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# Numerical Simulation of Some Non-Classical Elasticity Problems for the Half-Space by the Boundary Element Method

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In the present work non-classical elasticity problems for the homogeneous isotropic elastic half-space are stated and solved. The article considers the plane deformation. Namely, non-classical problems are considered, which are formulated in the following way: what normal stress is supposed to be applied to the part of the half-plane boundary to obtain the pre-given stress or displacement at the segment inside the body. The problems are solved with a boundary element method. There are test examples given showing the value of normal stress supposed to apply to the section of the half-plane boundary to obtain the pre-given stress or displacement at the segment inside the body. By using MATLAB software, the numerical results are obtained and corresponding graphics are constructed.

**Keywords:** Non-Classical elasticity problem, Boundary element method, Homogeneous isotropic half space, Half plane.

**AMS Subject Classification**: 74B05, 65N38, 74S15.

## 1. Introduction

In the theory of elasticity, there are a number of problems [1]-[13] that could be called non-classical due to the fact that boundary conditions on a part of the boundary surface or on the entire boundary surface are either overdetermined or underdetermined, or the conditions on the boundary are connected with the conditions inside the body (so called non-local problems).

The author's earlier work [14] deals with the solution of a non-classical three-dimensional thermoelasticity problem. The problem is to define the temperature on the upper and lower faces of the parallelepiped so that on some two planes inside the body that are parallel to the bases normal displacements or tangential ones would take a priori defined values.

The current article sets and solves non-classical two-dimensional elasticity problems for the homogeneous isotropic elastic half-space by the boundary element method (BEM) [15]. The considered problems do not coincide with the above-mentioned non-classical problems and are of a great applicable importance and are totally different from the above-mentioned problems, the method of solution (BEM) and the mathematical formulation.

Finally, there are test examples given showing the value of normal stress supposed to apply to the section of the half-plane boundary to obtain the pre-given stress

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or displacement at the segment inside the body. The numerical results of these problems are obtained and appropriate graphics with discussion are presented.

## 2. Statement of the problems

Let us set some non-classical static problems for the homogeneous isotropic half plane (See. Fig.1).

It is known that a homogeneous system of elastic static equilibrium in displacements in the Cartesian system of coordinates has the form [16]

$$\begin{cases} (\lambda+\mu)\,\theta_{,x}+\mu\Delta u=0 \\ (\lambda+\mu)\,\theta_{,y}+\mu\Delta v=0 \end{cases} \quad \text{in } S \tag{1}$$

where $\lambda=\frac{\nu E}{(1-2\nu)(1+\nu)}$, $\mu=\frac{E}{2(1+\nu)}$ are Lamé constants, $E$ is elasticity modulus, and $\nu$ Poissons's ratio; $\Delta(\cdot)=(\cdot)_{,xx}+(\cdot)_{,yy}$ is a Laplacian, $\theta=div\vec{U}=u_{,x}+v_{,y}$; $\vec{U}=(u,v)$ is the displacement vector; $(\cdot)_{,x}=\frac{\partial(\cdot)}{\partial x}$, $(\cdot)_{,y}=\frac{\partial(\cdot)}{\partial y}$; $(\cdot)_{,xx}=\frac{\partial^2(\cdot)}{\partial x^2}$; $(\cdot)_{,yy}=\frac{\partial^2(\cdot)}{\partial y^2}$.

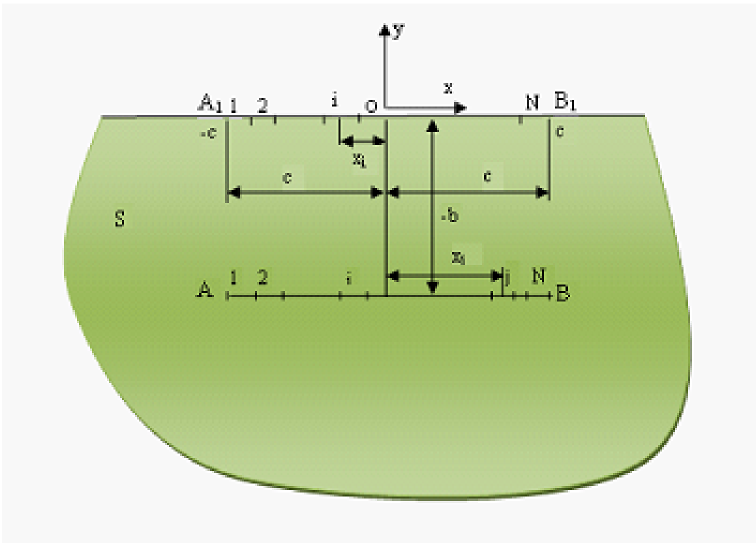


Figure 1. Illustration of localization problems of stresses and displacements for the elastic half plane.

### 2.1. *Statement of a problem when normal stress is applied to the segment inside a half plane*

Let us consider a non-classical problem for the half plane $S$ (see Fig. 1), when the tangent stress along the entire border and normal stress along boundary segment $|x|>c,\ y=0$ are equal to zero. Along the segment $|x|\le c,\ y=-b$ inside the body, the value of normal stress $\sigma_{yy}$ is known. So, let us find the solutions to the system of equilibrium equations (1) satisfying the following boundary conditions:

$$\begin{array}{l} \text{for } |x| < \infty \ \text{ and } \ y = 0 : \quad \sigma_{yx} = 0, \\ \text{for } \ |x| > c \ \text{ and } \ y = 0 : \quad \sigma_{yy} = 0, \\ \text{for } |x| \le c \ \text{ and } \ y = -b : \sigma_{yy} = -P_0(x), \end{array} \tag{2}$$

where $P_0(x)$ is the sufficiently smooth function given along the segment $[-c; c]$ .

We can formulate the stated problem as follows: let us find the kind of distribution of normal stress $\sigma_{yy}$ along section $|x| \le c,\ y = 0$ of the boundary of a half plane (see Fig. 1) so that the normal stress along segment $|x| \le c,\ y = -b$ inside the body is equal to the values of given function $P_0(x)$.

### 2.2. *Statement problem when normal displacement is given on the segment lying inside a half plane*

Let us consider a non-classical problem, when along the entire border of the half plane $S$ (see Fig. 1) the tangent stress is equal to zero, and normal displacement $u_y$ on segment $|x| \le c, \quad y = -b$ lying inside the body is known. Besides, the normal stress along the part $|x| \ge c, \quad y = 0$ of the boundary is equal to zero. Thus, we have the following boundary conditions:

$$\begin{array}{l} \text{for } |x| < \infty \ \text{ and } \ y = 0 : \ \sigma_{yx} = 0, \\ \text{for } \ |x| > c \ \text{ and } \ y = 0 : \quad \sigma_{yy} = 0, \\ \text{for } |x| \le c \ \text{ and } \ y = -b : u_y = -U_0(x), \end{array} \tag{3}$$

where $U_0(x)$ is the sufficiently smooth function given along the segment $[-c; c]$.

We can formulate this problem as follows: let us find the distribution of normal stress $\sigma_{yy}$ along the part $|x| \le c,\ y = 0$ of the boundary of the half plane when the normal displacement along the segment $|x| \le c,\ y = -b$ lying inside the half plane $S$ equals $-U_0(x)$.

## 3. Solving stated problems

Let us solve the stated problems by BEM. When solving the boundary value problems for the half plane by BEM, we use a singular solution of the Flamant problem (see Appendix A).

### 3.1. *Solving problem (1), (2)*

Let us divide segments $|x| \le c,\ y = 0$ and $|x| \le c,\ y = -b$ into $N$ segments (elements) of the same size $2a$ and smaller sizes (i.e. $a = c/N$ ). We mean that constant normal stresses $P_y^j$ act on each $j^{th}$ element of length $2a$ with center $(x^j, 0)$ of segment $|x| \le c,\ y = 0$. We need to find such values of these stresses, for which the values of the normal stresses in middle points $(x^i, \ -b)$ of each $i^{th}$ segment with a length of $2a$ along segment $|x| \le c,\ y = -b$ inside the body will be equal to the given value of $-P_0(x^i)$.

Normal stress in the center of the $i^{th}$ element lying on segment $|x| \le c,\ y = -b$ caused by the action of constant normal load $P_y^j$ on the $j^{th}$ element of segment

$|x| \le c,\ y = 0$ will be found by inserting $y = -b$, $x = x^i - x^j$ in the fourth formula of (A.1).

$$\sigma_{yy}(x^i,\,-b) = \frac{1}{\pi}\left[\left(\arctan\frac{b}{x^i-x^j-a} - \arctan\frac{b}{x^i-x^j+a}\right)\right.$$
$$\left. + \frac{b(x^i-x^j+a)}{(x^i-x^j+a)^2+b^2} - \frac{b(x^i-x^j-a)}{(x^i-x^j-a)^2+b^2}\right] P_y^j\,.$$

The normal stress in the center of the $i^{th}$ element lying on segment $|x| \le c,\ y = -b$ will be equal to the following sum:

$$\sigma_{yy}(x^i,\,-b) = \sum_{j=1}^{N} A^{ij} P_y^j, \qquad i = 1,\,2,\,...\,,N,$$

where for the influence coefficients $A^{ij}$ has the following formula

$$A^{ij} = \frac{1}{\pi}\left[\left(\arctan\frac{b}{x^i-x^j-a} - \arctan\frac{b}{x^i-x^j+a}\right)\right.$$
$$\left. + \frac{b(x^i-x^j+a)}{(x^i-x^j+a)^2+b^2} - \frac{b(x^i-x^j-a)}{(x^i-x^j-a)^2+b^2}\right].$$

Thus, we obtain the following system of $N$linear algebraic equations with $N$ unknown quantities $P_y^j$, $j = 1, 2, \ldots, N$.

$$\sum_{j=1}^{N} A^{ij} P_y^j = P_0\left(x^i\right)\,, \qquad i = 1,\ 2,\ ...\ ,\ N. \tag{4}$$

If solving (4) system in relation to the unknown quantities $P_y^j$ by means of any standard method of numerical analysis (by method of Gauss in our case), then we can assume that the set problem is solved and $\sigma_{yy}^j = P_y^j, \quad j = 1, \ldots, N$ (see Appendix ).

After solving these equations, we can express the displacements and stresses at any point $\left(x^i, y^k\right)$ of the body by means of other linear combination of load $P_y^j$. For example, the stresses and displacements have the following form:

$$\sigma_{xx}(x^i,\,y^k) = -\frac{1}{\pi}\sum_{j=1}^{N}\left[\left(\arctan\frac{y^k}{x^i-x^j-a} - \arctan\frac{y^k}{x^i-x^j+a}\right)\right.$$
$$\left. - \frac{y^k(x^i-x^j+a)}{(x^i-x^j+a)^2+(y^k)^2} + \frac{y^k(x^i-x^j-a)}{(x^i-x^j-a)^2+(y^k)^2}\right] P_y^j, \sigma_{yy}(x^i,\,y^k)$$
$$= -\frac{1}{\pi}\sum_{j=1}^{N}\left[\left(\arctan\frac{y^k}{x^i-x^j-a} - \arctan\frac{y^k}{x^i-x^j+a}\right)\right.$$
$$\left. + \frac{y^k(x^i-x^j+a)}{(x^i-x^j+a)^2+(y^k)^2} - \frac{y^k(x^i-x^j-a)}{(x^i-x^j-a)^2+(y^k)^2}\right] P_y^j, \sigma_{xy}(x^i,\,y^k)$$
$$= \frac{1}{\pi}\sum_{j=1}^{N}\left(y^k\right)^2\left[\frac{1}{(x^i-x^j+a)^2+(y^k)^2} - \frac{1}{(x^i-x^j-a)^2+(y^k)^2}\right] P_y^j, \tag{5}$$
$$i = 1,\,2,\,...\,,M_1, \quad k = 1,\,2,\,...\,,M_2.$$
$$u_x^j(x^i,\,y^k) = -\frac{1}{2\pi\mu}\sum_{j=1}^{N}\left\{(1-2\nu)\left[\left(x^i - x^j - a\right)\arctan\frac{y^k}{x^i-x^j-a}\right.\right.$$
$$\left. - \left(x^i - x^j + a\right)\arctan\frac{y^k}{x^i-x^j+a} - \pi a\right]$$
$$\left. + (1-\nu)\,y^k\frac{\ln\left((x^i-x^j-a)^2+(y^k)^2\right)}{\ln\left((x^i-x^j+a)^2+(y^k)^2\right)}\right\} P_y^j,$$

$$u_y^j(x^i,\ y^k) = \frac{1}{2\pi\mu}\sum_{j=1}^{N}\Big\{-y^k\,(1-2\nu)\Big(\arctan\frac{y^k}{x^i-x^j-a} - \arctan\frac{y^k}{x^i-x^j+a}\Big)$$
$$+\,(1-\nu)\Big[\big(x^i-x^j-a\big)\ln\Big((x^i-x^j-a)^2+\big(y^k\big)^2\Big)$$
$$-\big(x^i-x^j+a\big)\ln\Big((x^i-x^j+a)^2+\big(y^k\big)^2\Big)$$
$$+\big(L-x^j+a\big)\ln\big(L-x^j+a\big)^2-\big(L-x^j-a\big)\ln\big(L-x^j-a\big)^2\Big]\Big\}\,P_y^j.$$

## 3.2. *Solving problem (1), (3)*

Let us divide segments $|x| \le c,\ y = 0$ and $|x| \le c,\ y = -b$ into $N$ segments (elements) with equal $2a$ and smaller lengths. We mean that constant normal stresses $P_y^j$ act on each $j^{th}$ segment of segment $|x| \le c,\ y = 0$, each with the length of $2a$ and with the center $(x^j, 0)$. We must find such values of these stresses, for which the values of normal displacement in middle point $(x^i,\ -b)$ of each $i^{th}$ element with length $2a$ of $|x| \le c,\ y = -b$ segment inside the body should be equal to the given value of $-U_0(x^i)$.

Normal displacement $u_y^j(x^i,\ -b)$ in the center $(x^i,\ -b)$ of the $i^{th}$ element lying on the segment $|x| \le c,\ y = -b$ caused by the action of constant load $P_y^j$ on the $j^{th}$ element lying of segment $|x| \le c,\ y = 0$ will be found by inserting $y = -b$, $x = x^i - x^j$ and $L = L - x^j$ in the second formula from formula (A.1).

$$u_y^j(x^i,\ -b) = \frac{P_y^j}{2\pi\mu}\Big\{-b\,(1-2\nu)\Big(\arctan\frac{b}{x^i-x^j-a} - \arctan\frac{b}{x^i-x^j+a}\Big)$$
$$+\,(1-\nu)\Big[\big(x^i-x^j-a\big)\ln\big((x^i-x^j-a)^2+b^2\big)$$
$$-\big(x^i-x^j+a\big)\ln\big((x^i-x^j+a)^2+b^2\big)$$
$$+\big(L-x^j+a\big)\ln\big(L-x^j+a\big)^2-\big(L-x^j-a\big)\ln\big(L-x^j-a\big)^2\Big]\Big\}.$$

Next, the normal displacement in the center of the $i^{th}$ element lying on the segment $|x| \le c,\ y = -b$ will be computed with the following formula:

$$u_y(x^i,\ -b) = \sum_{j=1}^{N} B^{ij} P_y^i, \qquad i = 1,\ 2,\ \dots,\ N,$$

where we have the following formula for influence coefficients $B^{ij}$:

$$B^{ij} = \frac{1}{2\pi\mu}\Big\{-b\,(1-2\nu)\Big(\arctan\frac{b}{x^i-x^j-a} - \arctan\frac{b}{x^i-x^j+a}\Big)$$
$$+\,(1-\nu)\Big[\big(x^i-x^j-a\big)\ln\big((x^i-x^j-a)^2+b^2\big)$$
$$-\big(x^i-x^j+a\big)\ln\big((x^i-x^j+a)^2+b^2\big)$$
$$+\big(L-x^j+a\big)\ln\big(L-x^j+a\big)^2-\big(L-x^l-a\big)\ln\big(L-x^j-a\big)^2\Big]\Big\}.$$

Thus, the set problem is reduced to solving the following system of linear algebraic equations ($N$ equations with $N$ unknown values):

$$\sum_{j=1}^{N} B^{ij} P_y^i = -U_0(x^i)\,, \tag{6}$$

If we solve system (6) in relation to unknown values $P_y^j$, then the set problem can be considered as solved, like the problem set in 3.1.

## 4. Testable examples and discussion

### 4.1. *Numerical simulations of problems in stresses*

By using the MATLAB software, we obtained numerical values of the normal stresses along the segment AB (the given normal load) and distribution of normal stresses along the segment $\mathrm{A_1B_1}$ (the obtained normal stress ) shown in Fig. 1 for the following data: $c = 1m,\ \ 2m,\ \ 3m,\ \ 4m$ and $b = 5m,\ \ 6,5m,\ \ 8m,\ \ 10m$; $N = 120$; $P = 10kg/cm^2$. Below are represented graphics of some of the obtained result. Namely, Fig. 2 and Fig. 3 show normal load a) $P_0(x) = P\cdot x^2$ and b) $P_0(x) = P \cdot x^3$, respectively, along AB segment and distribution of obtained normal stress $\sigma_{yy} = P_y$ along $A_1B_1$ segment, when $c = 1m$ and $b = 5m,\ \ 6,5m,\ \ 8m,\ \ 10m$.

Moreover, represented 3D graphics of the distribution of stresses and displacements in the body according to the domain $-c < x < c,\ \ -30 < y < -10$, when $c = 1m$, $b = 30m$ for steel $E = 2\times 10^6 kg/cm^2$, $\nu = 0.3$ (see Fig. 4 - Fig.7) and technical rubber $E = 2\times 10^2 kg/cm^2$, $\nu = 0.42$ (see Fig.8 and Fig.9). Formula (5) evidences that the stresses in the stress problems do not depend on Young's modulus and Poison's ratio. As for the displacement, the normal displacement is less and tangential displacement is bigger in steel than in technical rubber.

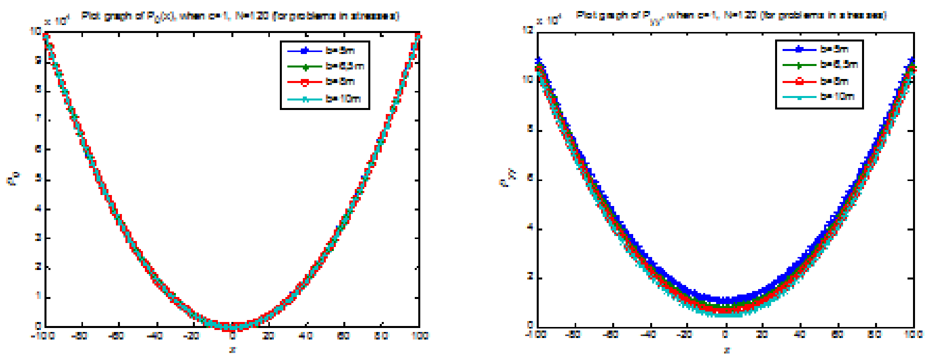


Figure 2. The load $P_0(x) = P \cdot x^2$ along the segment AB and distribution of obtained normal stress $P_y$ along the segment $A_1B_1$, when $c = 1m$.

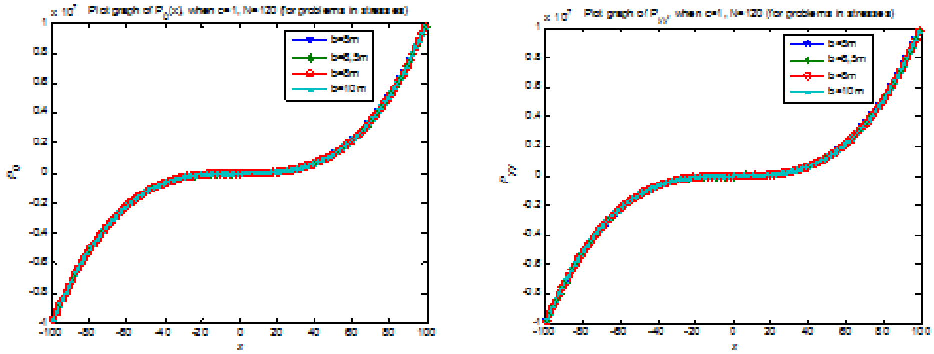


Figure 3. The load $P_0(x) = P \cdot x^3$ along the segment AB and distribution of obtained normal stress $P_y$ along the segment $A_1B_1$, when $c = 1m$.

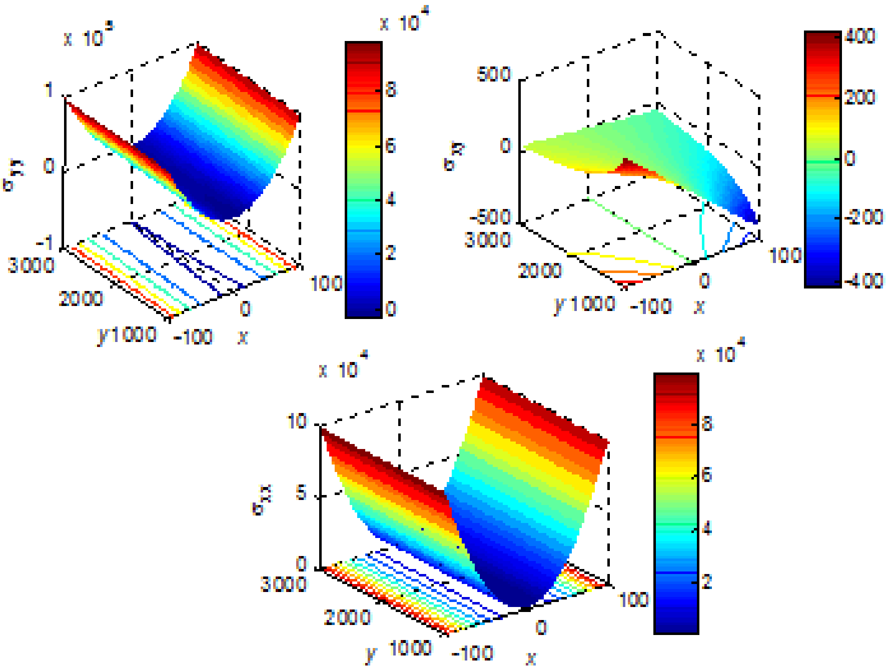


Figure 4. Distribution of stresses in the domain $-c < x < c, \quad -30 < y < -10$, when $c = 1m$, $b = 30m$, (in stresses for the problem, when $P_0(x) = P \cdot x^2$).

### 4.2. *Numerical simulations of problems in displacements*

For the following data: $E = 2 \times 10^2 kg/cm^2$, $\nu = 0.42$ or $E = 2 \times 10^6 kg/cm^2$, $\nu = 0.3$; $c = 1m, \ 2m, \ 3m, \ 4m$ and $b = 5m, \ 6,5m, \ 8m, \ 10m$; $N = 120$, $P = 10m$ numerical values of normal displacements at the segment $AB$ (the given normal displacement) and distribution normal stresses at the segment $A_1B_1$ (the obtained normal stress ) are obtained (see Fig. 1). Below graphics of some obtained results are represented. Namely, Fig. 10 and Fig.12 shows normal displacement a) $U_0(x) = P \cdot x^2$ and b) $U_0(x) = P \cdot x^3$, respectively, along AB segment and distribution of the obtained normal stress $\sigma_{yy} = P_y$ along $A_1B_1$ segment, when $c = 1m$ and $E = 2 \times 10^6 kg/cm^2$, $\nu = 0.3$, and on Fig. 11, Fig. 13, when $c = 1m$ and $E = 2 \times 10^2 kg/cm^2$, $\nu = 0.42$.

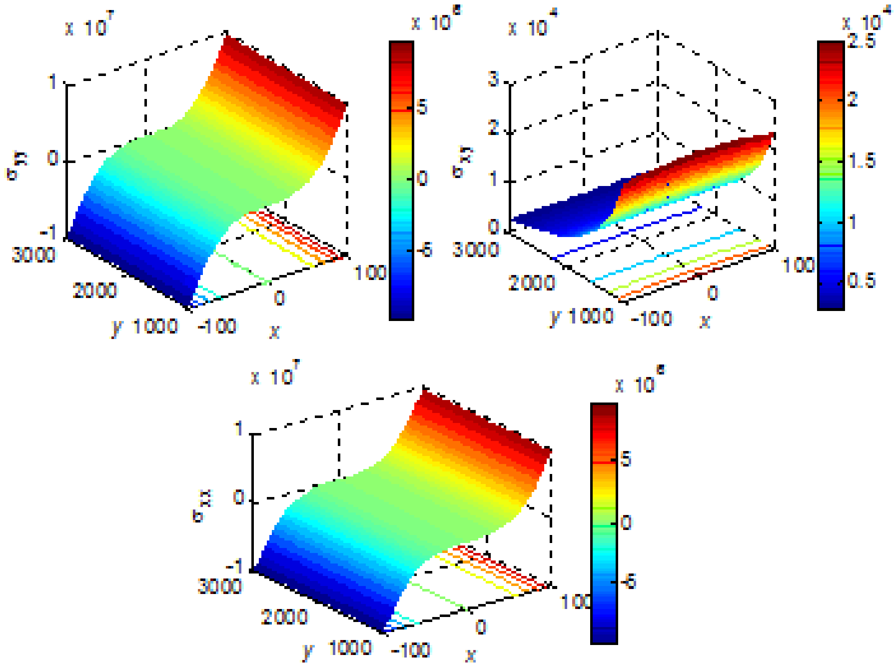


Figure 5. Distribution of stresses for steel or technical rubber in domain $-c < x < c, \quad -30 < y < -10$, when $c = 1m$, $b = 30m($ in stresses for the problem, when $P_0(x) = P \cdot x^3$.

Besides, 3D graphics of distribution of stresses and displacements in the body which occupies domain $-c < x < c, \quad -30 < y < -10$ are represented, when $c = 1m$, $b = 30m$ for the steel (see Fig. 14 - Fig.17) and the technical rubber (see Fig. 18 - Fig.21). In this case, stresses $\sigma_{yy}$ and $\sigma_{xx}$ in the steel are too big, while tangent stress $\tau_{xy}$ is less as compared to the technical rubber. In addition, the normal displacement in both materials are almost equal, while tangent displacement is bigger in steel than in technical rubber.

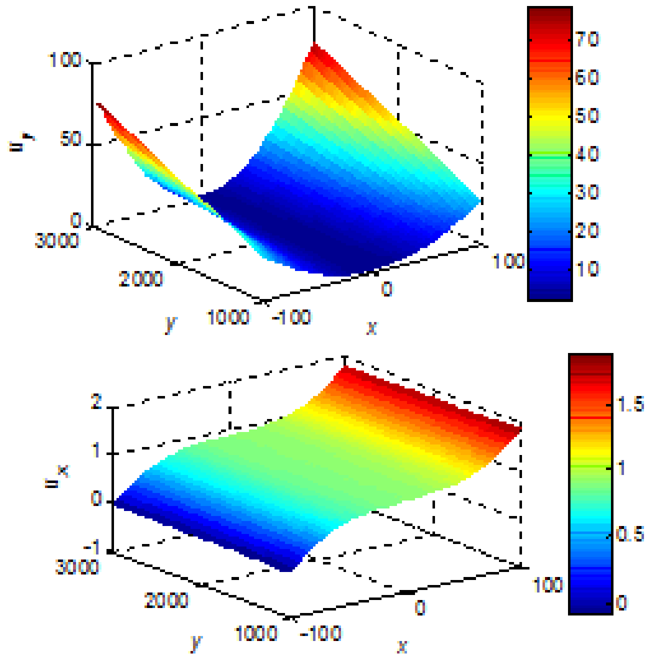


Figure 6. Distribution of displacements for steel in domain $-c < x < c, \quad -30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^6 kg/cm^2$, $\nu = 0.3$ (in stresses for the problem, when $P_0(x) = P \cdot x^2$).

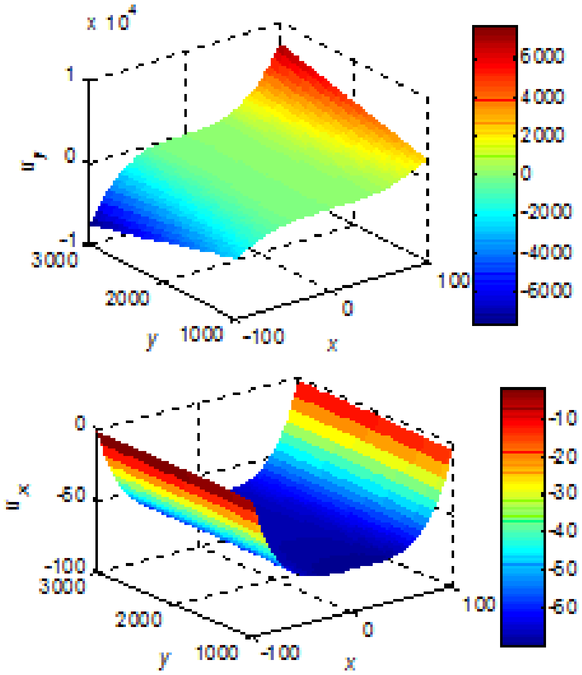


Figure 7. Distribution of displacements for steel in domain $-c < x < c, \quad -30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^6 kg/cm^2$, $\nu = 0.3$ (in stresses for the problem, when $P_0(x) = P \cdot x^3$.

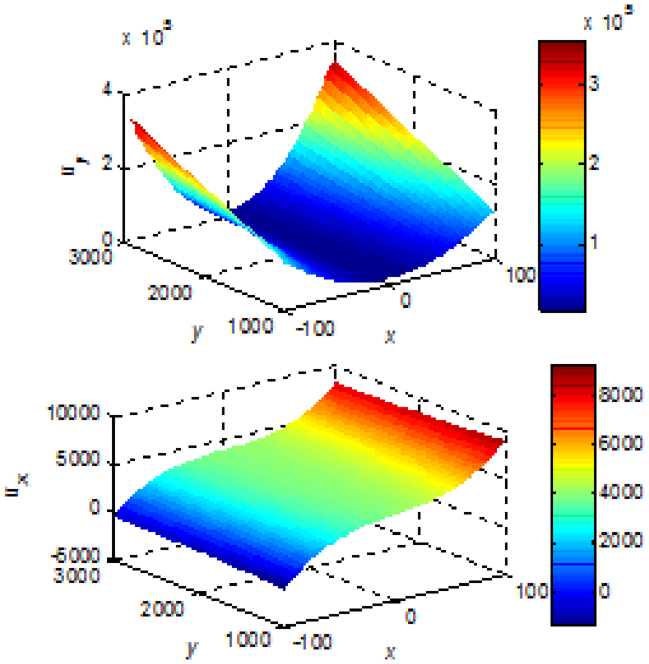


Figure 8. Distribution of displacements for technical rubber in domain $-c < x < c$, $\; -30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^2 kg/cm^2$, $\nu = 0.42$ (in stresses for the problem, when $P_0(x) = P \cdot x^2$).

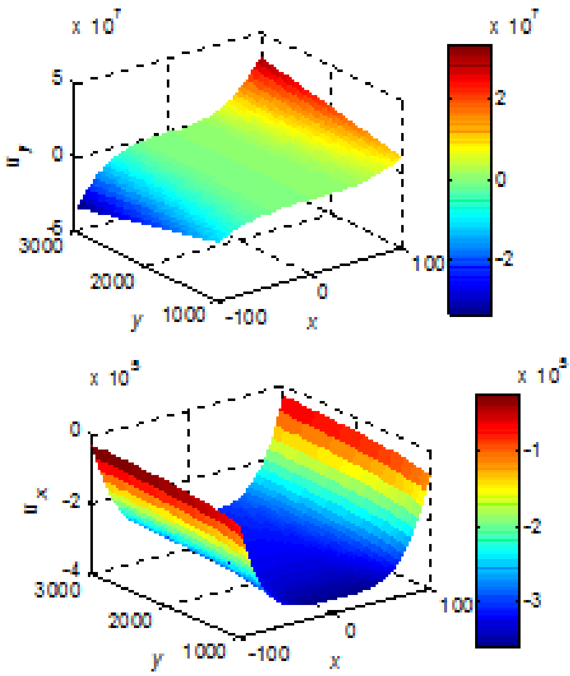


Figure 9. Distribution of displacements for technical rubber in domain $-c < x < c$, $\; -30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^2 kg/cm^2$, $\nu = 0.42$ (in stresses for the problem, when $P_0(x) = P \cdot x^3$).

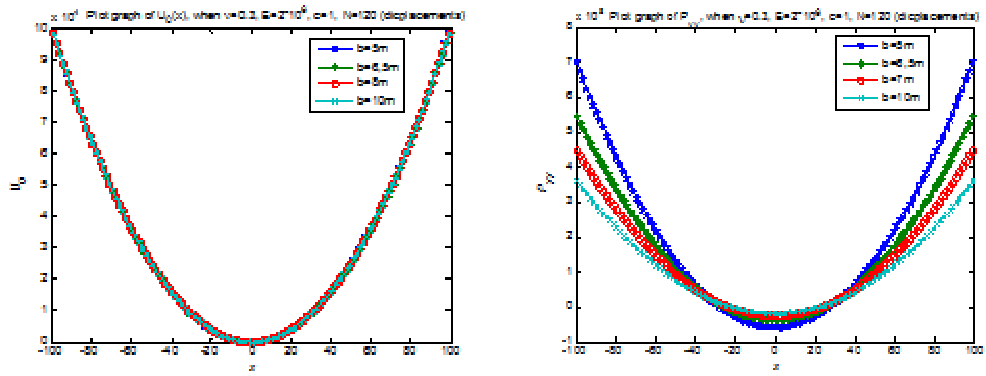


Figure 10. Displacement $U_0(x) = P \cdot x^2$ along segment AB and distribution of obtained normal stress $P_y$ along segment$A_1B_1$, when $c = 1m$and $E = 2 \times 10^6 kg/cm^2$, $\nu = 0.3$(steel).

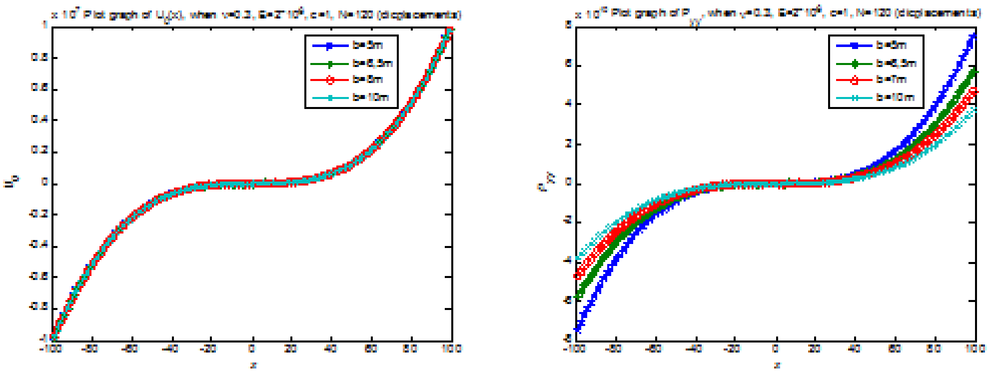


Figure 11. Displacement $U_0(x) = P \cdot x^3$ along segment AB and distribution of obtained normal stress $P_y$ along segment$A_1B_1$, when $c = 1m$and, $E = 2 \times 10^6 kg/cm^2$, $\nu = 0.3$(steel).

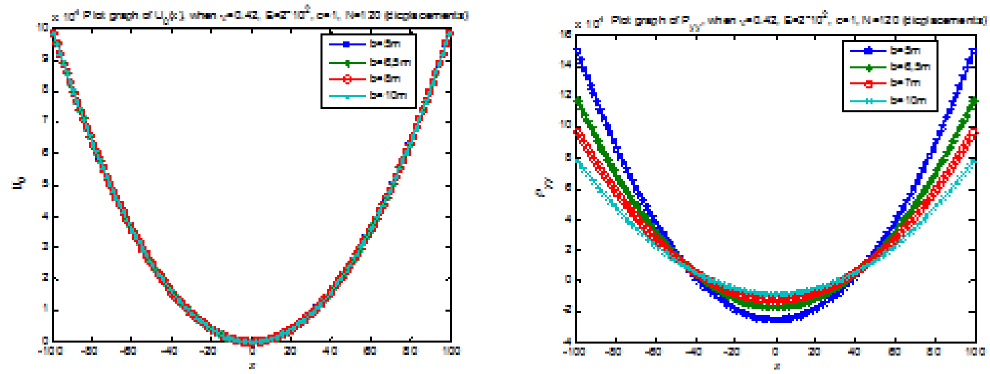


Figure 12. Displacement $U_0(x) = P \cdot x^2$ along segment AB and distribution of obtained normal stress $P_y$ along segment$A_1B_1$, when $c = 1m$and, $E = 2 \times 10^2 kg/cm^2$, $\nu = 0.42$(technical rubber).

## 5. Conclusions

In the paper some non-classical problems are stated. The essence of the problems is as follows: we must find the distribution of the normal stress along part $A_1B_1$

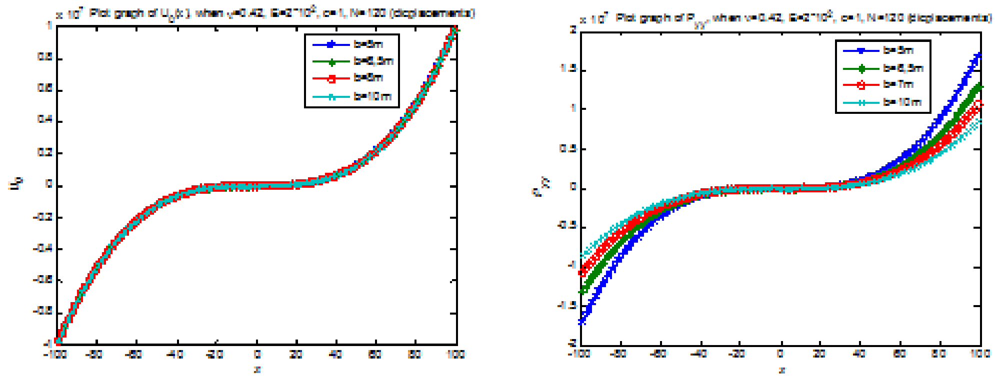

Figure 13. Displacement $U_0(x) = P \cdot x^3$ along segment AB and distribution of obtained normal stress $P_y$ along segment $A_1B_1$, when $c = 1m$ and, $E = 2 \times 10^2 kg/cm^2$, $\nu = 0.42$(technical rubber).

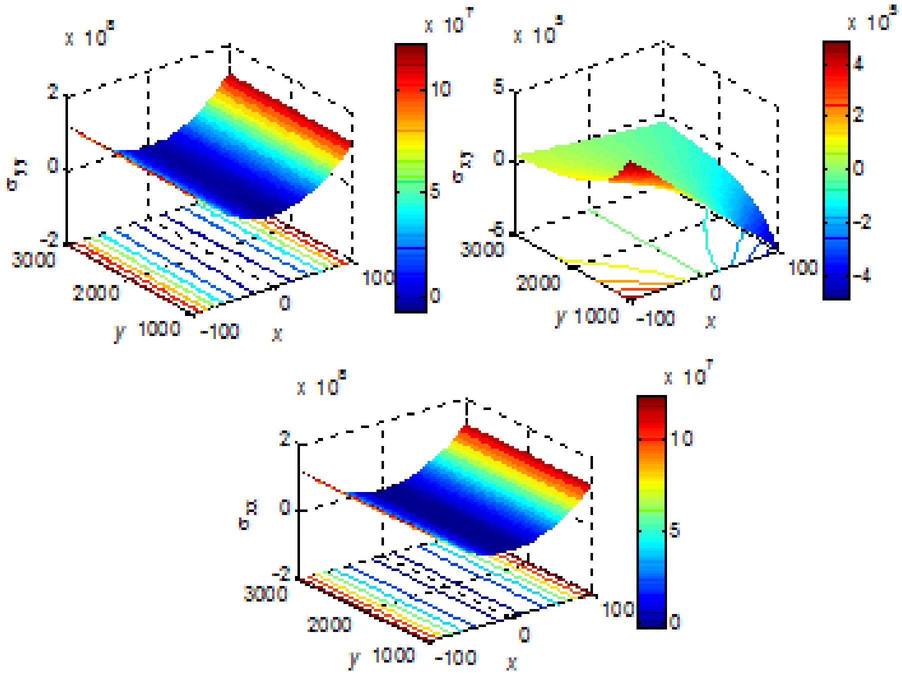

Figure 14. Distribution of stresses for steel in domain$-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^6 kg/cm^2$, $\nu = 0.3$ (in displacements for the problem, when $U_0(x) = P \cdot x^2$).

(see Fig. 1) of the border of the half plane so that normal stress $\sigma_{yy}$ or normal displacement $u_y$ along a segment parallel to the border of a given length distanced from the border by $b$ within the body should be equal to the value of the given function. The set problems are solved by BEM [15]. There are test examples given showing the value of normal stress supposed to apply to the section of the half-plane boundary to obtain the pre-given stress ($P_0(x) = P \cdot x^2$ or $P_0(x) = P \cdot x^3$) or displacement ($U_0(x) = P \cdot x^2$ or $U_0(x) = P \cdot x^3$, $P =$ constant) along the segment inside the body. Using the MATLAB's software the numerical results of these problems are obtained and appropriate graphics with discussion are presented.

The problems considered in the work can be used in practice. For instance, in soils and rocks, materials that are susceptible to cracking and faulting when

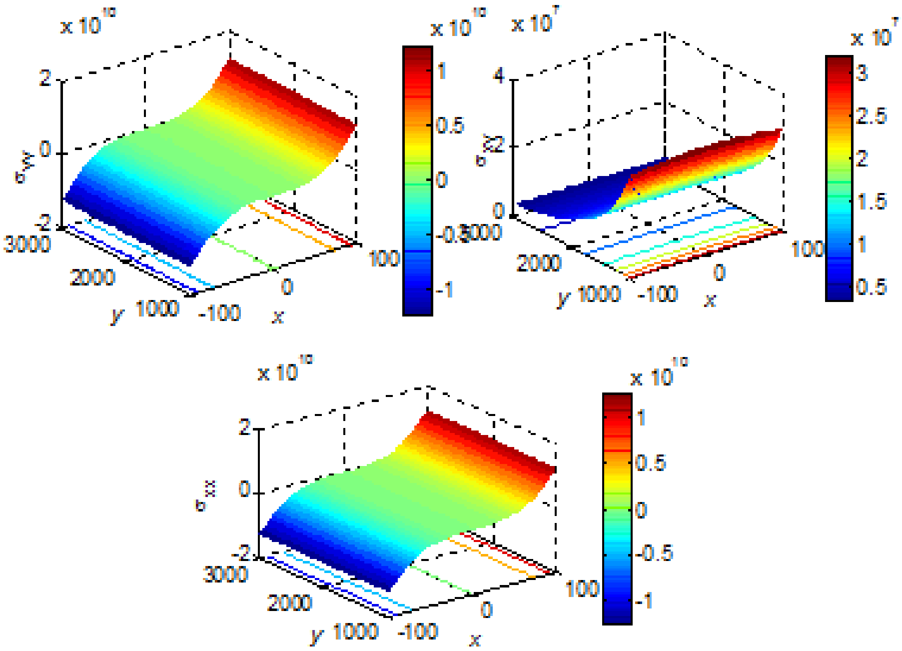

Figure 15. Distribution of stresses for steel in domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^6 kg/cm^2$, $\nu = 0.3$ (in displacements for the problem, when $U_0(x) = P \cdot x^3$).

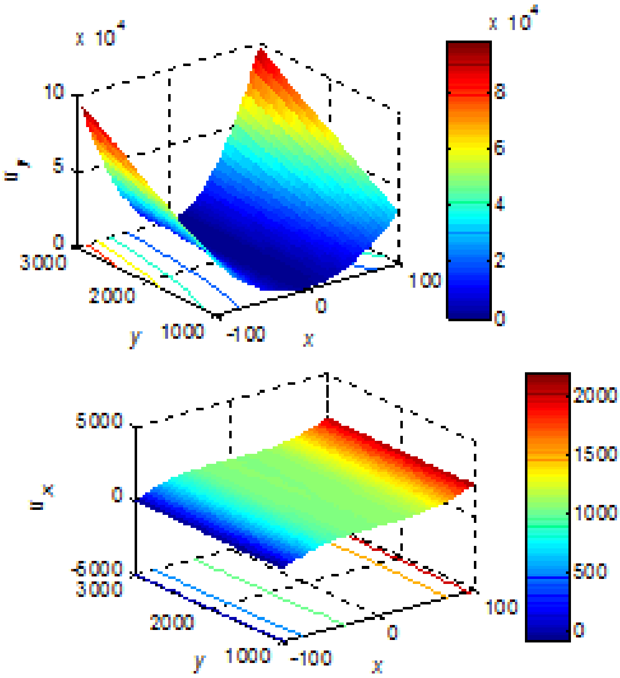

Figure 16. Distribution of displacements for steel in domain$-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^6 kg/cm^2$, $\nu = 0.3$ (in displacements for the problem, when $U_0(x) = P \cdot x^2$).

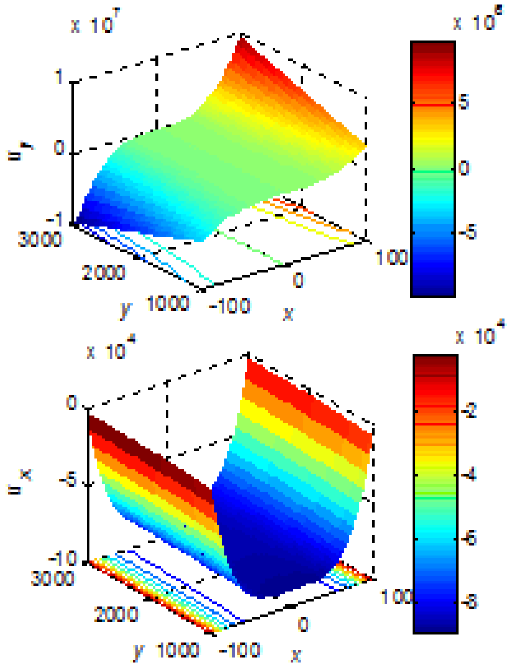


Figure 17. Distribution of displacements for steel in domain$-c < x < c, \quad -30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^6 kg/cm^2$, $\nu = 0.3$ (in displacements for the problem, when $U_0(x) = P \cdot x^3$).

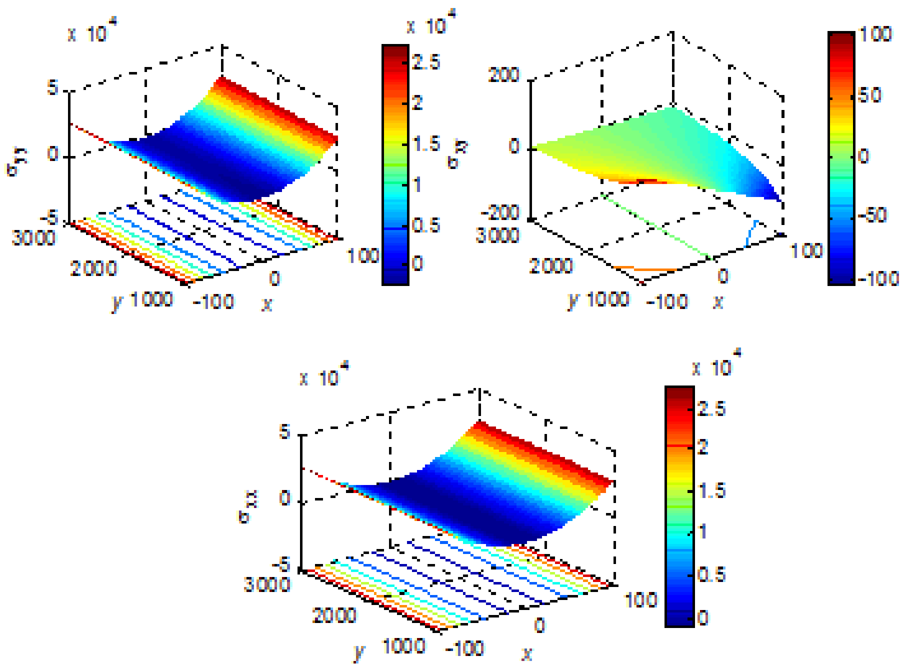


Figure 18. Distribution of stresses for technical rubber in domain $-c < x < c, \quad -30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^2 kg/cm^2$, $\nu = 0.42$ (in displacements for the problem, when $U_0(x) = P \cdot x^2$).

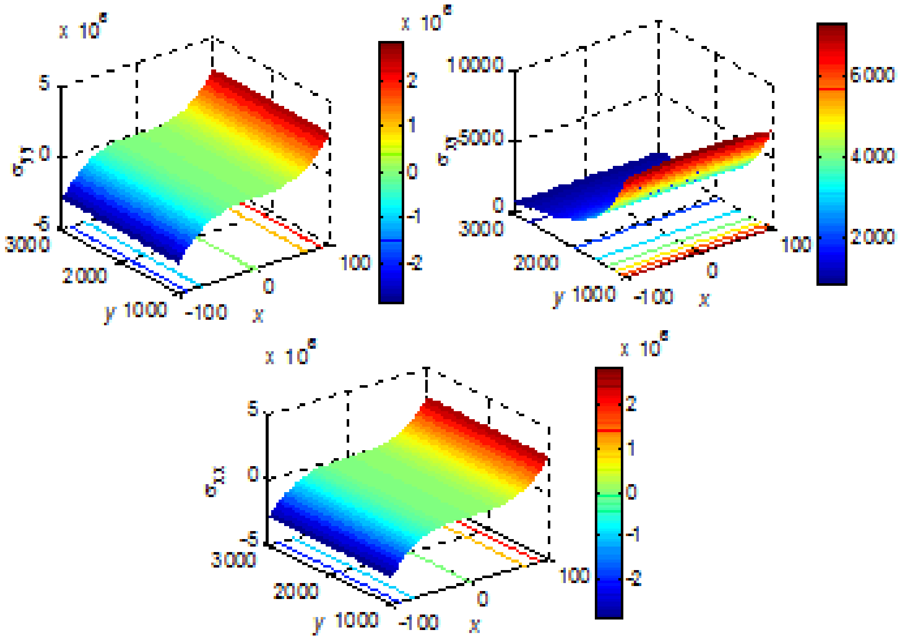


Figure 19. Distribution of stresses for technical rubber in domain $-c < x < c, \quad -30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^2 kg/cm^2$, $\nu = 0.42$ (in displacements for the problem, when $U_0(x) = P \cdot x^3$).

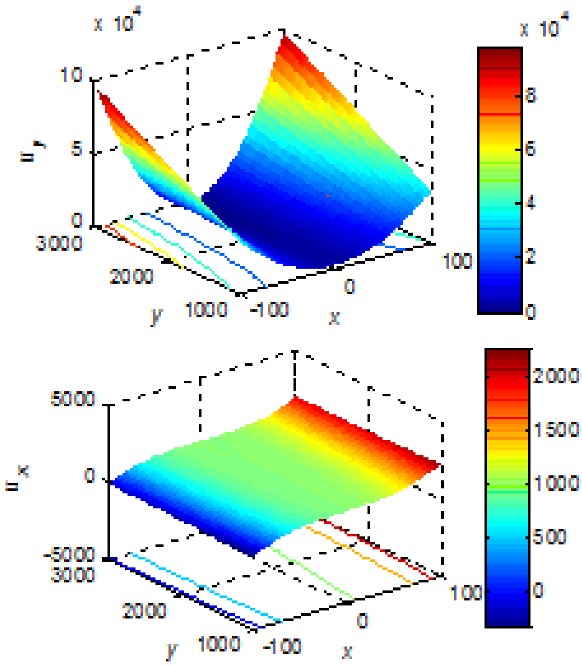


Figure 20. Distribution of displacements for technical rubber in domain $-c < x < c, \quad -30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^2 kg/cm^2$, $\nu = 0.42$ (in displacements for the problem, when $U_0(x) = P \cdot x^2$).

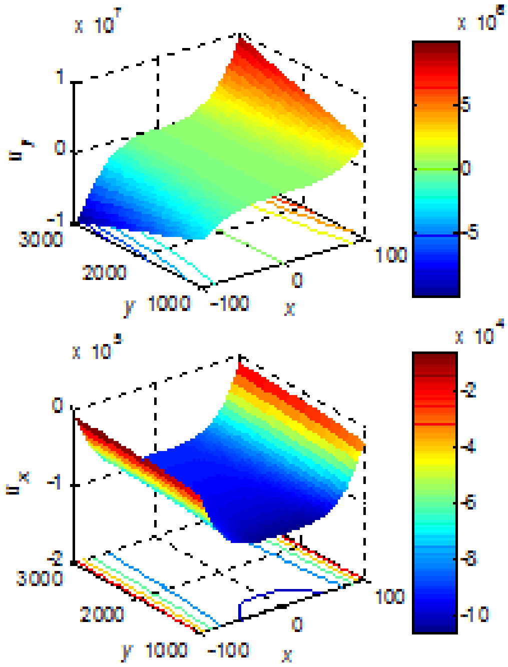


Figure 21. Distribution of displacements for technical rubber in domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2 \times 10^2 kg/cm^2$, $\nu = 0.42$ (in displacements for the problem, when $U_0(x) = P \cdot x^3$).

sheared, as well materials used to demolish military structures or in underground facilities.

## Appendix

## Flamant problem

The problem of a concentrated force applied perpendicular to the surface of an elastic isotropic half-plane is known as Flamant problem. The solution of the Flamant problem can be found in many courses of elasticity, for example, see [17, 18]. It is an example of singular solutions in the static theory of elasticity.

In the case of distribution of constant normal stresses $p_y(x) = P_y$ along the segment $-a \le x \le a$, $y = 0$ with a finite length we will have [15].

$$\begin{aligned}
u_x &= -\tfrac{P_y}{2\pi\mu}\left\{(1-2\nu)\left[(x-a)\theta_1-(x+a)\theta_2-\pi a\right]+(1-\nu)\,y\ln\left(\left(r_1^2/r_2^2\right)\right)\right\},\\
u_y &= \tfrac{P_y}{2\pi\mu}\left\{-(1-2\nu)\,y\,(\theta_1-\theta_2)+(1-\nu)\left[(x-a)\ln r_1^2-(x+a)\ln r_1^2+\right.\right.\\
&\qquad \left.\left.+(L+a)\ln(L+a)^2-(L-a)\ln(L-a)^2\right]\right\},\\
\sigma_{xx} &= -\tfrac{P_y}{\pi}\left[\theta_1-\theta_2+y\,(x-a)/r_1^2-y\,(x+a)/r_2^2\right],\\
\sigma_{yy} &= -\tfrac{P_y}{\pi}\left[\theta_1-\theta_2-y\,(x-a)/r_1^2+y\,(x+a)/r_2^2\right],\\
\sigma_{xy} &= -\tfrac{P_y}{\pi}y^2\left(1/r_1^2-1/r_2^2\right)
\end{aligned} \qquad (A.1)$$

where $\theta_1 = \arctan(y/(x-a))$, $\theta_2 = \arctan(y/(x+a))$, $r_1^2 = (x-a)^2 + y^2$, $r_2^2 = (x+a)^2 + y^2$, $L$ is any arbitrary constant and means that $u_y$ displacement will be measured relatively to the displacement of any $x = \pm L$ point of the boundary of a half-plane (reference point).

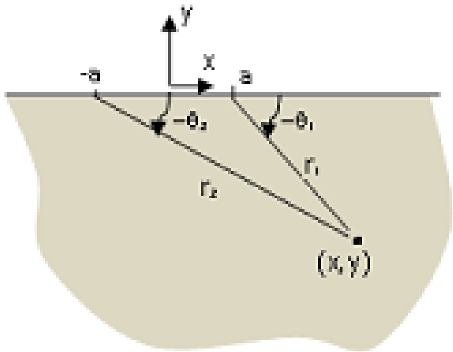


Figure 22. Distances to the extreme points and corresponding angles.

When $y = 0$, $\sigma_{yy}$ is as follows: $\sigma_{yy} = -P_y(\theta_1 - \theta_2)/\pi$.

According to Fig.22, when $y = 0$ then $\theta_1 = \theta_2$, with the exception of the segment $|x| < a$, where $\theta_1 = -\pi$ and $\theta_2 = 0$. Thus, we find that $\sigma_{yy} = 0$ when $|x| > a$, and $\sigma_{yy} = P_y$ when $|x| < a$.

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