Bulletin of TICMI
Vol. 18, No. 2, 2014, 67–81

On the Potential Theory in Cosserat Elasticity

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(*Received February 11, 2014; Revised September 12, 2014; Accepted November 14, 2014*)

In this paper we give an account of developments and applications of an indirect method to solve several BVPs proposed for the first time by one of the authors in 1988. In particular we describe some recent results obtained applying such a method in the theory of Cosserat continuum.

Key words: Boundary integral equations, Layer potentials, Cosserat theory, Differential forms, Simply and multiply connected domains, Reducible operators.

AMS Subject Classification: 35C15, 31B10, 35Q99.

1. Introduction

The boundary integral methods provide a powerful tool for studying boundary value problems for partial differential equations. In this paper we describe an indirect approach proposed for the first time in [1] for the Dirichlet problem for Laplace equation and, subsequently, generalized to different BVPs for other PDEs (see [2–8, 18–21]). This method hinges on the theory of reducible operators and on the theory of differential forms. Differently from other methods (see, e.g. [14], [17, Ch. 4]), it does not require the use of pseudo-differential operators nor the use of hypersingular integrals. After explaining the method (Section 2) and its generalizations (Section 3), in Section 4 we focus attention on the four basic BVPs related to the theory of Cosserat.

2. The method

Throughout this paper $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded simply connected domain (i.e. $\mathbb{R}^n \setminus \overline{\Omega}$ is connected) such that its boundary $\Sigma = \partial\Omega$ is a Lyapunov hypersurface (i.e. Σ has a uniformly Hölder continuous normal field of some exponent $l \in (0,1]$); $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ denotes the outwards unit normal vector at the point $x = (x_1, \dots, x_n) \in \Sigma$.

In the sequel p indicates a real number such that $p \in]1, +\infty[$. The symbol $L^p_k(\Sigma)$ stands for the space of the differential forms[1)] of degree k defined on Σ whose

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1) For details about differential forms, we refer the reader to [12, 13].

ISSN: 1512-0082 print

components belong to $L^p(\Sigma)$ in a coordinate system of class C^1 and then in every coordinate system of class C^1. By $W^{1,p}(\Sigma)$ we denote the usual Sobolev space.

In order to illustrate the method we consider the Dirichlet and the Neumann problems for the n-dimensional laplacian in Ω:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \Sigma, \end{cases} \tag{1}$$

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = f, & \text{on } \Sigma. \end{cases} \tag{2}$$

Concerning the Dirichlet problem, the classical indirect method of Fredholm seeks the solution of (1) in terms of a double layer potential

$$u(x) = \int_\Sigma \varphi(y) \frac{\partial}{\partial \nu_y} s(y-x) d\sigma_y, \tag{3}$$

s being the fundamental solution of the Laplace equation:

$$s(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & n = 2, \\ \dfrac{1}{(2-n)c_n} |x|^{2-n}, & n > 2 \end{cases}$$

(c_n is the hypersurface measure of the unit sphere in $\mathbb{R}^n$).

Looking for the solution of (1) in the form of a simple layer potential

$$u(x) = \int_\Sigma \varphi(y) s(y-x) d\sigma_y, \tag{4}$$

an integral equation of the first kind arises

$$\int_\Sigma \varphi(y) s(y-x) d\sigma_y = g(x), \qquad x \in \Sigma. \tag{5}$$

In the case $n = 2$, Muskhelishvili [23, p. 184] gave a method for solving (5). The idea is that, differentiating both sides of (5) with respect to the arc lenght s, one is led to a singular integral equation.

As Muskhelishvili's method is based on the theory of holomorphic functions of one complex variable, it is not readily extendable to higher dimensions. However, in [1] it was generalized to the case of n variables. The main idea consists in replacing holomorphic functions by conjugate differential forms and the derivative with respect to the arc length s by the exterior differential operator d.

Namely, let us assume $g \in W^{1,p}(\Sigma)$, $1 < p < \infty$. Applying d to both sides of (5) and observing that it is possible to differentiate under the integral sign (see [1]), we obtain the following singular integral equation

$$\int_{\Sigma} \varphi(y) d_x[s(y-x)] d\sigma_y = dg(x), \qquad x \in \Sigma. \tag{6}$$

Note that, if $n \geq 3$, the space in which we look for the solution of (6) and the space in which the datum is given are different: the unknown is a scalar function while the datum is a differential form of degree one.
The singular integral on the left-hand side of (6) can be viewed as a linear and continuous operator $S : L^p(\Sigma) \to L^p_1(\Sigma)$. In [1] it is shown that S can be reduced on the left[1)] by the following reducing operator $S' : L^p_1(\Sigma) \to L^p(\Sigma)$

$$S'\psi(x) = \underset{\Sigma}{*} \int_{\Sigma} \psi(y) \wedge d_x[s_{n-2}(y-x)], \qquad x \in \Sigma, \tag{7}$$

where $s_k(y-x) = \sum_{j_1<\ldots<j_k} s(y-x) dx^{j_1} \ldots dx^{j_k} dy^{j_1} \ldots dy^{j_k}$ is the double k-form introduced by Hodge and the symbol $\underset{\Sigma}{*}$ has the following meaning: if w is a $(n-1)$-form on Σ and $w = w_0 d\sigma$, then $\underset{\Sigma}{*} w = w_0$.

We have

$$S'S\varphi = -\frac{1}{4}\varphi + K^2\varphi, \tag{8}$$

where K is the compact operator

$$K\varphi(x) = \int_{\Sigma} \varphi(y) \frac{\partial}{\partial \nu_x} s(y-x) d\sigma_y, \qquad x \in \Sigma.$$

This implies that there exists a solution of (6) if, and only if, the compatibility conditions

$$\int_{\Sigma} dg \wedge h = 0 \tag{9}$$

are satisfied for any $h \in L^q_{n-2}(\Sigma)$ $\left(q = \frac{p}{p-1}\right)$ such that $S^*h = 0$, S^* being the adjoint of S. Moreover $S^*h = 0$ if, and only if, h is a weakly closed form ([1, pp.189–190]). Therefore the compatibility conditions (9) hold and hence there exists a solution $\varphi \in L^p(\Sigma)$ of (6). Then one can show the following result.

Theorem 2.1 : *Given $g \in W^{1,p}(\Sigma)$, the Dirichlet problem* (1) *has a unique solution representable by means of a simple layer potential* (4) *with density $\varphi \in L^p(\Sigma)$.*

Remark 1 : The left reduction (8) is not an equivalent reduction[2)]. However, we still have a kind of equivalence. Indeed in [5] is remarked that, under the assumption that $N(S'S) = N(S)$, if β is such that the equation $S\alpha = \beta$ is solvable, then

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[1)]We say that $S : B \to \tilde{B}$ (B, $\tilde{B}$ being Banach spaces) can be reduced on the left if there exists a continuous linear operator $S' : \tilde{B} \to B$ such that $S'S$ is a Fredholm operator (see, e.g., [11, 22]).
[2)]A left reduction is said to be equivalent if $\mathrm{N}(S') = \{0\}$, where $\mathrm{N}(S')$ denotes the kernel of S' (see, e.g., [22, p.19–20]). This means that $S\alpha = \beta$ if, and only if, $S'S\alpha = S'\beta$.

this equation is equivalent to $S'S\alpha = S'\beta$. Since $N(S'S) = N(S)$ and $S\varphi = dg$ admits always a solution, we deduce the equivalence between (6) and the Fredholm equation $S'S\varphi = S'(dg)$.

Let now $f \in L^p(\Sigma)$ $(1 < p < \infty)$ with $\int_\Sigma f\, d\sigma = 0$. As a consequence of Theorem 2.1 one can obtain the solution of the Neumann problem (2) by means of a double layer potential (3) with density belonging to $W^{1,p}(\Sigma)$ (see [5]).

The key to obtaining this result is the following formula, which was proved during the proof of [1, Theorem I, p.186]. For the reader convenience, we give it here with a direct proof.

Proposition 2.2: *For any $\psi \in W^{1,p}(\Sigma)$, we have*

$$\frac{\partial}{\partial \nu_z}\left(\int_\Sigma u(x)\frac{\partial}{\partial \nu_x}s(x-z)d\sigma_x\right)d\sigma_z = d_z\int_\Sigma du(x)\wedge s_{n-2}(x-z), \qquad z\in\Sigma. \tag{10}$$

Proof: Letting

$$U(z) = \int_\Sigma du(x)\wedge s_{n-2}(x-z),\ V(z) = \int_\Sigma u(x)\wedge d_z[s_{n-1}(x-z)], \quad z\notin\Sigma,$$

we have that

$$dU(z) = \delta V(z), \qquad z\notin\Sigma \tag{11}$$

(δ denotes the codifferential operator). Indeed, keeping in mind $(\delta_x d_x + d_x\delta_x)s_k(y-x) = 0$ and $\delta_x s_{k+1}(y-x) = d_y s_k(y-x)$ for $x\neq y$, we obtain

$$dU(z) = -\int_\Sigma u(x)\wedge d_z d_x[s_{n-2}(x-z)] = -\int_\Sigma u(x)\wedge d_z\delta_z[s_{n-1}(x-z)] =$$
$$= \int_\Sigma u(x)\wedge \delta_z d_z[s_{n-1}(x-z)] = \delta V(z), \qquad z\notin\Sigma.$$

On the other hand,

$$d_z[s_{n-1}(x-z)] = d_z\left[\sum_{j=1}^n s(x-z)dz_1\ldots\widehat{j}\ldots dz_n dx_1\ldots\widehat{j}\ldots dx_n\right] =$$
$$= \sum_{j=1}^n \frac{\partial}{\partial z_j}s(x-z)(-1)^{j-1}dz_1\ldots dz_n dx_1\ldots\widehat{j}\ldots dx_n =$$
$$= \sum_{j=1}^n \frac{\partial}{\partial z_j}s(x-z)\nu_j(x)\sigma_x dz_1\ldots dz_n = -\frac{\partial}{\partial \nu_x}s(x-z)d\sigma_x dz_1\ldots dz_n, \quad z\in\Omega, x\in\Sigma,$$

and thus

$$V(z) = -\int_\Sigma u(x)\frac{\partial}{\partial \nu_x}s(x-z)d\sigma_x dz_1\ldots dz_n.$$

Moreover, setting $V(z) = V_0(z)dz_1 \dots dz_n$, we have

$$\delta V(z) = (-1)^{n(n+1)+1} {*d*} V(z) = -{*d}V_0(z) = -\sum_{j=1}^{n} (-1)^{j-1} \frac{\partial}{\partial z_j} V_0(z) dz_1 \dots \widehat{j} \dots dz_n$$

and then the restriction on Σ of the form δV is given by

$$-\frac{\partial}{\partial \nu_z} V_0(z)\, d\sigma_z, \qquad z \in \Sigma. \tag{12}$$

Considering the restriction on Σ of (11), keeping in mind (12) and the fact that the restriction on Σ of the left hand side of (11) is dU, we get (10). □

Thanks to (10), by imposing the Neumann boundary condition we obtain

$$d_x \int_\Sigma d\psi(y) \wedge s_{n-2}(y-x) = f(x) d\sigma_x, \qquad x \in \Sigma. \tag{13}$$

Taking into account definition (7) of S', the integral equation (13) can be written as $S'(d\psi) = f$. On the other hand, since $\psi \in W^{1,p}(\Sigma)$, we can represent it as a simple layer potential (see Theorem 2.1). If we denote by φ its density, we can write (see (8)) $S'(d\psi) = S'S\varphi = -\frac{1}{4}\varphi + K^2\varphi$. In this way we get the Fredholm equation $-\frac{1}{4}\varphi + K^2\varphi = f$. In [5, p.29] it is proved that this equation always admits the solution.

3. Applications and generalizations

The method described in the previous section has been applied to different BVPs related to several PDEs.

Specifically, the method has been generalized to the Dirichlet ([5]) and the traction problem ([19]) for the Lamé system, to the Dirichlet ([5]) and the traction problem ([19]) for the Stokes system, to the four basic BVPs of the theory of thermoelastic pseudo-oscillations for an isotropic elastic body ([18]), to the four basic BVPs of the Cosserat theory of elasticity ([2, 3]) and to the two basic BVPs of the linear theory of viscoelasticity for Kelvin-Voigt materials with voids ([4]). All these problems have been considered in a bounded simply connected domain of $\mathbb{R}^3$. In Section 4 we shall describe with more details the results concerning the Cosserat theory.

This method has been applied also in multiply connected domains. We recall that an $(m+1)$-connected domain D of $\mathbb{R}^n$ $(n \geq 2)$ is a domain of the form $D = D_0 \setminus \bigcup_{j=1}^{m} \overline{D}_j$, where D_j $(j = 0, \dots, m)$ are $m+1$ bounded domains of $\mathbb{R}^n$ with connected boundaries $S_j \in C^{1,l}$ $(l \in (0,1])$ and such that $\overline{D}_j \subset D_0$ and $\overline{D}_j \cap \overline{D}_k = \emptyset$, $j, k = 1, \dots, m$, $j \neq k$. In particular, we have considered the Dirichlet and the Neumann problems for the Laplace equation ([7]), the Dirichlet and the traction problems for the Lamé system ([6]) and the Dirichlet problem for the Stokes system ([8]).

We observe that the case $n = 2$ requires some additional remarks. It is well known that there are some domains in which not every harmonic function can be represented by means of a harmonic simple layer potential; we say that the boundary of such domains is exceptional. For instance, on the unit disk we have

$$\int_{|y|=1} \log|y-x| ds_y = 0, \qquad |x| < 1.$$

Similar domains occur also in planar elasticity (see [6, Lemma 4.3]) and bidimensional Stokes system (see [8, Lemma 1]). Exceptional domains are usually avoided by scaling up (see, e.g., [15, 23]). Such domains do not appear in higher dimensions.

Also for $(m+1)$-connected domains there are particular boundaries for which the solution of the Dirichlet problem (1) cannot be represented by a simple layer potential. Such particular cases occur if, and only if, the exterior boundary S_0 (considered as the boundary of the simply connected domain D_0) is exceptional. In the case of the Laplace equation this statement is contained in the next theorem ([7, Theorem 3.1]).

Theorem 3.1: *Let $D \subset \mathbb{R}^2$ be an $(m+1)$-connected domain. The following conditions are equivalent:*

(i) there exists a Hölder continuous function $\varphi \neq 0$ such that

$$\int_S \varphi(y) \log|y-x| ds_y = 0, \qquad \forall\, x \in S;$$

(ii) a non zero constant cannot be represented by a simple layer potential;

(iii) S_0 is exceptional;

(iv) if $\varphi_0, \dots, \varphi_m$ are linearly independent functions of

$$\mathcal{P} = \left\{ \varphi \in L^p(S) \, : \, \int_S \varphi(y) \frac{\partial}{\partial s_x} \log|y-x| ds_y = 0 \right\}$$

and

$$\int_S \varphi_j(y) \log|y-x| ds_y = c_{jk}, \qquad x \in S_k, \qquad j,k = 0,1,\dots,m,$$

we have $\det\{c_{jk}\}_{j,k=0,1,\dots,m} = 0$;

(v) for every $\varphi \in \mathcal{P}$ the simple layer potential with density φ vanishes on S_0:

$$\int_S \varphi(y) \log|y-x| ds_y = 0, \qquad x \in S_0.$$

Similar results occur also in the theory of elastostatic (see [6, Section 4]) and in the theory of incompressible fluid flow (see [8, Section 4]).

Consider now the Neumann problem for the Laplace equation

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ \dfrac{\partial u}{\partial \nu} = f, & \text{on } S, \end{cases} \tag{14}$$

where $f \in L^p(S)$ satisfies the compatibility condition

$$\int_S f d\sigma = 0. \tag{15}$$

Generally speaking, a solution of (14) cannot be represented by a double layer potential. In fact, we have ([7, Theorem 5.2])

Theorem 3.2 : *Given $f \in L^p(S)$, a solution of the Neumann problem* (14) *can be represented by means of a double layer potential if and only if,*

$$\int_{S_j} f d\sigma = 0, \qquad j = 0, 1, \ldots, m.$$

The solution is uniquely determined up to an additive constant.

If f satisfies the only condition (15), we have to modify the integral representation of the solution (see [7, Theorem 5.4]):

Theorem 3.3 : *Given $f \in L^p(S)$ satisfying* (15)*, the Neumann problem* (14) *admits a solution given by*

$$u(x) = \int_S \varphi(y) \frac{\partial}{\partial \nu_y} s(y-x) d\sigma_y - \sum_{j=1}^{m} \frac{1}{|S_j|} \int_{S_j} f(t) d\sigma_t \int_{S_j} s(y-x) d\sigma_y, \quad x \in D,$$

where $\varphi \in W^{1,p}(S)$ and $|S_j|$ is the measure of S_j, $j = 1, \ldots, m$.
The solution is uniquely determined up to an additive constant.

Similar theorems hold for the traction problem for the Lamé system (see [6, Section 6]).

4. Cosserat linear theory

The couple-stress or Cosserat theory of elasticity has emerged from the work of the brothers François and Eugène Cosserat at the turn of the last century [9]. In the theory of classical elasticity, a material point has only three degrees of freedom corresponding to its position in the Euclidean space. In the Cosserat theory there are three additional independent degrees of freedom, related to the rotation of each particle. The theory of Cosserat continuum is involved in different branches of applied sciences like, for instance, elasto-plasticity, civil engineering, geo-mechanics, micropolar fluid flow and bio-mechanics.

The main boundary value problems of the Cosserat theory are four. These problems have been studied by means of the potential theory (see, e.g., [10, 16], [24–26] and the reference therein for the first and second problems in plane, anti-plane deformations and in the bending of plates); in particular the representability of the solution of the first and the second boundary value problem have been obtained by means of a double layer potential and a simple layer potential, respectively. In this section we show how to apply the method explained in Section 2 to the four basic problems, obtaining in this way integral representations for the solution different from the ones contained in [16].

4.1. *Notations and preliminaries*

Given the set of constants $\lambda, \mu, \alpha, \varepsilon, v, \beta$ satisfying the conditions

$$\alpha, \beta, \mu, v > 0; \quad 3\lambda + 2\mu > 0; \quad 3\varepsilon + 2v > 0,$$

the homogeneous static equation of a Cosserat continuum has the form [16, p. 50]

$$\begin{cases} (\mu + \alpha)\,\Delta u + (\lambda + \mu - \alpha)\,\mathrm{grad}\ \mathrm{div} u + 2\alpha\,\mathrm{rot}\ \omega = 0, & \text{in } \Omega, \\ (v + \beta)\,\Delta\omega + (\varepsilon + v - \beta)\,\mathrm{grad}\ \mathrm{div}\omega + 2\alpha\,\mathrm{rot}\ u - 4\alpha\omega = 0, & \text{in } \Omega, \end{cases} \tag{16}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector, $\omega = (\omega_1, \omega_2, \omega_3)$ is the rotation vector and Ω is a bounded simply connected domain of $\mathbb{R}^3$ with Lyapunov boundary Σ of exponent $l \in (0, 1]$. It is convenient to write the basic equations (16) in a matrix form:

$$M\mathcal{U} = 0, \tag{17}$$

where $\mathcal{U} = (u, \omega)'$ is a six-components column vector; M is the following block-matrix

$$M = \begin{pmatrix} M^1 & M^2 \\ M^3 & M^4 \end{pmatrix}$$

whose entries are (3×3)-matrices of differential operators given by

$$M^1_{ij} = (\mu + \alpha)\delta_{ij}\Delta + (\lambda + \mu - \alpha)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$M^2_{ij} = M^3_{ij} = -2\alpha \sum_{k=1}^{3} \delta_{ijk}\frac{\partial}{\partial x_k},$$

$$M^4_{ij} = \delta_{ij}[(v + \beta)\Delta - 4\alpha] + (\varepsilon + v - \beta)\frac{\partial^2}{\partial x_i \partial x_j}$$

for $i, j = 1, 2, 3$ (δ_{kj} and δ_{jkp} denote the Kronecker delta and the Levi-Civita symbol, respectively).

We recall that the block-matrix of the fundamental solution of the homogeneous system (17) is given by

$$\Psi(x) = \begin{pmatrix} \Psi^1(x) & \Psi^2(x) \\ \Psi^3(x) & \Psi^4(x) \end{pmatrix}, \qquad x \in \mathbb{R}^3 \setminus \{(0, 0, 0)\},$$

where $\Psi^i(x) = \left(\Psi^i_{kj}(x)\right)$, $k, j = 1, 2, 3$, $i = 1, \ldots, 4$, are the following (3×3)-

matrices (see [16, p.93]):

$$\Psi^1_{kj}(x) = \frac{\delta_{kj}}{2\pi}\left[\frac{1}{\mu|x|} - \frac{\alpha}{\mu(\alpha+\mu)}\frac{e^{-\sigma|x|}}{|x|}\right] + \\ + \frac{1}{2\pi\mu}\frac{\partial^2}{\partial x_k \partial x_j}\left[-\frac{(\lambda+\mu)}{2(\lambda+2\mu)}|x| + \frac{\beta+v}{4\mu}\frac{e^{-\sigma|x|}-1}{|x|}\right],$$

$$\Psi^2_{kj}(x) = \Psi^3_{kj}(x) = \frac{1}{4\pi\mu}\sum_{p=1}^{3}\delta_{jkp}\frac{\partial}{\partial x_p}\frac{1-e^{-\sigma|x|}}{|x|},$$

$$\Psi^4_{kj}(x) = \frac{\delta_{kj}}{2\pi(\beta+v)}\frac{e^{-\sigma|x|}}{|x|} + \frac{1}{8\pi}\frac{\partial^2}{\partial x_k \partial x_j}\left[\frac{e^{-\rho|x|}-e^{-\sigma|x|}}{\alpha|x|} - \frac{e^{-\sigma|x|}-1}{\mu|x|}\right],$$

$\sigma = \sqrt{\dfrac{4\alpha\mu}{(\mu+\alpha)(v+\beta)}}$ and $\rho = \sqrt{\dfrac{4\alpha}{\varepsilon+2v}}$.

We can rewrite the matrix $\Psi(x)$ as

$$\Psi^1_{kj}(x) = \frac{1}{4\pi}\left[\frac{\lambda+3\mu+\alpha}{(\mu+\alpha)(\lambda+2\mu)}\frac{\delta_{kj}}{|x|} + \frac{\lambda+\mu-\alpha}{(\mu+\alpha)(\lambda+2\mu)}\frac{x_k x_j}{|x|^3}\right] + C_{kj}(x),$$

$$\Psi^2_{kj}(x) = \Psi^3_{kj}(x) = \mathcal{O}(1),$$

$$\Psi^4_{kj}(x) = \frac{1}{4\pi}\left[\frac{\varepsilon+3v+\beta}{(v+\beta)(\varepsilon+2v)}\frac{\delta_{kj}}{|x|} + \frac{\varepsilon+v-\beta}{(v+\beta)(\varepsilon+2v)}\frac{x_k x_j}{|x|^3}\right] + D_{kj}(x),$$

where the functions $C_{kj}(x)$ and $D_{kj}(x)$ are bounded (see [2, Lemma 3.2]).

We denote by T the stress operator (see [16, p.59])

$$T = \begin{pmatrix} T^1 & T^2 \\ 0 & T^4 \end{pmatrix}, \qquad T^i = \left(T^i_{kj}\right), \qquad k,j = 1,2,3, \quad i = 1,2,4. \tag{18}$$

Note that

$$T^1 u = \lambda(\text{div } u)\nu + (2\mu)\frac{\partial u}{\partial \nu} + (\mu-\alpha)(\nu \wedge \text{rot } u),$$

$$T^2 u = 2\alpha(\nu \wedge u),$$

$$T^4 u = \varepsilon(\text{div } u)\nu + (2v)\frac{\partial u}{\partial \nu} + (v-\beta)(\nu \wedge \text{rot } u).$$

We now introduce the following block-matrix

$$S = \begin{pmatrix} S^1 & S^2 \\ 0 & S^4 \end{pmatrix}, \qquad S^i = (S^i_{kj}), \quad k,j = 1,2,3, \quad i = 1,2,4,$$

where each entry is a (3×3)-matrix given by

$$\begin{aligned}
&S^1u = (\lambda + \mu - \xi)(\text{div } u)\nu + (\mu + \xi)\frac{\partial u}{\partial \nu} + (\xi - \alpha)(\nu \wedge \text{rot } u),\\
&S^2u = 2\alpha(\nu \wedge u),\\
&S^4u = (\varepsilon + v - \chi)(\text{div } u)\nu + (v + \chi)\frac{\partial u}{\partial \nu} + (\chi - \beta)(\nu \wedge \text{rot } u),
\end{aligned}$$

ξ, χ being real parameters. S can be interpreted as the generalization of the stress operator T. In fact, if $\xi = \mu$ and $\chi = v$, then $S = T$. Further when

$$\xi = \left[\frac{2(\mu + \alpha)(\lambda + 2\mu)}{\lambda + 3\mu + \alpha} - \mu\right] \text{ and } \chi = \left[\frac{2(v + \beta)(\varepsilon + 2v)}{\varepsilon + 3v + \beta} - v\right], \tag{19}$$

we call S a pseudostress operator and we denote it by T^0.

Finally, we recall that the matrix $S\Psi$ can be written as

$$S\Psi = \begin{pmatrix} (S\Psi)^1 & (S\Psi)^2 \\ (S\Psi)^3 & (S\Psi)^4 \end{pmatrix},$$

where, for $k, j = 1, 2, 3$,

$$\begin{aligned}
&(S\Psi)^1_{kj}(y - x) = \frac{1}{4\pi}\left[\frac{(\mu + \xi)(\lambda + 3\mu + \alpha)}{(\mu + \alpha)(\lambda + 2\mu)} - 2\right] M^{jk}_x\left(\frac{1}{|y - x|}\right) + \mathcal{O}\left(\frac{1}{|y - x|^{2-l}}\right),\\
&(S\Psi)^2_{kj}(y - x) = \mathcal{O}\left(\frac{1}{|y - x|}\right), \quad (S\Psi)^3_{kj}(y - x) = \mathcal{O}\left(\frac{1}{|y - x|}\right),\\
&(S\Psi)^4_{kj}(y - x) = \frac{1}{4\pi}\left[\frac{(\chi + v)(\varepsilon + 3v + \beta)}{(v + \beta)(\varepsilon + 2v)} - 2\right] M^{jk}_x\left(\frac{1}{|y - x|}\right) + \mathcal{O}\left(\frac{1}{|y - x|^{2-l}}\right),
\end{aligned} \tag{20}$$

where $M^{ih} = \nu_i\dfrac{\partial}{\partial x_h} - \nu_h\dfrac{\partial}{\partial x_i}$, $1 \le i, h \le 3$ (see [2, Lemma 3.3]).

4.2. *BVPs in the Cosserat theory*

The basic problems of statics consist in finding a six-component vector $\mathcal{U}$ solution of (17) and satisfying one of the following boundary conditions, where f is an assigned vector function:

- for the first internal basic problem or Problem $(I)^+$:

$$\mathcal{U}^+(y) = \lim_{\Omega \ni x \to y} \mathcal{U}(x) = f(y), \qquad \forall\, y \in \Sigma;$$

- for the second internal basic problem or Problem $(II)^+$:

$$[T\mathcal{U}]^+(y) = f(y), \qquad \forall\, y \in \Sigma,$$

where T is given by (18);

- for the third internal basic problem or Problem $(III)^+$:

$$[H\mathcal{U}]^+(y) = f(y), \qquad \forall\, y \in \Sigma,$$

where

$$H = \begin{pmatrix} I & 0 \\ 0 & -T^4 \end{pmatrix},$$

I being the 3×3 identity matrix;

- for the fourth internal basic problem or Problem $(IV)^+$:

$$[R\mathcal{U}]^+(y) = f(y), \qquad \forall y \in \Sigma,$$

where

$$R = \begin{pmatrix} T^1 & T^2 \\ 0 & I \end{pmatrix}.$$

Let us define some potential-type integrals:

$$\mathcal{W}[\Phi](x) = \int_\Sigma [T_y \Psi(y-x)]' \Phi(y) d\sigma_y; \tag{21}$$

$$\mathcal{U}[\Phi](x) = \int_\Sigma \Psi(y-x) \Phi(y) d\sigma_y; \tag{22}$$

$$\mathcal{R}[\Phi](x) = \int_\Sigma [R_y \Psi(y-x)]' \Phi(y) d\sigma_y; \tag{23}$$

$$\mathcal{H}[\Phi](x) = \int_\Sigma [H_y \Psi(y-x)]' \Phi(y) d\sigma_y. \tag{24}$$

Integrals (21) and (22) are the double and simple layer potential, respectively.

4.3. *Reduction and representation theorems*

Here the symbol $\mathcal{S}^p$ stands for the class of simple layer potentials (22) with density in $[L^p(\Sigma)]^6$, $1 < p < \infty$.
In order to solve the Dirichlet problem $(I)^+$ in the class $\mathcal{S}^p$ with datum $f \in [W^{1,p}(\Sigma)]^6$ we first have to show that the singular integral system

$$\int_\Sigma d_x[\Psi_{ij}(y-x)]\gamma_j(y)\, d\sigma_y = df_i(x), \qquad i = 1, \ldots, 6 \tag{25}$$

can be reduced to an equivalent Fredholm one. In order to prove this claim we enunciate the following result.

Proposition 4.1: *Let us introduce the matrix*

$$J = \begin{pmatrix} J^1 \\ J^4 \end{pmatrix}$$

where $J^1, J^4 : [L^p(\Sigma)]^6 \to [L^p_1(\Sigma)]^3$ *are defined, for every* $\Phi = (\varphi_1, \varphi_2, \varphi_3, \vartheta_1, \vartheta_2, \vartheta_3)$, *by*

$$(J^1\Phi)_j(x) = \int_\Sigma \big[d_x\big[\Psi^1_{jh}(y-x)\big]\,\varphi_h(y) + d_x\big[\Psi^2_{jh}(y-x)\big]\,\vartheta_h(y)\big]\,d\sigma_y,$$
$$(J^4\Phi)_j(x) = \int_\Sigma \big[d_x\big[\Psi^3_{jh}(y-x)\big]\,\varphi_h(y) + d_x\big[\Psi^4_{jh}(y-x)\big]\,\vartheta_h(y)\big]\,d\sigma_y.$$

Let $\widetilde{J}$ *be the matrix*

$$\widetilde{J} = \begin{pmatrix} \widetilde{J}^1 & 0 \\ 0 & \widetilde{J}^4 \end{pmatrix}$$

where $\widetilde{J}^1, \widetilde{J}^4 : [L^p_1(\Sigma)]^3 \to [L^p(\Sigma)]^3$ *are defined as*

$$(\widetilde{J}^1\psi)_i = (\lambda+\mu-\xi)\mathcal{K}_{jj}(\psi)\nu_i + (\mu+\alpha)\mathcal{K}_{ij}(\psi)\nu_j + (\xi-\alpha)\mathcal{K}_{ji}(\psi)\nu_j,$$
$$(\widetilde{J}^4\psi)_i = (\varepsilon+v-\chi)\mathcal{F}_{jj}(\psi)\nu_i + (v+\beta)\mathcal{F}_{ij}(\psi)\nu_j + (\chi-\beta)\mathcal{F}_{ji}(\psi)\nu_j,$$

with

$$\mathcal{K}_{js}(\psi)(x) = 2\Theta_s(\psi_j)(x) - \delta^{123}_{pkq}\int_\Sigma \frac{\partial}{\partial x_s}\big[F^1_{jp}(y-x)\big] \wedge \psi_k(y) \wedge dy^q, \quad \psi \in [L^p_1(\Sigma)]^3,$$

$$\mathcal{F}_{js}(\varphi)(x) = 2\Theta_s(\varphi_j)(x) - \delta^{123}_{pkq}\int_\Sigma \frac{\partial}{\partial x_s}\big[F^2_{jp}(y-x)\big] \wedge \varphi_k(y) \wedge dy^q, \quad \varphi \in [L^p_1(\Sigma)]^3,$$

$$F^1_{jp}(y-x) = \frac{1}{4\pi}\left[\frac{(\mu+\xi)(\lambda+3\mu+\alpha)}{(\mu+\alpha)(\lambda+2\mu)} - 2\right]\frac{\delta_{jp}}{|y-x|}$$
$$+\frac{1}{4\pi}\frac{(\mu+\xi)(\lambda+\mu-\alpha)}{(\mu+\alpha)(\lambda+2\mu)}\frac{1}{|y-x|}\frac{\partial}{\partial y_j}|y-x|\frac{\partial}{\partial y_p}|y-x| + (\mu+\xi)C_{jp}(y-x),$$

$$F^2_{jp}(y-x) = \frac{1}{4\pi}\left[\frac{(\chi+v)(\varepsilon+3v+\beta)}{(v+\beta)(\varepsilon+2v)} - 2\right]\frac{\delta_{jp}}{|y-x|}$$
$$+\frac{1}{4\pi}\frac{(\chi+v)(\varepsilon+v-\beta)}{(v+\beta)(\varepsilon+2v)}\frac{1}{|y-x|}\frac{\partial}{\partial y_j}|y-x|\frac{\partial}{\partial y_p}|y-x| + (\chi+v)D_{jp}(y-x),$$

$$\Theta_s(\psi)(x) = *\int_\Sigma d_x[s_1(y-x)] \wedge \psi(y) \wedge dx^s, \qquad \psi \in L^p_1(\Sigma).$$

Then $\widetilde{J}J\Phi = -\Phi + P^2\Phi + Q\Phi$, *where* P *is the integral operator*

$$P\Phi(x) = \int_\Sigma S_x[\Psi(y-x)]\Phi(y)d\sigma_y,$$

and Q *is a compact operator from* $[L^p(\Sigma)]^6$ *into itself.*

Keeping in mind (20), if we choose ξ and χ as in (19), the kernel of P has only a weak singularity: $T^0_x[\Psi(y-x)] = \mathcal{O}\left(|y-x|^{l-2}\right)$ and then P is a compact operator from $[L^p(\Sigma)]^6$ into itself.

We deduce that the operator J can be reduced on the left and then the integral system (25) admits a solution if, and only if,

$$\int_\Sigma \gamma_i \wedge df_i = 0, \qquad i = 1, \ldots, 6, \tag{26}$$

for any $\gamma = (\gamma_1, \ldots, \gamma_6) \in [L_1^q(\Sigma)]^6$ such that $J^*\gamma = 0$.

By virtue of [2, Theorem 5.1], $J^*\gamma = 0$ if, and only if, all the components γ_i, $i = 1, \ldots, 6$, are weakly closed 1-forms. Thus conditions (26) hold and we obtain the following theorem ([2, Theorem 5.2]).

Theorem 4.2: *Given $f \in [W^{1,p}(\Sigma)]^6$, the BVP*

$$\begin{cases} \mathcal{U} \in \mathcal{S}^p, & \\ M\mathcal{U} = 0, & \text{in } \Omega, \\ d\mathcal{U} = df, & \text{on } \Sigma \end{cases}$$

admits solution. It is given by (22) *where the density Φ solves the singular integral system $J\Phi = df$.*

Since any solution of a Dirichlet problem with constant datum can be represented by means of a simple layer potential (see [2, Lemma 5.1]), Theorem 4.2 implies the following representation result for Problem $(I)^+$.

Theorem 4.3: *Given $f \in [W^{1,p}(\Sigma)]^6$, the following Dirichlet BVP*

$$\begin{cases} \mathcal{U} \in \mathcal{S}^p, & \\ M\mathcal{U} = 0, & \text{in } \Omega, \\ \mathcal{U} = f, & \text{on } \Sigma \end{cases}$$

admits a unique solution $\mathcal{U}$. In particular, the density Φ of $\mathcal{U}$ can be written as $\Phi = \Phi_0 + \Gamma_0$, where Φ_0 solves the singular integral system

$$\int_\Sigma d_x[\Psi_{ij}(y-x)]\, \Phi_{0j}(y) d\sigma_y = df_i(x), \qquad i = 1, \ldots, 6, \ \text{a.e. } x \in \Sigma$$

and Γ_0 is the density of a simple layer potential constant on Σ.

We remark that, as for the Laplace equation (see Remark **1**), the obtained reduction is not an equivalent one. Also in this case we have the equivalence in the sense indicated in Remark **1** (see [2, Theorem 5.4]).

Theorem 4.3 allows to obtain integral representations of the other three BVPs in the Cosserat theory different from the usual ones (see, e.g., [16]).

We recall that the classical compatibility conditions for the second BVP are

$$\int_\Sigma f_k(y) d\sigma_y = 0, \qquad k = 1, 2, 3; \tag{27}$$

$$\int_\Sigma \Big[f_{3+k}(y) + \sum_{i,j=1}^{3} \delta_{kij} y_i f_j(y) \Big] d\sigma_y = 0, \qquad k = 1, 2, 3. \tag{28}$$

For the second and the fourth BVPs we have the following results ([3, Theorems 3.1 and 5.2]).

Theorem 4.4: *Given* $f \in [L^p(\Sigma)]^6$, *the second BVP admits a solution in the form of the potential* (21) *if, and only if,* (27) *and* (28) *hold.*

The density of (21) *is given by a simple layer potential* $\mathcal{U}[\Phi]$, $\Phi \in [L^p(\Sigma)]^6$ *being a solution of the singular integral system*

$$-\Phi + K^2\Phi = f\,,$$

where

$$K\Phi(x) = \int_\Sigma T_x[\Psi(y-x)]\Phi(y)d\sigma_y, \quad x \in \Sigma\,.$$

Moreover, the solution is determined up to an additive rigid displacement $(u,\omega)'$, *where* $u = a \wedge x + b$ *and* $\omega = a$ $(a, b \in \mathbb{R}^3)$.

Theorem 4.5: *Given* $f \in [L^p(\Sigma)]^3 \times [W^{1,p}(\Sigma)]^3$, *the fourth BVP admits a solution in the form of the potential* (23) *if, and only if,* (27) *holds.*

The density of (23) *is given by a simple layer potential* $H\mathcal{H}[\Phi]$, $\Phi \in [L^p(\Sigma)]^3 \times [W^{1,p}(\Sigma)]^3$ *being a solution of the singular integral system*

$$-\Phi + L^2\Phi = f\,,$$

where

$$L\Phi(x) = \int_\Sigma R_x[H_y\Psi(y-x)]'\Phi(y)d\sigma_y, \quad x \in \Sigma\,.$$

Moreover, the solution is determined up to an additive rigid translation $(u,\omega)'$, *where* $u = b$ *and* $\omega = 0$ $(b \in \mathbb{R}^3)$.

The third BVP is more delicate; indeed we have the following result ([3, Proposition 4.2]).

Proposition 4.6: *Given* $f \in [W^{1,p}(\Sigma)]^3 \times [L^p(\Sigma)]^3$, *the third BVP admits a unique solution in the form of the potential* (24) *if the following conditions*

$$\int_\Sigma f(x)\psi^{(h)}(x)d\sigma_x = 0, \quad h = 1,2,3 \tag{29}$$

are satisfied, $\{\psi^{(h)}\}$ *being a complete system of linearly independent solutions of*

$$\phi(z) + \int_\Sigma R_z[H_y\Psi(y-z)]'\phi(y)d\sigma_y = 0.$$

In order to remove conditions (29), we have to modify the representation of the solution by adding an additional term to $\mathcal{H}$ (see [3, Theorem 4.3]).

Theorem 4.7: *Given* $f \in [W^{1,p}(\Sigma)]^3 \times [L^p(\Sigma)]^3$, *the third BVP admits a unique solution represented by* $\mathcal{U}[\Phi] = \mathcal{H}[\Phi] + \mathcal{R}\left[-\frac{f}{2}\right]$, *where* $\Phi \in [L^p(\Sigma)]^3 \times [W^{1,p}(\Sigma)]^3$,

$\mathcal{H}$ is the potential (24) *and* $\mathcal{R}\left[-\frac{f}{2}\right]$ *is the potential* (23).

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