahakian, *Convergence of double Fourier series and generalized $\Lambda$-variation*, Georgian Math. J. **19**, 3 (2012), 497-509

[15] U. Goginava, A. Sahakian, *On the convergence of multiple Fourier series of functions of bounded partial generalized variation*, Anal. Math., **39**, 1 (2013), 45-56

[16] U. Goginava, A. Sahakian, *On the summability of multipleFourier series of functions of bounded partial generalized variation*, Proc. Steklov Inst. Math., **280** (2013), 150-161

[17] U. Goginava, A. Sahakian, *On the convergence and summability of double Walsh-Fourier series of functions of bounded generalized variation*, Journal of Contemporary Mathematical Analysis **48**, 6 (2014)

[18] U. Goginava, A. Sahakian, *Convergence of multiple fourier series of functions of bounded generalized variation* (submitted)

[19] G.H. Hardy, *On double Fourier series and especially which represent the double zeta function with real and incommensurable parameters*, Quart. J. Math. Oxford Ser., **37** (1906), 53-79

[20] C. Jordan, *Sur la series de Fourier*, *C.R. Acad. Sci. Paris*, **92** (1881), 228-230

[21] J. Marcinkiewicz, *On a class of functions and their Fourier series*, Compt. Rend. Soc. Sci. Warsowie, **26** (1934), 71-77

[22] A.I. Sablin, *$\Lambda$-variation and Fourier series* (Russian), Izv. Vyssh. Uchebn. Zaved. Mat., **10** (1987), 66-68; English transl. in Soviet Math., (Iz. VUZ) **31** (1987)

[23] O.G. Sargsyan, *On the convergence and the Gibbs phenomenon of multiple Fourier series for functions of bounded harmonic variation* (Russian), Izv. Akad. Nauk Armenii Mat., **28**, 3 (1993), 3-20, English transl. in J. Contemp. Math. Anal., **28**, 3 (1993)

[24] A.A. Sahakian, *On the convergence of double Fourier series of functions of bounded harmonic variation* (Russian), Izv. Akad. Nauk Armyan. SSR Ser. Mat., **21**, 6 (1986), 517-529, English transl., Soviet J. Contemp. Math. Anal., **21**, 6 (1986), 1-13

[25] D. Waterman, *On the summability of Fourier series of functions of $\Lambda$-bounded variation*, Studia Math., **55** (1976), 87-95

[26] D. Waterman, *On convergence of Fourier series of functions of generalized bounded variation*, Studia Math., **44**, 1 (1972), 107-117

[27] N. Wiener, *The quadratic variation of a function and its Fourier coefficients*, Massachusetts J. Math., **3** (1924), 72-94

[28] L.V. Zhizhiashvili, *Trigonometric Fourier Series and their Conjugates*, Kluwer Academic Publishers, Dobrecht, Boston, London, 1996

[29] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, 1959.