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Convergence of the Logarithmic Means of Two-Dimensional Trigonometric Fourier Series

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Abstract. We discuss on some convergence and divergence properties of two-dimensional (Nörlund) logarithmic means of Fourier series.

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1. Main Results

Let $f \in \left(T^2\right)$, $T^2 = [-\pi, \quad \pi]^2$ be a 2π-periodic functions with respect to each variable. The two-dimensional Fourier series of f with respect to the trigonometric system is the series

$$s\,[f] = \sum_{m,n=-\infty}^{+\infty} \widehat{f}(m,n)\, e^{imx} e^{iny},$$

where

$$\widehat{f}(m,n) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) e^{-imx} e^{-iny} dxdy$$

are the Fourier coefficients of the function f.

Let $C(T^2)$be the space of continuous functions are 2π-periodic with respect to each variable with the norm

$$\|f\|_c = \sup_{x,y \in T^2} |f(x,y)|\,.$$

Let $f \in C(T^2)$. The expression

$$\omega(\delta, f)_c = \sup \left\{ \|f(\cdot + u, \cdot + v) - f(\cdot, \cdot)\|_c : u^2 + v^2 \leq \delta^2 \right\}$$

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is called the total modulus of continuity of the function f.

The partial modulus of continuity are defined by

$$\omega_1(\delta, f)_c = \sup\left\{\|f(\cdot+u,\cdot) - f(\cdot,\cdot)\|_c : |u| \le \delta\right\},$$

$$\omega_2(\delta, f)_c = \sup\left\{\|f(\cdot,\cdot+v) - f(\cdot,\cdot)\|_c : |v| \le \delta\right\}.$$

We also use the notion of a mixed modulus of continuity. They are defined as follows:

$$\omega_{1,2}(\delta_1, \delta_2, f)_c = \sup\Big\{\|f(\cdot+u,\cdot+v) - f(\cdot+u,\cdot) - f(\cdot,\cdot+v) + f(\cdot,\cdot)\|_c : |u| \le \delta_1, |v| \le \delta_2\Big\}, \quad f \in C(T^2).$$

The Riesz's means of the Fourier series has been studied by a lot of authors. We mention for instance the papers of Szasz [11] and Yabuta [12], devoted to the logarithmic means. Similar means with respect to the Walsh and Vilenkin systems were discussed by Simon [10], and Gat [5]. The Norlund logarithmic means has been studied in ([1-7],[10-12]).

In this paper we investigate the approximation properties of two-dimensional logarithmic means of double trigonometric Fourier series of f defined as follows:

$$t_{n,m}(f,x,y) = \frac{1}{l_n l_m}\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\frac{s_{i,j}(f,x,y)}{(n-i)(m-j)}, \quad l_n = \sum_{k=1}^{n}\frac{1}{k},$$

where $S_{M,N}(f,x,y)$ is the partial sum of double Fourier series of f defined by

$$s_{M,N}(f,x,y) = \sum_{m=-M}^{M}\sum_{n=-N}^{N}\widehat{f}(m,n)e^{imx}e^{iny}.$$

It is evident that

$$t_{n,m}(f,x,y) - f(x,y) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[f(x+t,y+s) - f(x,y)\right]F_n(t)F_m(s)dtds,$$

where

$$F_n(t) = \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k(t)}{n-k}$$

and $D_k(t)$is Dirichlet kernel.

For one dimensional trigonometric Fourier series Goginava and Tkebuchava [6] proved that the following are true

Theorem A [6]. Let $f \in C(T)$ and

$$\omega(\delta, f)_c = o\Big(\frac{1}{\log(1/\delta)}\Big)$$

then

$$||t_n(f) - f||_c \to oasn \to \infty.$$

Theorem B [6]. There exists a function $f \in C(T)$ such that

$$\omega(\delta, f)_c = O\left(\frac{1}{\log(1/\delta)}\right)$$

and $t_n(f, 0)$ diverges.

It is well-known that the following statement is true [13].

Theorem C (Zhizhiashvili). Let $f \in C(T^2)$, then

$$\|S_{n,m}(f) - f\|_c \le c\Big\{\omega_1\big(\frac{1}{n}, f\big)_c \log(n+1) + \omega_2\big(\frac{1}{m}, f\big)_c \log(m+1)$$
$$+\omega_{1,2}\big(\frac{1}{n}, \frac{1}{m}, f\big)_c \log(n+1)\log(m+1)\Big\}.$$

From (1) and (2)Let A=(a_{mnjk}) denote a positive rectangular matrix, i. e., a_{mnjk}=0 for $j > m$ or $k > n$, a $a_{mnjk} > 0$ for each $0 \le j \le m, 0 \le k \le n$ and

$$\sum_{j=0}^{m}\sum_{k=0}^{n} a_{mnjk} = 1.$$

For any double sequence (S_{jk}), define

$$t_{mn} = \sum_{j=0}^{m}\sum_{k=0}^{n} a_{mnjk} \cdot s_{jk}, \;\; m, n = 0,\ 1,\ 2,\ \ldots$$

The sequence (S_{jk}) is said to be summable by A if t_{mn} tends to a finite limit as $m, n \to \infty$.

A double rectangular matrix A is said to be regular if it sums every bounded convergent double sequence (S_{jk}) to the same limit. Necessary and sufficient conditions for the matrix A to be regular are known (see, e.g. [9]):

$$\lim_{m,n\to\infty} \sum_{j=0}^{m} a_{mnjk} = 0 \;\; (k = 0, 1, ...), \tag{1}$$

$$\lim_{m,n\to\infty} \sum_{k=0}^{n} a_{mnjk} = 0 \;\; (j = 0, 1, ...). \tag{2}$$

Since

$$\|t_{n,m}(f)-f\|_c \le \frac{1}{l_n l_m}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\frac{\|S_{i,j}(f)-f\|_c}{(n-i)(m-j)},$$

from (1) and (2) we conclude that the following theorem is true.

Theorem 1.1: *Let $f \in C(T^2)$ and*

$$\omega(\delta, f)_c = o\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right).$$

Then

$$\|t_{n,m}(f)-f\|_C \to 0 \quad as \quad m,n \to \infty.$$

In the paper we investigate sharpness of Theorem 1.1. In particular, the following is true

Theorem 1.2: *There exist a function $f \in C(T^2)$ such that*

$$\omega(\delta, f)_c = O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right),$$

and $t_{n,n}(f,0,0)$ diverges.

Proof: (of Theorem 1.2) We choose a monotonically increasing sequence of positive integers$\{n_k; k \ge 1\}$ such that

$$n_1 \ge 2,$$

$$n_k^2 \le n_{k+1}, \tag{3}$$

$$\sum_{l=1}^{k-1}\frac{2^{2n_l}}{n_l^2} < \frac{2^{2n_k}}{n_k^2}, \tag{4}$$

$$\left(\frac{n_k}{2^{2n_k}}\right)^2 \sum_{i=0}^{k-1}\left(\frac{2^{2n_i}}{n_i}\right)^2 < \frac{1}{k}. \tag{5}$$

We construct a function f defined as follows. Set

$$f(x,y) = \sum_{k=1}^{\infty}\frac{f_k(x)\cdot f_k(y)}{n_k^2},$$

where

$$f_k(x) = \sin\left(2^{2n_k} + \frac{1}{2}\right)x \cdot 1_{\left[6\cdot\gamma_{n_k,}\, 6\cdot m(n_k)\cdot\gamma_{n_k}\right]}(x),$$

$$k = 1, 2, \dots\ , \qquad x \in [-\pi, \pi],$$

where 1_A is the characteristic function of a set A and

$$m(n_{n_k}) = \max\left\{s : s\gamma_{n_k} \leq \gamma_{n_{k-1}}\right\}, \gamma_{n_k} = \frac{\pi}{6(2^{2n_k} + 1/2)}..$$

First we prove that

$$\omega(\delta, f)_C = O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right). \tag{6}$$

For every sufficiently small $\delta > 0$ there exists a positive integer k such that

$$\frac{\pi}{2^{2n_k} + 1/2} \leq \delta < \frac{\pi}{2^{2n_{k-1}} + 1/2}.$$

Since

$$|f_{n_l}(x+\delta) - f(x)| = O(\delta 2^{2n_l}), l = 1, 2..., k-1,$$

from (3) and (4) we get

$$|f(x+\delta, y) - f(x,y)| \leq \sum_{l=1}^{k-1} \frac{1}{n_l^2} \cdot |f_{n_l}(x+\delta) - f_{n_l}(x)| + 2\sum_{l=k}^{\infty} \frac{1}{n_l^2}$$
$$= O\left(\delta \sum_{l=1}^{k-1} \frac{2^{2n_l}}{n_l^2}\right) + O\left(\frac{1}{n_k^2}\right) = O\left(\delta \frac{2^{2n_{k-1}}}{n_{k-1}^2}\right) + O\left(\frac{1}{n_k^2}\right)$$
$$= O\left(\left(\frac{1}{\log(1/\delta}\right)^2\right).$$

Consequently,

$$\omega_1(\delta, f)_C = O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right). \tag{7}$$

Analogously, we obtain

$$\omega_2(\delta, f)_C = O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right). \tag{8}$$

Since

$$\omega(\delta, f)_C \le \omega_1(\delta, f)_C + \omega_2(\delta, f)_C$$

from (7) and (8) we get (6)

Next, we shall prove that $t_{2^{2n_k},2^{2n_k}}(f,0,0)$ diverges.

It is clear that

$$\left|t_{2^{2n_k},2^{2n_k}}(f,0,0) - f(0,0)\right| = \left|t_{2^{2n_k},2^{2n_k}}(f,0,0)\right|$$

$$= \left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,s)F_{2^{2n_k}}(t)F_{2^{2n_k}}(s)dtds\right|$$

$$\ge \frac{c}{n_k^2}\left(\int_{-\pi}^{\pi} f_{n_k}(t)\,F_{2^{2n_k}}(t)dt\right)^2 - \sum_{i=1}^{k-1}\frac{c}{n_i^2}\left(\int_{-\pi}^{\pi} f_{n_i}(t)\,F_{2^{2n_k}}(t)dt\right)^2$$

$$-\sum_{i=k+1}^{\infty}\frac{c}{n_i^2}\left(\int_{-\pi}^{\pi} f_{n_i}(t)F_{2^{2n_k}}(t)dt\right)^2 = I - II - III. \tag{9}$$

Since (see [6])

$$\begin{aligned}
&l_{2^{2n}}F_{2^{2n}}(x)\\
&= \frac{\sin(2^{2n}+\frac{1}{2})x}{2\sin\frac{x}{2}}\sum_{k=1}^{2^{2n}-2}\frac{2}{k(k+1)(k+2)}\frac{\sin^2\left((k+1)\frac{x}{2}\right)}{2\sin^2(x/2)}\\
&+\frac{1}{2^{2n}(2^{2n}-1)}\times\frac{\sin(2^{2n}+\frac{1}{2})}{2\sin(x/2)}\frac{\sin^2 2^{2n-1}x}{2\sin^2(x/2)}\\
&+\frac{1}{2^{2n}}\frac{\sin^2(2^{2n}+\frac{1}{2})x}{4\sin^2(x/2)} - \frac{3}{4}\frac{\sin(2^{2n}+\frac{1}{2})x}{2\sin(x/2)}\\
&-\frac{\cos(2^{2n}+\frac{1}{2})x}{2\sin(x/2)}\left(\sum_{k=1}^{n}\frac{\sin kx}{k}\right),
\end{aligned}$$

we have

$$I = \frac{c}{n_k^2}\left(\int_{-\pi}^{\pi} f_{n_k}(t)F_{2^{2n_k}}(t)dt\right)^2 \geq$$

$$\geq \left(\frac{c}{n_k^2}\left|\int_{\frac{\pi}{2^{2n_k+1/2}}}^{\frac{\pi m(n_k)}{2^{n_k+1/2}}} \frac{\sin^2(2^{2n_k}+1/2)t}{2\sin(t/2)} \sum_{i=1}^{2^{2n_k-2}} \frac{2}{i(i+1)(i+2)} \cdot \frac{\sin^2(i+1)\frac{t}{2}}{2\sin^2(t/2)}dt\right|\right.$$

$$-\frac{c}{n_k^2}\frac{1}{2^{2n_k}(2^{2n_k}-1)}\left|\int_{\frac{\pi}{2^{2n_k+1/2}}}^{\frac{\pi\cdot m(n_k)}{2^{2n_k+1/2}}} \frac{\sin^2(2^{2n_k}+1/2)t}{2\sin(t/2)}\frac{\sin^2 2^{n_k-1}t}{2\sin^2(t/2)}dt\right|$$

$$-\frac{c}{n_k^2}\frac{1}{2^{2n_k}}\left|\int_{\frac{\pi}{2^{2n_k+1/2}}}^{\frac{\pi m(n_k)}{2^{2n_k+1/2}}} \sin\left(2^{2n_k}+1/2\right)t\frac{\sin^2(2^{2n_k}+1/2)t}{4\sin^2(t/2)}dt\right|$$

$$-\frac{c}{n_k^2}\left|\int_{\frac{\pi}{2^{2n_k+1/2}}}^{\frac{\pi m(n_k)}{2^{2n_k+1/2}}} \frac{\sin^2\left(2^{2n_k}+1/2\right)t}{2\sin\left(t/2\right)}dt\right|$$

$$\left.-\ \frac{c}{n_k^2}\left|\int_{\frac{\pi}{2^{2n_k+1/2}}}^{\frac{\pi m(n_k)}{2^{2n_k+1/2}}} \frac{\sin\left(2^{2n_k}+1/2\right)t\cos\left(2^{2n_k}+1/2\right)t}{2\sin\left(t/2\right)}\left(\sum_{i=1}^{2^{2n_k}}\frac{\sin it}{i}\right)dt\right|\right)^2 =$$

$$= (I_1 - I_2 - I_3 - I_4 - I_5)^2. \tag{10}$$

It is evident that

$$I_2,\ I_3,\ I_4,\ I_5 = 0\left(\frac{1}{n_k^2}\cdot\int_{\frac{\pi}{2^{2n_k+1/2}}}^{\frac{\pi m(n_k)}{2^{2n_k+1/2}}} \frac{1}{t}dt\right) = 0\left(\frac{1}{n_k}\right). \tag{11}$$

Since (see [14])

$$\sin(i+1)\cdot\frac{t}{2} \geq \frac{2}{\pi}\frac{i+1}{2}t, \quad i = 1, 2, ..., 2^{n_k-1}-1,$$

$$\text{for} \quad t \in I_{n_k}, \quad I_n = \bigcup_{m=1}^{2^{n-1}} \left[\alpha_{mn},\ \beta_{mn}\right],$$

where

$$\alpha_{mn} = \frac{\pi \cdot (12m+1)}{6 \cdot (2^{2n}+1/2)}, \beta_{mn} = \frac{\pi \cdot (12m+5)}{6 \cdot (2^{2n}+1/2)}, m,\ n = 1, 2, ...$$

and

$$\sin\left(2^{2n_k}+1/2\right)t \geq 1/2, \qquad \left|\sum_{k=1}^{2^{2n}} \frac{\sin kx}{k}\right| \leq c < \infty,$$

for I_1 we have

$$I_1 \geq \frac{c}{n_k^2} \sum_{i=1}^{2^{n_k-1}-1} \frac{(i+1)^2}{i\,(i+1)\,(i+2)} \sum_{m=1}^{2^{n_k-1}} \int_{\alpha_{m,n_k}}^{\beta_{m,n_k}} \frac{1}{t}\,dt \geq c > 0. \tag{12}$$

Combining (11) and (12) we conclude that

$$I \geq c > 0. \tag{13}$$

Now, we estimate II. Since [6]

$$\|t_n(f) - f\|_c \leq c \cdot \omega\,(1/n,\ \ f)_c \log(n+1)$$

and

$$\omega\left(f_{n_i}, \frac{1}{2^{2n_k}}\right)_c = 0\left(\frac{2^{2n_i}}{2^{2n_k}}\right), \quad i = 1, 2, ..., k-1,$$

from (4) and (5) we get

$$II \leq C \sum_{i=1}^{k-1} \frac{1}{n_i^2} \left\|t_{2^{2n_k}}(f_{n_i}) - (f_{n_i})\right\|_c^2 \leq C \sum_{i=1}^{k-1} \left(\frac{1}{n_i}\omega\left(f_{n_i}, \frac{1}{2^{2n_k}}\right) n_k\right)^2 \tag{14}$$

$$\leq \ \ C \cdot \sum_{i=1}^{k-1} (\frac{1}{n_i}\frac{2^{2n_i}}{2^{2n_k}} n_k)^2 \leq C(\frac{n_k}{2^{2n_k}})^2 \sum_{i=1}^{k-1} (\frac{2^{2n_i}}{n_i})^2 \leq \frac{c}{k} = o(1) \text{ as } \ k \to \infty.$$

It is obvious that

$$\|F_n\|_L = O\left(\frac{1}{\log n} \cdot \sum_{i=1}^{n-1} \frac{\|D_i\|_1}{n-i}\right)$$

$$= O\left(\frac{1}{\log n} \cdot \sum_{i=1}^{n-1} \frac{\log(i+1)}{n-i}\right) = O\left(\log(n+1)\right).$$

Then we have

$$III = O\left(\sum_{i=k+1}^{\infty} \frac{1}{n_i^2} \cdot \|F_{2^{2n_k}}\|_1^2\right) = O(\sum_{i=k+1}^{\infty} \frac{1}{n_i^2} n_k^2) \tag{15}$$

$$= O\left(\left(\frac{n_k}{n_{k+1}}\right)^2\right) = O(\frac{n_k^2}{n_k^4}) = O(\frac{1}{n_k^2}) = o(1) \quad as \quad k \to \infty.$$

After substituting 13, (14) and (15) in (9) we obtain

$$\overline{\lim_{k\to\infty}} \left|t_{2^{2n_k},\, 2^{2n_k}}(f,0,0) - f(0,0)\right| > 0.$$

□

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