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On the Generalized Cesáro Means of Trigonometric Fourier Series

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The behavior of generelized Cesáro (C,α_n)-means $(\alpha_n \in (-1,d), d > 0)$ of trigonometric Fourier series of H^ω classes in the space of continuous functions is studied. The unimprovement of the obtained results is given.
In 1953 Nash [20] introduced the class of functions Φ. In this paper the behaviour of generalized Cesáro (C,α_n)-means $(\alpha_n \in (-1,0))$ of trigonometric Fourier series of $H^\omega \cap \Phi$ classes in the space of continuous functions is investigated. The sharpness of the results obtained is formulated.
Furthermore, analog of theorem (2.9) for the multiple case is given.

Keywords: Trigonometric system, Cesáro means, H^ω classes, Classes Φ, Mixed and particular modulus of continuity.

AMS Subject Classification: 42A24, 42A16.

1. Introduction

Let f be a 2π-periodic Lebesgue integrable function and

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos ix dx \quad \text{and} \quad b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin ix dx$$

be its Fourier coefficients. Also let

$$S_n(x,f) = \frac{a_0}{2} + \sum_{i=1}^{n}(a_i \cos ix + b_i \sin ix) \tag{1.1}$$

be partial sums of the Fourier series of f with respect to the trigonometric system. Let $C([0,2\pi])$ denotes the space of 2π-periodic continuous functions with the norm $||f||_{C([0,2\pi])} := max_{x\in[0,2\pi]}$. If $f \in C[0,2\pi]$ then

$$\omega(\delta,f) = \max\{|f(x_1) - f(x_2)| \ : \ |x_1 - x_2| \le \delta, \ x_1, x_2 \in [0,2\pi]\}$$

is called the modulus of continuity of *f*.

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If a modulus of continuity ω (see [1]) is given then H^ω denotes the class of functions $f \in C([0,2\pi])$ for which

$$\omega(\delta, f) \leq \omega(\delta), \quad \delta \in [0, 2\pi).$$

If $\omega(\delta) = C_0 \cdot \delta$, where C_0 is a positive constant, then $H^\omega \equiv Lip_{C_0} 1$.

We consider a generalized Cesáro method (see [2]). Let $(\alpha_n)$and (S_n) be sequences of real numbers, where $\alpha_n > -1$, $n = 1, 2, \ldots$. Suppose

$$\sigma_n^{\alpha_n} = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha_n - 1} S_\nu / A_n^{\alpha_n}, \tag{1.2}$$

where

$$A_k^{\alpha_n} = (\alpha_n + 1)(\alpha_n + 2)...(\alpha_n + k)/k! \ .$$

If (α_n) is a constant sequence $(\alpha_n = \alpha,\ n = 1, 2, ...)$ then $\sigma_n^{\alpha_n}$ coincides with the usual Cesáro σ_n^α-means ([3], Ch.III). If in (1.2) instead of S_ν we substitute $S_\nu(x, f)$ (see (1.1)) then the corresponding means $\sigma_n^{\alpha_n}$ are denoted by $\sigma_n^{\alpha_n}(x, f)$.

Many authors have considered the convergence behaviour of $\sigma_n^\alpha(x, f)$ for functions from various classes (Fejér [4], Riesz [5], Zygmund [6], Natanson [7], Izumi [8], Satô ([9],[10]), Taberski [11], Stechkin [12], Zamansky [13], Efimov [14], Uljanoff [15], Zhzhiashvili [16], Totik ([17],[18])).

It is well-known (cf. [19] and [3] (Ch. III, Theorem (1.2)) that a summation method defined by a matrix (a_{ij}) $(i, j = 0,\ 1,\ ...)$ is regular if and only if

$$\begin{cases} 1.\ \lim\limits_{n\to\infty} a_{n\nu} = 0, \quad \nu = 0,\ 1,\ \ldots, \\ 2.\ N_n \equiv |a_{n0}| + |a_{n1}| + \cdots + |a_{nn}| + \cdots \ \text{is a bounded sequence}, \\ 3.\ \lim\limits_{n\to\infty} a_{n\nu} = 0, \ \text{where } A_n \equiv a_{n0} + a_{n1} + \cdots + a_{nn} + \cdots . \end{cases}$$

In particular, the (C, α)-summation method is regular if and only if $\alpha \geq 0$ (see [3], Ch. III, Theorem (1.21)).

In 1953 Nash [20] introduced the class of functions Φ.

Definition 1.1: Let Φ be a positive sequence with $\lim\limits_{n\to\infty} \Phi(n) = +\infty$. We say that a function $f \in C([0, 2\pi])$ belongs to the class Φ $(f \in \Phi)$ if for every real number $a,\ b$ $(|b - a| \leq 2\pi)$ and uniformly in x

$$\left| \int_a^b f(x+t) \cos ntdt \right| \leq \frac{1}{\Phi(n)}.$$

Nash [20] established the fact that if $f \in C([0, 2\pi]) \cap \Phi$ and

$$\overline{\lim_{n\to\infty}} \Phi(n)/n = +\infty$$

then $f \equiv 0$. Therefore, it is natural to assume that $\Phi(n) = O(n)$.

Furthermore, Nash [20] proved the theorem from which various tests for uniform convergence of Fourier series turn out.

Later Satô [21] (see, also, [22], pp.299-302) gave more precise result and she established analogous of her early statement for Cesáro summability method of negative order. In [23] we investigated Satô's [24] results for Fourier series and for its conjugate series; studied analagous problems for Cesáro summability method as well

Theorem 1.2: *(cf. [23]). Let $f \in C([0, 2\pi]) \cap \Phi$ and $0 < \alpha < 1$. Then there exists a positive constant $c(f)$ such that*

$$\left\|\sigma_n^{-\alpha}(\cdot, f) - f(\cdot)\right\|_C \leq c(f) \left[\omega^{1-\alpha}(1/n, f) \left(\frac{n}{\Phi(n)}\right)^{\alpha} + \frac{1}{n} \int_{\pi/n}^{\pi} \frac{\omega(t, f)}{t^2} dt\right].$$

The second term on the right side of the last estimation can be omitted (cf. [25]), i.e. under the conditions of the last theorem the following estimation is valid

$$||\sigma_n^{-\alpha}(\cdot,\, f) \;-\; f(\cdot)||_{C([0,2\pi])} \leq c(f)\omega^{1-\alpha}(1/n,\, f) \left(\frac{n}{\Phi(n)}\right)^{\alpha}.$$

In [25] the unimprovement of this statement is proved.

2. Formulation of the results

Theorem 2.1: *Let (α_n) be any sequence on the interval $(-1, d]$, where d is a real number $(d \in R)$. The summation method defined by (1.2) is a regular method if and only if*

$$\liminf_{n \to \infty} (\alpha_n \ln n) > -\infty.$$

Corollary 2.2: *If (α_n) is any sequence with $\alpha_n \geq -C/\ln n$, where C is a positive constant, then (C, α_n) is a regular method.*

Theorem 2.3: *If $f \in H^\omega$ and $\alpha_n \in (0, 1]$, $n = 3, 4, ...$, then*

$$||\sigma_n^{\alpha_n}(\cdot, f) \;-\; f(\cdot)||_C \leq C \max\left\{\frac{n^{\alpha_n} - 1}{\alpha_n \cdot n^{\alpha_n}}\omega(1/n), \frac{\alpha_n}{n}\int_{\pi/n}^{\pi} \frac{\omega(t)}{t^2} dt\right\},$$

where C is an absolute constant.

Corollary 2.4: *Let $f \in H^w$. Then*

$$||\sigma_n^{\alpha_n}(\cdot, f) - f(\cdot)||_C \leq C\omega(1/n) \ln n, \quad n = 3.4 \ldots.$$

Corollary 2.5: *If* $f \in H^w$ *and* $0 < \delta < \alpha_n < 1,\ n = 1, 2, ...,$ *where* δ *is a constant, then*

$$||\sigma_n^{\alpha_n}(\cdot, f) - f(\cdot)||_C \leq C(\delta, \omega)\omega(1/n)\ln\left(1/\omega(1/n)\right).$$

Theorem 2.6: *Let* ω *be a modulus of continuity and* $\alpha_n \in (0, 1]$, *then*

$$\sup_{f\in H^\omega}\limsup_{n\to\infty}(||\sigma_n^{\alpha_n}(\cdot, f) - f(\cdot)||_C/d_n) > 0,$$

where

$$d_n = \max\left\{\frac{n^{\alpha_n} - 1}{\alpha_n \cdot n^{\alpha_n}}\omega(1/n), \frac{\alpha_n}{n}\int\limits_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt\right\}.$$

Theorem 2.7: *Suppose* $f \in H^w$ *and for all natural* n $1 < \alpha_n \leq d$ (d *is a positive constant*). *Then*

$$||\sigma_n^{\alpha_n}(\cdot, f) - f(\cdot)||_C \leq \frac{C(d)}{n}\int\limits_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt.$$

It is well-known that in the case where $\alpha_n \equiv 1$, for all natural n, the correctness of the last estimation was established by Natanson [7] (see also [15]).

Theorem 2.8: *There exists a function* $f \in H^w$ *such that for every sequence* (α_n) $(\alpha_n \in (1, d],\ n = 1, 2, ...,\ d > 1)$ *and for all natural* n

$$||\sigma_n^{\alpha_n}(\cdot, f) - f(\cdot)||_C \geq \frac{\widetilde{C}}{n}\int\limits_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt,$$

where $\widetilde{C}$ *is a positive constant.*

Some of these results were announced in [26] without proof.

Theorem 2.9: *Let* (α_n) *be any sequence on the interval* $(0, 1),\ n = 3,\ 4, ...,$ *and* $f \in H^\omega$ *then*

$$||\sigma_n^{-\alpha_n}(\cdot, f)\ -\ f(\cdot)||_C \leq C_\omega\omega(1/n)\frac{n^{\alpha_n} - 1}{\alpha_n \cdot (1 - \alpha_n)}. \tag{2.1}$$

For the class of functions $Lip_{C_0}1$ in the case $\liminf\limits_{n\to\infty}\alpha_n > 0$ we can get more precise estimation than the last one is.

Theorem 2.10: *If for all natural* n $\alpha_n \in (0, 1)$ *and* $\liminf\limits_{n\to\infty}\alpha_n > 0$. *Then for every function* $f \in Lip_{C_0}1$ *there exists a positive infinitesimal sequence* (ε_n), *such that*

$$||\sigma_n^{-\alpha_n}(\cdot,f)\ -\ f(\cdot)||_C \le \frac{\varepsilon_n}{(1-\alpha_n)n^{1-\alpha_n}}.$$

The estimations in theorems 2.9 and 2.10 are senseless if there exist a number $\varepsilon_0(0<\varepsilon_0<1)$ and a sequence of natural numbers (m_k) such that

$$\alpha_{m_k} \ge 1-\varepsilon_0\frac{\ln\ln m_k}{\ln m_k},\ \ k=1,2,....$$

Indeed,

$$\frac{m_k^{\alpha_{m_k}}}{1-\alpha_{m_k}} \ge \frac{\ln m_k}{\varepsilon_0 \ln\ln m_k}\cdot m_k^{1-\varepsilon_0\frac{\ln\ln m_k}{\ln m_k}}$$

$$=\frac{\ln m_k}{\varepsilon_0\ln\ln m_k}\cdot\frac{m_k}{(e^{\ln\ln m_k})^{\varepsilon_0}}=\frac{(\ln m_k)^{1-\varepsilon_0}\cdot m_k}{\varepsilon_0\ln\ln m_k}.$$

Therefore, it is natural to assume

$$0<\alpha_n\le 1-\frac{\ln\ln n}{\ln n},\ n=3,4,....$$

Corollary 2.11: *Let* $f\in H^\omega$ *and there exists a positive constant* C *such that* $0<\alpha_n<C/\ln n,\ n=3,4,...,,\ \alpha_n\in(0,1)$, *then*

$$||\sigma_n^{-\alpha_n}(\cdot,f)\ -\ f(\cdot)||_C \le C_\omega\omega(1/n)\ln n.$$

In particular, if Dini-Lipschitz condition

$$\omega(1/n)=\overline{\overline{o}}(1/\ln n),\ n\to\infty,$$

is fulfilled then

$$||\sigma_n^{-\alpha_n}(\cdot,f)\ -\ f(\cdot)||_C = \overline{\overline{o}}(1),\ n\to\infty.$$

Therefore, Dini-Lipschitz condition is enough not only for the uniform convergence of the corresponding Fourier series, but it ensures the uniform convergence of $\sigma_n^{-\alpha_n}$-means for some negative sequence (α_n).

Corollary 2.12: *If* $f\in H^\omega$ *and* $\delta_1/\ln n\le\alpha_n\le\delta_2<1,\ n=3,4,...,$ *where* δ_1 *and* δ_2 *are positive constants, then*

$$||\sigma_n^{-\alpha_n}(\cdot,f)\ -\ f(\cdot)||_C \le C_\omega(\delta_1,\delta_2)\omega(1/n)\frac{n^{\alpha_n}}{\alpha_n},\ n=3,4,....$$

Corollary 2.13: *Let* $f\in H^\omega,\ H^\omega\ne Lip_{C_0}1$ *(for any positive constant* C_0*) and* $0<\delta_1\le\alpha_n\le\delta_2<1,\ n=3,4,...,$ *where* δ_1 *and* δ_2 *are constants.*

Then

$$||\sigma_n^{-\alpha_n}(\cdot,f) \; - \; f(\cdot)||_C \leq C_\omega(\delta_1,\delta_2)\omega(1/n)n^{\alpha_n}.$$

The last corollary, for the constant sequence (α_n), implies the well-known Zygmund [27] statement.

Corollary 2.14: *Let* $f \in Lip_{C_0}1$ *and* $0 < \delta_1 \leq \alpha_n \leq \delta_2 < 1,\ n = 3,4,...,$ *where* δ_1 *and* δ_2 *are constants. Then*

$$||\sigma_n^{-\alpha_n}(\cdot,f) \; - \; f(\cdot)||_C = \overset{=}{o}(n^{\alpha_n-1}),\ n \to \infty.$$

From the Corollary 2.14 it follows our [28] earlier theorem.

Corollary 2.15: *If* $f \in H^\omega,\ H^\omega \neq Lip_{C_0}1$ *(for any positive constant* C_0*) and* $0 < \delta \leq \alpha_n < 1,\ n = 3,4,...,$ *where* δ *is a constant, then*

$$||\sigma_n^{-\alpha_n}(\cdot,f) \; - \; f(\cdot)||_C \leq C_\omega(\delta)\omega(1/n)\frac{n^{\alpha_n}}{1-\alpha_n}.$$

Corollary 2.16: *Let* $f \in Lip_{C_0}1$ *and* $0 < \delta < \alpha_n < 1,\ n = 3,4,...,$ *where* δ *is a constant. Then*

$$||\sigma_n^{-\alpha_n}(\cdot,f) \; - \; f(\cdot)||_C = \overset{=}{o}\left(\frac{n^{\alpha_n-1}}{1-\alpha_n}\right),\ n \to \infty.$$

It is clear that Corollary 2.14 implies directly from Corollary 2.16. Also, from Corollary 2.12 or from Corollary 2.16 it follows Corollary 2.13.

Formulated results and, in particular, Theorem 2.9 and Theorem 2.10 are the best possible.

Theorem 2.17: *Let* $\alpha_n \in (0,1), n \in N$. *If* $\liminf_{n\to\infty}\alpha_n = 0$ *then*

$$\sup_{f\in H^\omega}\limsup_{n\to\infty}\frac{||\sigma_n^{-\alpha_n}(\cdot,f) - f(\cdot)||_C}{\omega(1/n)\frac{n^{\alpha_n-1}}{\alpha_n\cdot(1-\alpha_n)}} > 0.$$

Theorem 2.18: *Let* (ε_n) *be any positive infinitesimal sequence and* $0 < \alpha_n < 1,\ n = 1,2,....$ *If* $\liminf_{n\to\infty}\alpha_n > 0$ *then*

$$\sup_{f\in Lip_{C_0}1}\limsup_{n\to\infty}\frac{||\sigma_n^{-\alpha_n}(\cdot,f) - f(\cdot)||_C}{\frac{n^{\alpha_n-1}\varepsilon_n}{1-\alpha_n}} > 0.$$

Consider the estimations of Corollaries 2.4 and 2.11. They are Dini-Lipschitz type estimations. It is natural because the following statements are true.

Theorem 2.19: *If* $-C_1/\ln n \leq \alpha_n \leq C_2/\ln n$, $n = 2, 3, ...$, *(* C_1 *and* C_2 *are positive numbers) then* $\sigma_n^{\alpha_n}(\cdot, f)$ *convergence at a point* x *if and only if* $S_n(x, f)$ *is convergent.*

Theorem 2.20: *If* $\alpha_n \geq C/(\varepsilon_n \ln n)$ *where* (ε_n) *is a positive null sequence and* C *is a positive constant, then there exists a continuous function* f *for which* $\sigma_n^{\alpha_n}(0, f)$ *convergence and* $S_n(0, f)$ *is a divergent sequence.*

Theorem 2.21: *If* $\alpha_n \leq -C/(\varepsilon_n \ln n)$ *where* C *is a positive constant,* $\varepsilon_n > 0$ *and* $\lim\limits_{n\to\infty} \varepsilon_n = 0$. *Then there exists a continuous function* f *such that* $S_n(0, f)$ *convergence but* $\sigma_n^{\alpha_n}(0, f)$ *does not.*

Remark 1: Using Kolmogorov's well-known theorem we can conclude that there exists an integrable 2π-periodic function f generalized $\sigma_n^{\alpha_n}(\cdot, f)$ means $(0 < \alpha_n \leq C/\ln n)$ of which are divergent at each point.

Theorem 2.22: *a) Let* (α_n) *be a sequence on the interval* $(0, 1)$ *and* $f \in H^\omega \cap \Phi$. *Then for every sufficiently large natural number* n

$$\left\|\sigma_n^{-\alpha_n}(\cdot, f) - f(\cdot)\right\|_C \leq \frac{c_\Phi}{\alpha_n \cdot (1-\alpha_n)}\omega\left(\frac{1}{n}\right)\left[\left(\frac{n}{\Phi(n)\omega(1/n)}\right)^{\alpha_n} - 1\right]. \qquad (2.2)$$

b) For every sequence (α_n) *of the interval* $(0, 1)$ *and for arbitrary modulus of continuity* ω $(H^\omega \neq Lip1)$ *and a positive sequence* Φ $(\lim\limits_{n\to\infty} \Phi(n) = +\infty,\ \Phi(n) = O(n))$ *there are a function* $f_0 \in H^\omega \cap \Phi$, *a sequence* (n_k) *of natural numbers and a positive constant* c_0 *, such that*

$$\left\|\sigma_{n_k}^{-\alpha_{n_k}}(\cdot, f_0) - f_0(\cdot)\right\|_C \geq \frac{c_0}{\alpha_{n_k} \cdot (1-\alpha_{n_k})}\omega\left(\frac{1}{n_k}\right)\left[\left(\frac{n_k}{\Phi(n_k)\omega(1/n_k)}\right)^{\alpha_{n_k}} - 1\right].$$

The case $H^\omega = Lip_{c_0}1$ is studied in [6] (see Theorems 2, 3 and 5).

Corollary 2.23: *Under the assumptions of the last theorem we have for a sufficiently large number* n :

a) $\|\sigma_n^{-\alpha_n}(\cdot, f) - f(\cdot)\|_C \leq \frac{c(\Phi,\eta)}{\alpha_n}\omega\left(\frac{1}{n}\right)\left[\left(\frac{n}{\Phi(n)\omega(1/n)}\right)^{\alpha_n} - 1\right]$ *if* $\alpha_n \in (0, \eta),\ 0 < \eta < 1$;

b) $\|\sigma_n^{-\alpha_n}(\cdot, f) - f(\cdot)\|_C \leq \frac{c(\Phi,\eta)}{1-\alpha_n}\omega^{1-\alpha_n}\left(\frac{1}{n}\right)\left(\frac{n}{\Phi(n)}\right)^{\alpha_n}$ *if* $\alpha_n \in (\eta, 1),\ 0 < \eta < 1$;

c) $\|\sigma_n^{-\alpha_n}(\cdot, f) - f(\cdot)\|_C \leq c(\Phi, \eta, \gamma)\omega^{1-\alpha_n}\left(\frac{1}{n}\right)\left(\frac{n}{\Phi(n)}\right)^{\alpha_n}$ *if* $\alpha_n \in (\eta, \gamma),\ 0 < \eta < \gamma < 1$;

d) $\|\sigma_n^{-\alpha_n}(\cdot, f) - f(\cdot)\|_C \leq c_\Phi\omega\left(\frac{1}{n}\right)\ln\frac{n}{\Phi(n)\omega(1/n)}$ *if* $\lim\limits_{n\to\infty}\left(\frac{n}{\Phi(n)\omega(1/n)}\right)^{\alpha_n} = 1$.

The proof of Corollary 2.23 is evident.

Corollary 2.24: *Let* (α_n) *be any sequence on the interval* $(0, 1)$ *and* $f \in H^\omega$. *Then for every sufficiently large natural number* n (2.1) *is correct.*

Proof: It is enough to prove the last statement for function f with $\|f\|_C \leq 1$. Since $f \in H^\omega$ by Lemma 1 of [23] we may enclose that $f \in \Phi$, where $\Phi(n) = c_\omega/\omega(1/n)$. Thus from (2.2) we obtain

$$\left\|\sigma_n^{-\alpha_n}(\cdot, f) - f(\cdot)\right\|_C \leq \frac{c_\omega}{\alpha_n \cdot (1-\alpha_n)} \omega\left(\frac{1}{n}\right)\left[(c_\omega \cdot n)^{\alpha_n} - 1\right].$$

Thus

$$\left\|\sigma_n^{-\alpha_n}(\cdot, f) - f(\cdot)\right\|_C \leq \frac{c_\omega}{\alpha_n \cdot (1-\alpha_n)} \omega\left(\frac{1}{n}\right) n^{\alpha_n}.$$

If $\alpha_n \geq 1/\ln(c_\omega n)$ then for sufficiently large n

$$n^{\alpha_n} \geq n^{1/\ln(c_\omega n)} = \frac{1}{c_\omega^{1/\ln(c_\omega n)}} \cdot (c_\omega n)^{1/\ln(c_\omega n)}$$

$$= \frac{1}{c_\omega^{1/\ln(c_\omega n)}} \cdot e > \frac{9}{10} e.$$

Hence in the examined case we obtain the validity of (2.1).

Now let's examine the case $\alpha_n < 1/\ln(c_\omega n)$. We shall prove that

$$(c_\omega n)^{\alpha_n} - 1 \leq 2(n^{\alpha_n} - 1)$$

i.e.

$$\frac{1}{n^{\alpha_n}} \leq 2 - c_\omega^{\alpha_n}.$$

For this purpose we consider the function

$$f(x) = 2 - (c_\omega)^x - 1/n^x, \quad x > 0,$$

and

$$f'(x) = \frac{\ln n}{n^x} - (c_\omega)^x \ln c_\omega.$$

Since $f(0) = 0$ and $f'(x) > 0$ on the interval $(0, 1/\ln(c_\omega n)]$ for sufficiently large n, we obtain (2.1). □

Now we shall formulate the analog of Theorem 2.9 for a multiple case. First we formulate some necessary notations.

Let $C([0,2\pi]^n)$ be the space of continuous on $T^n = [0,2\pi]^n$, 2π-periodic relative to each variable functions f with the norm:

$$||f||_C = ||f||_{C([0,2\pi]^n)} = \max_{x \in [0,2\pi]^n} |f(x)|.$$

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space, $M = \{1, 2, ..., n\}$ let $(n \in \mathbb{N}, n \geq 2)$, let B be an arbitrary subset of M, and $|B|$ be a number of elements of B.
For any $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $B \subseteq M$ let $x_B = (u_1, u_2, ..., u_n)$, where $u_i = x_i$ if $i \in B$ and $u_i = 0$ if $i \in B' = M \backslash B$. Let $B = \{s_1, s_2, ..., s_r\}$ then

$$\Delta^{\{s_i\}}(f, x, h_{\{s_i\}}) = f(x + h_{\{s_i\}}) - f(x).$$

The expression we get by successive application of operations $\Delta^{\{s_1\}}, ..., \Delta^{\{s_r\}}$ will be denoted by $\Delta^B(f, x, h_B)$.

The expression

$$\omega_B(\delta, f) = \sup_{|h_i| \leq \delta_i;\, i \in B} ||\Delta^B(f, \cdot, h_B)||_C \ \ (\delta_i \in (0, \pi])$$

is called a mixed or a particular modulus of continuity of a function f when $|B| \in [2, n]$ or $|B| = 1$ respectively. Let ω_B be mixed or a particular modulus of continuity (see, for example, [30], Ch. II, 1.1). If $\delta(B) = \{\delta_{i_1}, \delta_{i_2}, ..., \delta_{i_r}\}$ then

$$H(\omega_B, C) = \{f : \omega_B(\delta, f) \leq \omega_B(\delta(B)), \quad \delta_{i_j} \in (0, \pi],\, j = \overline{1, r}\}\,;$$

$$H(M, C) = \bigcap_{B \subseteq M} H(\omega_B, C)\,.$$

Suppose $S_p(x, f)$ is a rectangular partial trigonometric sums of a function f (see, for example, [30], Ch. II, 2.1) and

$$\sigma_m^{\alpha_m}(x, f) = \left(\prod_{i=1}^{n} A_{m_i}^{\alpha_{m_i}^{(i)}} \right)^{-1} \sum_{p \geq 0}^{m} \prod_{i=1}^{n} A_{m_i - p_i}^{\alpha_{m_i}^{(i)} - 1} S_p(x, f)\,,$$

where $m = \{m_1, m_2, ..., m_n\}$, $\alpha_m = \{\alpha_{m_1}^{(1)}, \alpha_{m_2}^{(2)}, ..., \alpha_{m_n}^{(n)}\}$, $p = \{p_1, p_2, ..., p_n\}$ and

$$A_k^l = (l + 1)(l + 2)...(l + k)/k!.$$

Notation $p \geq 0$ means that $p_i \geq 0,\ i = \overline{1, n}$.

Theorem 2.25 : *Let $f \in H(M, C)$ and $\alpha_m = \{\alpha_{m_1}^{(1)}, \alpha_{m_2}^{(2)}, ..., \alpha_{m_n}^{(n)}\}$ is a sequence in $\mathbb{R}^n$, $\alpha_{m_i} \in (0; 1)$, $i = \overline{1, n}$. There exists a positive constant (which doesn't depend on f and the sequence (α_m)) such that*

$$||\sigma_m^{-\alpha_m}(\cdot, f) - f(\cdot)||_C \leq C_\omega \sum_{B \subseteq M} \omega_B \left(\frac{1}{m_{i_1}}, \frac{1}{m_{i_2}}, ..., \frac{1}{m_{i_r}} \right) \times$$

$$\times \prod_{i_k \in B} \frac{m_{i_k}^{\alpha_{m_{i_k}}^{(i_k)}} - 1}{\alpha_{m_{i_k}}^{(i_k)} (1 - \alpha_{m_{i_k}}^{(i_k)})}.$$

Corollary 2.26: *If $f \in C([0,2\pi]^n)$ and for some i_0 $(1 \le i_0 \le n)$*

$$\omega_{\{i_0\}}\left(\frac{1}{k}, f\right) \cdot \left(\frac{k^{\alpha_k^{(i_0)}}}{\alpha_k^{(i_0)}(1-\alpha_k^{(i_0)})}\right)^{-n} \to 0, \quad k \to \infty,$$

and

$$\omega_{\{i\}}\left(\frac{1}{k}, f\right) = O\left(\left(\frac{k^{\alpha_k^{(i)}}-1}{\alpha_k^{(i)}(1-\alpha_k^{(i)})}\right)^{n}\right) \quad (i=\overline{1,n},\ i \ne i_0)$$

then

$$||\sigma_m^{-\alpha_m}(\cdot, f) - f(\cdot)||_C \to 0, \;\; m_i \to \infty, \;\; (i=\overline{1,n}).$$

References

[1] S.M. Nikol'skii, *The Fourier series with the given modulus of continuity* (Russian), Dokladi Akad. Nauk SSSR, **53** (1946), 191-197

[2] I.B. Kaplan, *Cesáro means of variable order*, Izv, Vyz. Uchebn. Zaved. Mathematica, **18**, 5 (1960), 62-73

[3] A. Zygmund, *Trigonometric Series*, Cambrige University Press, **1** (1959)

[4] L. Fejér, *Untersuchungen iiber Fouriersche Reihen*, Math. Ann., **58** (1904), 501-569

[5] M. Riesz, *Sur la sommation des séries de Fourier*, Acta Sci. Math. (Szeged), **1** (1923), 104-113

[6] A. Zygmund, *Sur la sommabilite des séries de Fourier des fonctions vérifiant la condition de Lipschitz*, Bull. de I'Acad. Polonaise, (1925), 1-9

[7] I.P. Natanson, *On the accurancy of representation of continuous periodic functions by the singular integrals* (Russian), Dokladi Akad. Nauk SSSR, **73** (1950), 273-276

[8] S. Izumi, *Some trigonometrical series*, Tohoku Math. J., **6** (1954), 30-34

[9] Satô, *Uniform convergence of Fourier series*, Proc. Jap. Acad., **30** (1954), 698-701

[10] Satô, *Uniform convergence of Fourier series*, Proc. Jap. Acad., **31** (1955), 261-263.

[11] R. Taberski, *On the convergence of singular integrals*, Zeszyty Naukowe Universytety im A. Mickiewicza, Math. Fiz., **2** (1960), 33-51

[12] S.B. Stechkin, *On the approximation of periodic functions by the sums of Fejér* (Russian), Trudy Mat. Inst. Steklov, **62** (1961), 48-60

[13] M. Zamansky, *Classes de saturation de certains pocds d'approximation des séries de Fourier*, Annali di Scuola Norm, Sup. di Pisa, **66** (1949), 19-93

[14] A.V. Efimov, *On the approximation of some classes of continuous functions by the Fourier and the Fejér sums* (Russian), Isvestija Akad. Nauk SSSR, Math. Ser., **22** (1958), 81-116

[15] P.L. Uljanoff, *On the approximation of the functions* (Russian), Siberian Math. J., **5** (1964), 418-437

[16] L. Zhizhiashvili, *On the trigonometric Fourier series* (Russian), Mat. Sbornik, **100** (1976), 580-609

[17] V. Totik, *On the strong approximation by the (C,α)-means of Fourier series*, I, Analysis Mathematica, **6** (1980), 57-85

[18] V. Totik, *On the strong approximation by the (C,α)-means of Fourier series*, II, Analysis Mathematica, **6** (1980), 165-184

[19] O. Toeplitz, *Uber allgemeine lineare mittelbildungen, Prace Matematyczno-Fyzyczne*, **22** (1911), 113-119

[20] J.P. Nash, *Uniform convergence of Fourier series*, Rice Inst. Pamplet (1953), 31-57

[21] M. Satô, *Uniform convergence of Fourier series*, Proc. Japan. Acad., **30** (1954), 528-531

[22] N.K. Bari, *Trigonometric Series* (Russian), Fizmatgiz, 1961

[23] T. Akhobadze, *On convergence and summability of Fourier series*, Analyses Mathematica, **8** (1982), 79-102

[24] M. Satô, *Uniform convergence of Fourier series*, II, Proc. Japan. Acad., **30** (1954), 698-701

[25] T. Akhobadze, *On the problem of convergence and Cesáro summability of trigonometric Fourier series*, Acta Math. Hung. **55** (1990), 3-31

[26] T. Akhobadze, *On generalized cesáro summability of trigonometric Fourier series*, Bull. Georgian Acad. Sci., **170** (2004), 23-24

[27] A. Zygmund, *Sur la′ sommabilitédes Séries de Fourier des functions vérifiant la condition de Lipschits*, Bull. de I′ Acad, Polon., (1925), 1-9

[28] T. Akhobadze, *On the uniform convergence and (C,α)-summability of trigonometric Fourier series* (Russian), Bull. Georgian Acad. Sci., **128** (1987), 249-252

[29] T. Akhobadze,*On the convergence of generalized cesáro means of trigonometric Fourier series*, II, Acta Math. Hung. **115** (2007), 79-100

[30] L. Zhizhiashvili, *Trigonometric Series and their Conjugates*, Kluwer Acad. Publ., 1996