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Green's Function for the Light Scattering Equations

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Abstract. The aim of this paper is to construct Green's function in an infinite medium for the light scattering equation. To this end the method of spectral resolution of the solutions by the eigenfunctions of the corresponding characteristic equation is used.

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We consider the equation which occurs when investigating one important problem of mathematical physics, namely the equation which describes the light scattering. The considered equation has the form

$$\mu \frac{\partial I(\tau,\mu,x)}{\partial \tau} = \alpha(x)I(\tau,\mu,x) - \frac{\lambda}{2}A\alpha(x)\int_{-\infty}^{+\infty}\alpha(x')dx'\int_{-1}^{+1}I(\tau,\mu',x')d\mu' \qquad (1)$$

$$\tau, x \in (-\infty,+\infty), \mu \in (-1,+1)$$

where $\alpha(x)$ is a continuous, integrable, positive function, A is a normalizing multiplier

$$A\int_{-\infty}^{+\infty}\alpha(x)dx = 1,$$

Many weel known authors [1-3] investigated this equation. We seek the solution of this equation in the following form

$$I(\tau,\mu,x) = e^{\tau/\nu}\varphi_\nu(\mu,x).$$

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With this assumption, Eq. (1) becomes

$$(\nu\alpha(x)-\mu)\varphi_\nu(\mu,x)=\frac{\lambda}{2}A\nu\int_{-\infty}^{+\infty}\int_{-1}^{+1}\alpha(x)\alpha(x')\varphi_\nu(\mu',x')d\mu'dx' \qquad (2)$$

where ν is a parameter, so-called characteristic equation. It is very convenient to normalize φ_ν so that

$$\int_{-\infty}^{+\infty}\int_{-1}^{+1}\alpha(x)\varphi_\nu(\mu,x)d\mu dx=1.$$

Then the above becomes

$$(\nu\alpha(x)-\mu)\varphi_\nu(\mu,x)=\frac{\lambda}{2}A\nu\alpha(x)$$

From this point the conventional argument runs as follows

$$\varphi_\nu(\mu,x)=\frac{\lambda}{2}A\frac{\nu\alpha(x)}{\nu\alpha(x)-\mu}.$$

Inserting this result into Eq.(2) yields the condition

$$\Lambda(\nu)\equiv 1-\frac{\lambda}{2}A\int_{-\infty}^{+\infty}\int_{-1}^{+1}\frac{\nu\alpha^2(x)}{\nu\alpha(x)-\mu}d\mu dx=0. \qquad (3)$$

There are, as is known [4], for $\lambda>1$, two regular purely imaginary eigenvalues of the characteristic equation (2). Here they will be denoted by $\pm\nu_0$. For $\lambda<1$ the regular eigenvalues is absent. The two roots $\pm\nu_0$ occur. With the normalization the corresponding solutions of the initial equation are

$$I_{0^\pm}(\tau,\mu,x)=e^{\pm\tau/\nu_0}\varphi_{0^\pm}(\mu,x). \qquad (4)$$

where

$$\varphi_{0\pm}(\mu,x)=\frac{\lambda}{2}A\frac{\nu_0\alpha(x)}{\nu_0\alpha(x)\mp\mu} \qquad (5)$$

The argument has given the usual solutions of the homogeneous light scattering equation. However, there are others. It is to see that

$$I(\tau,\mu,x)=\int_0^{+\infty}\int_{M(\nu)}e^{\tau/\nu}\varphi_{\nu,(\zeta)}(\mu,x)u(\nu,\zeta)d\zeta d\nu \qquad (6)$$

$$\tau\in(-\infty,0)\quad \mu\in(-1,+1),\quad x\in(-\infty,+\infty).$$

where

$$\varphi_{\nu,(\zeta)}(\mu,x)=\frac{\lambda}{2}A\frac{\nu\rho(\nu)\alpha(x)\alpha(\zeta)}{\nu\alpha(x)-\mu} \tag{7}$$

$$+\left(\delta(\zeta-x)-\frac{\lambda}{2}A\int_{-1}^{+1}\frac{\nu\rho(\nu)\alpha(x)\alpha(\zeta)}{\nu\alpha(x)-\mu'}d\mu'\right)\delta(\nu\alpha(x)-\mu)$$

$$\nu\in(-\infty,+\infty),\quad \zeta\in M(\nu).$$

is solution of the initial equation (1). Here,

$$\rho^{-1}(\nu)=1-\frac{\lambda}{2}A\int_{-\infty}^{+\infty}\int_{-1}^{+1}\frac{\nu\alpha^2(x)\theta(|\nu|\alpha(x)-1)}{\nu\alpha(x)-\mu}d\mu dx,$$

$M(\nu)=\{x\in(-\infty,+\infty):|\nu|\alpha(x)<1\}$, $u(\nu,\zeta)$ is a continuous, integrable function satisfying H^* condition [6] with respect to variable ν
To summarize: There are, when $\lambda>1$, two discrete eigenfunctions given by Eqs. (5) and also the class of the so-called singular eigenfunctions given by Eqs. (7).

Now consider the equation which is named the adjoint of the characteristic equation

$$(\nu\alpha(x)-\mu)\varphi_\nu^*(\mu,x)=\frac{\lambda}{2}A\nu\int_{-\infty}^{+\infty}\int_{-1}^{+1}\alpha^2(x)\varphi_\nu^*(\mu',x')d\mu'dx' \tag{8}$$

where ν is a parameter.

Now, it is very convenient to normalize φ_ν^* so that

$$\int_{-\infty}^{+\infty}\int_{-1}^{+1}\varphi_\nu^*(\mu,x)d\mu dx=1.$$

Then

$$(\nu\alpha(x)-\mu)\varphi_\nu^*(\mu,x)=\frac{\lambda}{2}A\nu\alpha^2(x)$$

From this gives

$$\varphi_\nu^*(\mu,x)=\frac{\lambda}{2}A\frac{\nu\alpha^2(x)}{\nu\alpha(x)-\mu}.$$

Inserting this result into Eq.(8) we have also the condition

$$\Lambda(\nu)\equiv 1-\frac{\lambda}{2}A\int_{-\infty}^{+\infty}\int_{-1}^{+1}\frac{\nu\alpha^2(x)}{\nu\alpha(x)-\mu}d\mu dx=0.$$

Consequently, also for the adjoint characteristic equation, when $\lambda > 1$, there are, two regular eigenfunctions and a class of singular eigenfunctions

$$\varphi^*_{\nu,(\zeta)}(\mu,x) = \frac{\lambda}{2}A\frac{\nu\rho(\nu)\alpha^2(x)}{\nu\alpha(x)-\mu} \tag{9}$$

$$+\left(\delta(\zeta-x) - \frac{\lambda}{2}A\int_{-1}^{+1}\frac{\nu\rho(\nu)\alpha^2(x)}{\nu\alpha(x)-\mu'}d\mu'\right)\delta(\nu\alpha(x)-\mu)$$

$$\nu \in (-\infty,+\infty), \quad \zeta \in M(\nu), \qquad M(\nu) = \{x \in (-\infty,+\infty) : |\nu|\alpha(x) < 1\}$$

The usefulness of these functions arises from the fact that they are both bi-orthogonal and complete. This can be stated in the form of theorems

Theorem 1**:**

$$\int_{-\infty}^{+\infty}\int_{-1}^{+1}\frac{\mu}{\alpha(x)}\varphi^*_\nu(\mu,x)\varphi_{\nu'}(\mu,x)d\mu dx = 0, \quad \nu' \neq \nu \tag{10}$$

Proof: φ^*_ν and $\varphi_{\nu'}$ satisfy the equations

$$\left(1 - \frac{1}{\nu}\frac{\mu}{\alpha(x)}\right)\varphi^*_\nu(\mu,x) = \frac{\lambda}{2}A\nu\int_{-\infty}^{+\infty}\int_{-1}^{+1}\alpha(x)\varphi^*_\nu(\mu',x')d\mu' dx'$$

$$\left(1 - \frac{1}{\nu'}\frac{\mu}{\alpha(x)}\right)\varphi_{\nu'}(\mu,x) = \frac{\lambda}{2}A\nu\int_{-\infty}^{+\infty}\int_{-1}^{+1}\alpha'(x)\varphi_{\nu'}(\mu',x')d\mu' dx'$$

Multiplying the first of these by $\varphi_{\nu'}(\mu,x)$, the second by $\varphi^*_\nu(\mu,x)$, subtracting, and integrating, we get

$$\left(\frac{1}{\nu'} - \frac{1}{\nu}\right)\int_{-\infty}^{+\infty}\int_{-1}^{+1}\frac{\mu}{\alpha(x)}\varphi^*_\nu(\mu,x)\varphi_{\nu'}(\mu,x)d\mu dx = 0.$$

□

It is seen that if $g(\nu,\zeta,\zeta)$ is an arbitrary integrable function then

$$\tilde{\varphi}^*_{\nu,(\zeta)}(\mu,x) = \varphi^*_{\nu,(\zeta)}(\mu,x) + \int_{M(\nu)} g(\nu,\zeta,\zeta')\varphi^*_{\nu,(\zeta')}(\mu,x)d\zeta'$$

is also a singular eigenfunction of the adjoint equation. Moreover, if

$$S(\nu,\zeta,x) = -\pi^2\nu^2\int_{M(\nu)}\frac{\lambda^2}{4}A^2\rho^2(\nu)\alpha(x)\alpha^2(\zeta')d\zeta'$$

$$-\frac{\lambda}{2}A\int_{M(\nu)}\int_{-1}^{+1}\frac{\nu\rho(\nu)\alpha(x)\alpha(\zeta')}{\nu\alpha(\zeta')-\mu'}d\mu'$$

$$\times\frac{\lambda}{2}A\int_{M(\nu)}\int_{-1}^{+1}\frac{\nu\rho(\nu)\alpha^2(\zeta')}{\nu\alpha(\zeta')-\mu'}d\mu' d\zeta'$$

$$+\frac{\lambda}{2}A\int_{-1}^{+1}\frac{\nu\rho(\nu)\alpha(x)\alpha(\zeta)}{\nu\alpha(\zeta)-\mu'}d\mu'$$

$$+\frac{\lambda}{2}A\int_{-1}^{+1}\frac{\nu\rho(\nu)\alpha^2(x)}{\nu\alpha(x)-\mu'}d\mu',$$

and $g(\nu,\zeta,x)$ is the solution of the equation

$$g(\nu,\zeta,x)-\int_{M(\nu)}S(\nu,\zeta',x)g(\nu,\zeta,\zeta')d\zeta'=S(\nu,\zeta,x)$$

$$\nu\in(-\infty,+\infty)\quad \zeta\in M(\nu)$$

then $\tilde{\varphi^*}_{\nu,(\zeta)}(\mu,x)$ will also be singular eigenfunction of the adjoint characteristic equations and the following equality (cf.[5])

$$\int_{-\infty}^{+\infty}\int_{-1}^{+1}\frac{\mu}{\alpha(x)}\tilde{\varphi^*}_{\nu_0,(\zeta_0)}(\mu,x)\varphi_{\nu,(\zeta)}(\mu,x)d\mu dx=\delta(\nu_0-\nu)\delta(\zeta_0-\zeta)$$

holds.

Theorem 2: *The functions $\varphi_{0^\pm}$ and $\varphi_{\nu,(\zeta)}$, $-\infty,<\nu<+\infty$ are complete for functions $\psi(\mu,x)$ defined in $-1<\mu<+1$, $-\infty<x<+\infty$ satisfying H^* condition [6] with respect to variable μ, integrable with respect to x, i.e.*

$$\psi(\mu,x)=a_{0^\pm}\varphi_{0^\pm}(\mu,x)+\int_{-\infty}^{+\infty}\int_{M(\nu)}\varphi_{\nu,(\zeta)}(\mu,x)u(\nu,\zeta)d\zeta d\nu$$

where

$$a_{0^\pm}=\frac{1}{N_{0^\pm}}\int_{-\infty}^{+\infty}\int_{-1}^{+1}\frac{\mu}{\alpha(x)}\varphi^*_{0^\pm}(\mu,x)\psi(\mu,x)d\mu dx,$$

$$u(\nu,\zeta)=\int_{-\infty}^{+\infty}\int_{-1}^{+1}\frac{\mu}{\alpha(x)}\tilde{\varphi}^*_{\nu,(\zeta)}(\mu,x)\psi(\mu,x)d\mu dx.$$

From this theorem there follows correctness of the formula

$$\alpha(x)\delta(\mu-\mu_0)\delta(x-x_0)=\mu\frac{1}{N_{0^\pm}}\varphi_{0^\pm}(\mu,x)\varphi^*_{0^\pm}(\mu_0,x_0)$$

$$+\mu\int_{-\infty}^{+\infty}\int_{M(\nu)}\varphi_{\nu,(\zeta)}(\mu,x)\tilde{\varphi}^*_{\nu,(\zeta)}(\mu_0,x_0)d\zeta d\nu,$$

$$|\mu\mu_0|\in(0,1)\quad x,x_\in(-\infty,+\infty)$$

which can be useful when investigating the problems of the point sources.

Theorem 3 : *Every solution of the equation (1) can be represented in the form*

$$I(\tau,\mu,x)=c_{0^\pm}e^{\pm\tau/\nu_0}\varphi_{0^\pm}(\mu,x) \tag{11}$$

$$+\int_{-\infty}^{+\infty}\int_{M(\nu)}e^{\tau/\nu}\varphi_{\nu,(\zeta)}(\mu,x)v(\nu,\zeta)d\zeta d\nu$$

$$\tau\in(-\infty,+\infty)\quad \mu\in(-1,+1),\quad x\in(-\infty,+\infty)$$

where $c_{0^\pm}$ and v are defined uniquely by I, and Vice versa, this expansion is a formal solution of equation (1) for arbitrary $c_{0^\pm}$ and v, moreover this expansion gives us a general presentation of solutions of equation (1) for arbitrary $c_{0^\pm}$ and v guaranteeing convergence of the integrals in the right part of formula (11).

As an illustration of the applicability of the results Green's function for the light scattering equation will be constructed. To be definite $\lambda<1$ is assumed here. Green's function $I_g(\tau,\mu,x)$ satisfies the equation

$$\mu\frac{\partial I_g(\tau,\mu,x)}{\partial\tau}=\alpha(x)I_g(\tau,\mu,x)-\frac{\lambda}{2}A\alpha(x)\int_{-\infty}^{+\infty}\alpha(x')dx'\int_{-1}^{+1}I_g(\tau,\mu',x')d\mu' \tag{12}$$

$$+\delta(\tau)\delta(\mu-\mu_0)\delta(x-x_0)$$

$$\tau,x\in(-\infty,+\infty),\mu\in(-1,+1)$$

Integrating across the plane $\tau=0$ shows that I_g satisfies the homogeneous equation (1) for $\tau\neq0$ and the jump condition

$$\mu(I_g(0^+,\mu,x)-I_g(0^-,\mu,x))=\alpha(x)\delta(\mu-\mu_0)\delta(x-x_0)$$

(see [1]).

Let us look for the solution I_g which vanishes as $|\tau| \to \infty$. It is sufficient to expand I_g in the form

$$I_g = \int_0^{+\infty} \int_{M(\nu)} e^{\tau/\nu} \varphi_{\nu,(\zeta)}(\mu, x) u(\nu, \zeta) d\zeta d\nu, \quad \tau < 0$$

or

$$I_g = -\int_{-\infty}^{0} \int_{M(\nu)} e^{\tau/\nu} \varphi_{\nu,(\zeta)}(\mu, x) u(\nu, \zeta) d\zeta d\nu, \quad \tau > 0$$

The jump condition then gives an integral equation to determine the expansion coefficients. It is

$$\alpha(x)\delta(\mu - \mu_0)\delta(x - x_0) = \mu \int_{-\infty}^{+\infty} \int_{M(\nu)} \varphi_{(\nu,(\zeta)}(\mu, x) u(\nu, \zeta) d\zeta d\nu.$$

The solution obtained using the orthogonality relations is

$$u(\nu, \zeta) = \tilde{\varphi}^*_{\nu,(\zeta)}(\mu_0, x_0)$$

Hence I_g can be written in the typical normal mode expansion

$$I_g = \int_0^{+\infty} \int_{M(\nu)} e^{\tau/\nu} \tilde{\varphi}^*_{\nu,(\zeta)}(\mu_0, x_0) \varphi_{\nu,(\zeta)}(\mu, x) d\zeta d\nu \quad \tau < 0$$

$$\mu_0, \mu \in (-1, +1) \quad x_0, x \quad \in (-\infty, +\infty)$$

$$I_g = -\int_{-\infty}^{0} \int_{M(\nu)} e^{\tau/\nu} \tilde{\varphi^*}_{\nu,(\zeta)}(\mu_0, x_0) \varphi_{\nu,(\zeta)}(\mu, x) d\zeta d\nu \quad \tau > 0$$

$$\mu_0, \mu \in (-1, +1) \quad x_0, x \quad \in (-\infty, +\infty).$$

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