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Hierarchical Models for Biofilms Occupying Thin Prismatic Domains

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In this paper hierarchical models of biofilms occupying a thin prismatic domain are considered.

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1. Introduction

1.1. *Some Remarks on Biofilms*

A biofilm is a complex gel-like aggregation of microorganisms like bacteria, cyanobacteria, algae, protozoa and fungi. They stick together, they attach to a surface and they embed themselves in a self-produced extracellular matrix of polymeric substances, called EPS. Even if a biofilm contains water, it is mainly in a solid phase. Biofilms can develop on surfaces which are in permanent contact with water, i.e. on solid/liquid interfaces or on different types of interfaces such as air/solid, liquid/liquid or air/liquid (see [1] and references therein).

To describe the complex structure of biofilms, we consider, four different phases: Live cells (B), Dead cells (D), Extra cellural matrix of polymetric substances – EPS (E), and Liquid (L). We denote the concentration of biomass by $C_\phi = \rho_\phi \phi$, where $\rho_\phi$is the mass density of the phase in [g/cm^3] and $\phi = B, D, E, L$is the volume fraction of the phases. We assume that the biomasses are incompressible and Newtonian, then ρ_B, ρ_D, ρ_L, and ρ_E are positive constants, and also that the phases have all the same constant density. Since EPS encompasses the cells, we can assume that live cells, dead cells, and EPS have the same transport velocityv_s. We denote instead by v_L the velocity of liquid, and by Γ_ϕ, with ($\phi = B, D, E, L$), the mass exchange rate. The equations expressing mass balance with the equations for

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the velocity and pressure P give the following system (see [1])

$$\partial_t B + \nabla \cdot (B v_s) = B(L k_B - k_D),$$

$$\partial_t D + \nabla \cdot (D v_s) = \alpha B k_D - D k_N,$$

$$\partial_t E + \nabla \cdot (E v_s) = B L k_E - \in E,$$

$$\partial_t L + \nabla \cdot (L v_L) = B\left[(1-\alpha)k_D - L k_B - L k_E\right] + D k_N + \in E,$$

$$\partial_t[(1-L)v_s] + \nabla \cdot [(1-L)v_s \otimes v_s] + (1-L)\nabla P = \nabla \Sigma + (M - \Gamma_L)v_L - M v_s,$$

$$\partial_t(L v_L) + \nabla \cdot (L v_L \otimes v_L) + L \nabla P = -(M - \Gamma_L)v_L + M v_s,$$

$$-\Delta P = \nabla \cdot [\nabla \cdot ((1-L)v_s \otimes v_s + L v_L \otimes v_L)] - \Delta \Sigma,$$

where $k_B$and k_D are respectively a birth term and a death term for the active bacterial cells, α is the fraction of active cells that gives rise to dead cells (the remaining proportion becoming liquid), k_N is the natural decay of dead cells, k_E represents the production of EPS, and $\in E$, with $\in$ constant, is the natural decay of EPS. We assume, for simplicity, that k_B, k_D, k_N, k_E are constants. M is a Darsy constant and Σ is a stress function

$$\Sigma := -\gamma(1-L), \qquad \gamma = const.$$

Assuming the volume constraint (see [1])

$$B + D + E + L = 1.$$

Γ_L is given by the expression

$$\Gamma_L := B\left[(1-\alpha)k_D - L k_B - L k_E\right] + D k_N + \in E.$$

On the boundary, we impose Neumann conditions for the volume ratios and no-flux boundary conditions for the velocities:

$$\nabla B \cdot n|_{\partial\Omega^b} = \nabla E \cdot n|_{\partial\Omega^b} = \nabla D \cdot n|_{\partial\Omega^b} = 0,$$

$$v_s \cdot n|_{\partial\Omega^b} = v_L \cdot n|_{\partial\Omega^b} = 0.$$

2. 2D Problem for Biofilm Occupying Thin Prismatic Domain

Assume that biofilm occupy the following domain

$$\Omega^b := \left\{(x_1, x_2, x_3) : \ -\infty < x_1 < +\infty, \ 0 < x_2 < l, \ \ 0 \le x_3 \le h^b, \ \ h^b = const\right\}.$$

For the sake of simplicity let all the physical and geometrical quantities under consideration are independent of x_1 and let $D = 0$, $E = 0$; so we arrive at the two-dimensional case

$$\begin{aligned}
&\partial_t B + \nabla_2 \cdot (B v_s) = B(L k_B - k_D),\\
&\partial_t[(1-L)v_s] + \nabla_2 \cdot [(1-L)v_s \otimes v_s] + (1-L)\nabla_2 P\\
&\qquad = \nabla_2 \Sigma + (M - \Gamma_L) v_L - M v_s,\\
&\partial_t(L v_L) + \nabla_2 \cdot (L v_L \otimes v_L) + L \nabla_2 P = -(M - \Gamma_L) v_L + M v_s,\\
&-\Delta_2 P = \nabla_2 \cdot [\nabla_2 \cdot ((1-L) v_s \otimes v_s + L v_L \otimes v_L)] - \Delta_2 \Sigma.
\end{aligned} \tag{1}$$

We consider the linearized problem, the case when all the unknown functions B, L, v_s, v_L, P are slightly perturbed from the constant values $B^*, L^*, v_s^*, v_L^*, P^*$ respectively, i.e., they can be written in the following form

$$B := B^* + \tilde{B}, \quad L := L^* + \tilde{L}, \quad v_s := v_s^* + \tilde{v}_s, \quad v_L := v_L^* + \tilde{v}_L, \quad P := P^* + \tilde{P}.$$

Let us assume that

$$B^* := 1 - \frac{k_D}{k_B}, \quad L^* := \frac{k_D}{k_B}, \quad v_s^* = v_L^* = 0,$$

and $k_B > k_D$.
System (1) can be rewritten as follows

$$\begin{aligned}
&\partial_t \tilde{B} + \left(1 - \frac{k_D}{k_B}\right) \nabla_2 \cdot \tilde{v}_s = \left(1 - \frac{k_D}{k_B}\right) k_B \tilde{L},\\
&\left(1 - \frac{k_D}{k_B}\right) \partial_t \tilde{v}_s + \left(1 - \frac{k_D}{k_B}\right) \nabla_2 \tilde{P} = -\gamma \, \nabla_2 \tilde{B} + M(\, \tilde{v}_L - \, \tilde{v}_s),\\
&\frac{k_D}{k_B} \partial_t \tilde{v}_L + \frac{k_D}{k_B} \nabla_2 \tilde{P} = -M(v_L - v_s),\\
&-\Delta_2 \tilde{P} = \gamma \, \Delta_2 \tilde{B},
\end{aligned} \tag{2}$$

which we solve under the following initial and boundary conditions

$$\begin{aligned}
&\tilde{L}(x_2, x_3, 0) = \tilde{L}^0(x_2, x_3), \qquad \tilde{v}_s(x_2, x_3, 0) = \tilde{v}_L(x_2, x_3, 0) = 0,\\
&\nabla_2 \tilde{L}(x_2, x_3, t)\Big|_{\partial\Omega^b} = \nabla_2 \tilde{P}(x_2, x_3, t)\Big|_{\partial\Omega^b} = 0,\\
&\tilde{v}_s(x_2, x_3, t)|_{\partial\Omega^b} = \tilde{v}_L(x_2, x_3, t)|_{\partial\Omega^b} = 0,
\end{aligned} \tag{3}$$

where $\tilde{L}^0(x_2, x_3)$ is a prescribed function.

Using Vekua's dimension reduction method (for the method see, e.g., [2]-[4]) in the zero approximation from (2) and (3) we get

$$
\begin{aligned}
&-\partial_t \tilde{L}_0 + \left(1 - \frac{k_D}{k_B}\right) \tilde{v}_{s0,2} = \left(1 - \frac{k_D}{k_B}\right) k_B \tilde{L}_0, \\
&\left(1 - \frac{k_D}{k_B}\right) \partial_t \tilde{v}_{s0} + \left(1 - \frac{k_D}{k_B}\right) \tilde{P}_{0,2} = \gamma \, \tilde{L}_{0,2} + M(\, \tilde{v}_{L0} - \tilde{v}_{s0}), \\
&\frac{k_D}{k_B} \partial_t \tilde{v}_L + \frac{k_D}{k_B} \tilde{P}_{0,2} = -M(v_{L0} - v_{s0}), \\
&\tilde{P}_{0,22} = \gamma \, \tilde{L}_{0,22} \, ,
\end{aligned}
\tag{4}
$$

$$
\tilde{L}_0(x_2, 0) = \tilde{L}_0^0(x_2), \qquad \tilde{v}_{s0}(x_2, 0) = \tilde{v}_{L0}(x_2, 0) = 0, \tag{5}
$$

$$
\begin{aligned}
&\tilde{L}_{0,2}(0, t) = \tilde{L}_{0,2}(l, t) = \tilde{P}_{0,2}(0, t) = \tilde{P}_{0,2}(l, t) = 0, \\
&\tilde{v}_{s0}(0, t) = \tilde{v}_{s0}(l, t) = \tilde{v}_{L0}(0, t) = \tilde{v}_{L0}(l, t) = 0,
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
&\left(\tilde{L}_0, \, \tilde{P}_0, \, \tilde{v}_{s0}, \, \tilde{v}_{L0}\right)(x_2, t) := \int_0^{h^b} \left(\tilde{L}, \, \tilde{P}, \, \tilde{v}_s, \, \tilde{v}_L\right)(x_2, x_3, t) dx_3, \\
&\tilde{L}_0^0(x_2) := \int_0^{h^b} \tilde{L^0}(x_2, x_3) dx_3
\end{aligned}
$$

are so called zero moments of the corresponding quantities $\tilde{L}$, $\tilde{P}$, $\tilde{v}_s$, $\tilde{v}_L$, and $\tilde{L^0}(x_2, x_3)$ (see, e.g., [2]-[4]),

$$
\tilde{L}(x_2, x_3, t) \cong \frac{1}{h^b} \tilde{L}_0(x_2, t), \quad \tilde{P}(x_2, x_3, t) \cong \frac{1}{h^b} \tilde{P}_0(x_2, t),
$$

$$
\tilde{v}_s(x_2, x_3, t) \cong \frac{1}{h^b} \tilde{v}_{s0}(x_2, t), \quad \tilde{v}_L(x_2, x_3, t) \cong \frac{1}{h^b} \tilde{v}_{L0}(x_2, t),
$$

$$
\tilde{L^0}(x_2, x_3) \cong \frac{1}{h^b} \tilde{L}_0^0(x_2).
$$

Summing the second and third equations of the system (4) and taking into account the fourth equation and IBC (5) we get

$$
\left(1 - \frac{k_D}{k_B}\right) \partial_t \tilde{v}_{s0} + \frac{k_D}{k_B} \partial_t \tilde{v}_{L0} = 0, \;\; \Rightarrow \;\; \left(1 - \frac{k_D}{k_B}\right) \tilde{v}_{s0} + \frac{k_D}{k_B} \tilde{v}_{L0} = f(x)
$$

in view of IC $(\tilde{v}_{s0}(x, 0) = \tilde{v}_L(x, 0) = 0)$, we get

$$\left(1-\frac{k_D}{k_B}\right)\tilde{v}_{s0}+\frac{k_D}{k_B}\tilde{v}_{L0}=0 \quad\Rightarrow\quad \tilde{v}_{s0}+\frac{k_D}{k_B}\left(\tilde{v}_{L0}-\tilde{v}_{s0}\right)=0 \quad\Rightarrow$$
$$(\tilde{v}_{L0}-\tilde{v}_{s0})=-\frac{k_B}{k_D}\tilde{v}_{s0}. \tag{7}$$

Therefore, the second equation of the system (4) can be rewritten as follows

$$\left(1-\frac{k_D}{k_B}\right)\,\partial_t\tilde{v}_{s0}+M\frac{k_B}{k_D}\tilde{v}_{s0}=\frac{k_D}{k_B}\gamma\,\tilde{L}_{0,2}\,,$$

$$\partial_t\tilde{v}_{s0}+\frac{Mk_B^2}{k_D(k_B-k_D)}\tilde{v}_{s0}=\frac{k_D}{k_B-k_D}\gamma\,\tilde{L}_{0,2}\,,$$

whose solution has the following form

$$\tilde{v}_{s0}=\int_0^t\frac{k_D}{k_B-k_D}\gamma\,\tilde{L}_{0,2}\,e^{\frac{Mk_B^2}{k_D(k_B-k_D)}(\tau-t)}d\tau. \tag{8}$$

From the third equation of (4), by virtue of (7), we obtain

$$\frac{k_D}{k_B}\partial_t\tilde{v}_L=M\frac{k_B}{k_D}\tilde{v}_{s0}-\frac{k_D}{k_B}\gamma\,\tilde{L}_{0,2},$$

whence,

$$\partial_t\tilde{v}_L=M\frac{k_B^2}{k_D^2}\tilde{v}_{s0}-\gamma\,\tilde{L}_{0,2}=\frac{Mk_B^2}{k_D(k_B-k_D)}\gamma\int_0^t\,\tilde{L}_{0,2}\,e^{\frac{Mk_B^2}{k_D(k_B-k_D)}(\tau-t)}d\tau-\gamma\,\tilde{L}_{0,2}$$

and

$$\tilde{v}_L=\frac{Mk_B^2}{k_D(k_B-k_D)}\gamma\int\limits_0^t ds\int\limits_0^s\,\tilde{L}_{0,2}\,e^{\frac{Mk_B^2}{k_D(k_B-k_D)}(\tau-s)}d\tau-\gamma\int\limits_0^t\tilde{L}_{0,2}\,d\tau$$
$$=\frac{Mk_B^2}{k_D(k_B-k_D)}\gamma\int\limits_0^t\tilde{L}_{0,2}\,d\tau\int\limits_\tau^t\,e^{\frac{Mk_B^2}{k_D(k_B-k_D)}(\tau-s)}ds-\gamma\int\limits_0^t\tilde{L}_{0,2}\,d\tau$$
$$=\gamma\int\limits_0^t\tilde{L}_{0,2}\left(1-e^{\frac{Mk_B^2}{k_D(k_B-k_D)}(\tau-t)}\right)d\tau-\gamma\int\limits_0^t\tilde{L}_{0,2}\,d\tau$$
$$=-\gamma\int\limits_0^t\tilde{L}_{0,2}\,e^{\frac{Mk_B^2}{k_D(k_B-k_D)}(\tau-t)}d\tau.$$

Finally, from the first equation of (4) taking into account (8), we get

$$\partial_t \tilde{L}_0 = \frac{k_D}{k_B}\gamma \int_0^t \tilde{L}_{0,22}\, e^{\alpha(t-\tau)} d\tau - (k_B - k_D)\,\tilde{L}_0\,, \quad \alpha\colon = -\frac{M k_B^2}{k_D(k_B-k_D)}. \tag{9}$$

Using the Laplace transform, from (9) we have

$$s\,\hat{L}_0 - \tilde{L}_0^0(x_2) = \frac{k_D}{k_B}\gamma\,\hat{L}_{0,22}\,\frac{1}{s-\alpha} - (k_B - k_D)\,\hat{L}_0,$$

hence,

$$\frac{k_D\gamma}{k_B(s-\alpha)}\,\hat{L}_{0,22} = (s + k_B - k_D)\,\hat{L}_0 + \tilde{L}_0^0(x_2)$$

and taking into account the homogeneous BC (6) $\tilde{L}_{0,2}(0,t) = \tilde{L}_{0,2}(l,t) = 0$, we obtain

$$\hat{L}_0(x_2,s) = \frac{\sqrt{k_B(s-\alpha)}}{2\sqrt{k_D\gamma(s+k_B-k_D)}}$$

$$\times \int_0^{x_2} \tilde{L}_0^0(\xi)\left[e^{\sqrt{\frac{k_B(s+k_B-k_D)(s-\alpha)}{k_D\gamma}}(x_2-\xi)} - e^{-\sqrt{\frac{k_B(s+k_B-k_D)(s-\alpha)}{k_D\gamma}}(x_2-\xi)}\right] d\xi$$

$$-\frac{\sqrt{k_B(s-\alpha)}}{2\sqrt{k_D\gamma(s+k_B-k_D)}}\,\frac{e^{\sqrt{\frac{k_B(s+k_B-k_D)(s-\alpha)}{k_D\gamma}}x_2} + e^{-\sqrt{\frac{k_B(s+k_B-k_D)(s-\alpha)}{k_D\gamma}}x_2}}{e^{\sqrt{\frac{k_B(s+k_B-k_D)(s-\alpha)}{k_D\gamma}}l} - e^{-\sqrt{\frac{k_B(s+k_B-k_D)(s-\alpha)}{k_D\gamma}}l}}$$

$$\times \int_0^{l} \tilde{L}_0^0(\xi)\left[e^{\sqrt{\frac{k_B(s+k_B-k_D)(s-\alpha)}{k_D\gamma}}(l-\xi)} + e^{-\sqrt{\frac{k_B(s+k_B-k_D)(s-\alpha)}{k_D\gamma}}(l-\xi)}\right] d\xi,$$

where by $\hat{L}_0(x_2,s)$ we denote the Laplace transform of the function $\tilde{L}_0(x_2,t)$.
Thus,

$$\hat{P}_0(x_2,s) = \gamma\,\hat{L}_0(x_2,s) + C(s).$$

For bounded on $[0,l]$ function $\tilde{L}_0^0(x)$ it can be shown that the inverse Laplace transform $\hat{L}_0(x_2,s)$ exists.

Examples:
1. $\tilde{L}_0^0(x_2) = \Lambda = const$, then

$$\tilde{L}_0(x_2,t) = \Lambda e^{(k_D-k_B)t}.$$

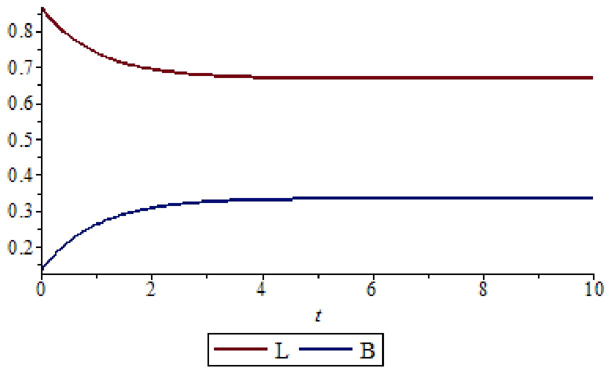

Figure 1. $\dfrac{k_D}{k_B} = \dfrac{2}{3}, \quad \widetilde{L}_0^0 = \dfrac{1}{5}$

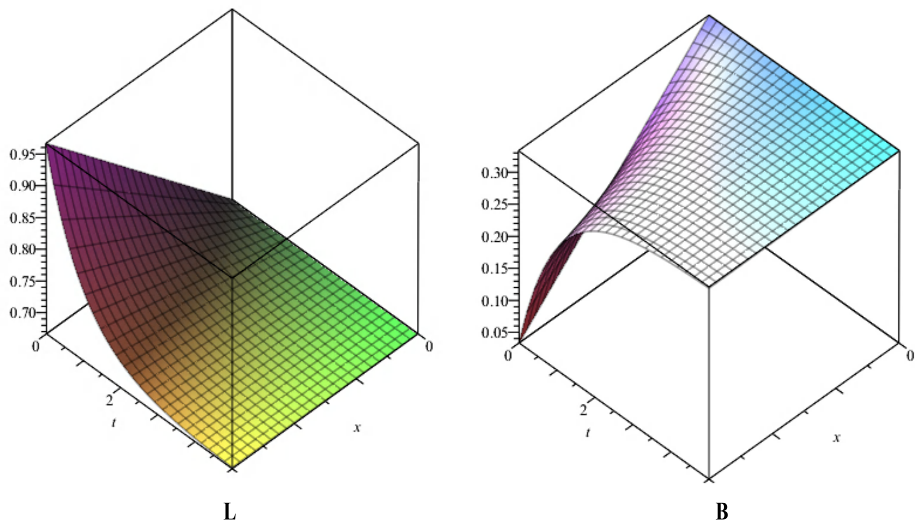

Figure 2. $\dfrac{k_D}{k_B} = \dfrac{2}{3}, \quad \widetilde{L}_0^0 = x_2$

2. $\tilde{L}_0^0(x_2) = x_2 + \Lambda$

$$\tilde{L}_0(x_2, t) = (x_2 + \Lambda)e^{(k_D - k_B)t}.$$

Corresponding plots of the functions for $\tilde{L}_0(x_2, t)$ and $\tilde{B}_0(x_2, t)$ are given in Figures 1-2.

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References

[1] F. Clarelli, C. DI Russo, R. Natalini, M. Ribot, *Mathematical models for biofilms on the surface of monuments*, Applied and Industrial Mathematics In Italy III, proceedings of SIMAI Conference 2008, Series on Advances in Mathematics for Applied Sciences - **82** (2009)

[2] I. N. Vekua, *On a way of calculating of prismatic shells* (in Russian), Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences. 21 (1955), 191-259

[3] I. N. Vekua, *Shell Theory: General Methods of Construction*, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1985

[4] G. Jaiani, *Cusped shell-like structures*, Springer-Briefs in Applied Science and Technology, Springer-Heidelberg-Dordrecht-London-New York, 2011