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On Generalized Measurability Properties of Certain Projective Sets

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Some pathological subsets of the real line $\mathbf{R}$ are considered from the viewpoint of their potential definability and their measurability properties with respect to different classes of measures on $\mathbf{R}$.

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Throughout this paper, E denotes a base (ground) set and $\mathbf{R}$ stands for the real line.

Let $f : E \to \mathbf{R}$ be a function and let $\mathcal{M}$ be a class of σ-finite measures defined on some σ-algebras of subsets of E.

According to the terminology adopted in [1], we shall say that:

(a) f is absolutely measurable with respect to $\mathcal{M}$ if, for any measure $\mu \in \mathcal{M}$, this f turns out to be μ-measurable;

(b) f is relatively measurable with respect to $\mathcal{M}$ if there exists at least one measure $\mu \in \mathcal{M}$ such that this f is μ-measurable;

(c) f is absolutely nonmeasurable with respect to $\mathcal{M}$ if there exists no measure from $\mathcal{M}$ for which this f becomes measurable.

Naturally, for a set $X \subset E$, we have some direct analogues of the introduced notions (a), (b), and (c).

Namely, we shall say that X is absolutely measurable (relatively measurable, absolutely nonmeasurable) with respect to the class $\mathcal{M}$ if the characteristic function of X is absolutely measurable (relatively measurable, absolutely nonmeasurable) with respect to $\mathcal{M}$.

More detailed information about the above concepts can be found in Chapter 5 of [1]. Actually, (a), (b), and (c) may be regarded as generalized measurability properties of functions acting from E into $\mathbf{R}$.

In the present paper we intend to discuss certain pathological subsets of $\mathbf{R}$ from the point of view of their generalized measurability and potential definability. As widely known, all projective sets in $\mathbf{R}$ (in the classical sense of Luzin and Sierpiński)

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can be treated as one of the possible realizations of the concept of definability, and in our further considerations we will be focused on projective sets with bad measurability properties. The following preliminary examples are relevant in this context. They serve to somehow illustrate the introduced notions (a), (b), and (c).

Example 1: Take $E = \mathbf{R}$ and consider the class $\mathcal{M}_1$ of the completions of all σ-finite Borel measures on $\mathbf{R}$. As well known, it is consistent with **ZFC** set theory that every projective subset of $\mathbf{R}$ is absolutely measurable with respect to $\mathcal{M}_1$. In particular, the Axiom of Projective Determinacy implies the absolute measurability of all projective sets in $\mathbf{R}$ with respect to the same class $\mathcal{M}_1$. Moreover, all analytic and co-analytic sets in $\mathbf{R}$ are absolutely measurable with respect to $\mathcal{M}_1$, without assuming any additional set-theoretical assertions.

Example 2: Take again $E = \mathbf{R}$ and let $\mathcal{M}_2$ be the class of the completions of all nonzero σ-finite diffused (i.e., vanishing at the singletons) Borel measures on $\mathbf{R}$. Consider any Bernstein type subset B of $\mathbf{R}$. This B and its complement $\mathbf{R} \setminus B$ meet each nonempty perfect set in $\mathbf{R}$ (see, for instance, [4], [7], [8]). It is not difficult to show that B is absolutely nonmeasurable with respect to $\mathcal{M}_2$. Conversely, any subset of $\mathbf{R}$ absolutely nonmeasurable with respect to $\mathcal{M}_2$ is a Bernstein set in $\mathbf{R}$ (see [1]). It should be noticed that in Gödel's Constructible Universe $\mathbf{L}$ there are certain Bernstein sets belonging to the projective class $\Delta^1_2(\mathbf{R}) = \Sigma^1_2(\mathbf{R}) \cap \Pi^1_2(\mathbf{R})$, so we have in $\mathbf{L}$ projective sets absolutely nonmeasurable with respect to $\mathcal{M}_2$.

Example 3: Once again put $E = \mathbf{R}$ and consider the class $\mathcal{M}_3$ of measures on $\mathbf{R}$ which extend the standard Lebesgue measure λ on $\mathbf{R}$ and are invariant under the group of all translations of $\mathbf{R}$. It is well known that there exist subsets of $\mathbf{R}$ which are absolutely nonmeasurable with respect to $\mathcal{M}_3$. One of such subsets is a classical Vitali set (recall that Vitali sets are usually defined as selectors of the quotient set $\mathbf{R}/\mathbf{Q}$, where $\mathbf{Q}$ stands for the field of all rational numbers). At the same time, there exists a Vitali subset of $\mathbf{R}$ relatively measurable with respect to the class of those measures on $\mathbf{R}$ which extend λ and are quasi-invariant under the group of all translations of $\mathbf{R}$ (see [1]). In Gödel's Universe $\mathbf{L}$ the real line $\mathbf{R}$ is endowed with a well-ordering $\preceq$ whose graph belongs to the projective class $\Delta^1_2(\mathbf{R}^2)$. Defining in $\mathbf{L}$ the set $V \subset \mathbf{R}$ by the formula

$$x \in V \Leftrightarrow (\forall y \in \mathbf{R})(x - y \in \mathbf{Q} \Rightarrow x \preceq y),$$

it is not difficult to verify that V is a Vitali set belonging to the projective class $\Delta^1_2(\mathbf{R})$. Consequently, we have in $\mathbf{L}$ projective sets absolutely nonmeasurable with respect to $\mathcal{M}_3$.

Example 4: Let E be a ground set, let μ be a σ-finite measure defined on some σ-algebra of subsets of E, and let $\mathcal{M}_\mu(E)$ denote the class of all those measures on E which extend μ. It is known that any set $X \subset E$ is relatively measurable with respect to $\mathcal{M}_\mu(E)$. In other words, there are no subsets of E absolutely nonmeasurable with respect to the same class.

By definition, a Hamel basis of $\mathbf{R}$ is any of its bases, when this $\mathbf{R}$ is treated as a vector space over $\mathbf{Q}$. The following statement is valid.

Theorem 1: *If there exists a projective Hamel basis of* $\mathbf{R}$*, then there exists a projective Vitali set in* $\mathbf{R}$ *which is a vector space over* $\mathbf{Q}$*.*

Proof: Let $H = \{e_i : i \in I\}$ be a projective Hamel basis of $\mathbf{R}$. For $1 \in \mathbf{Q}$, we have its unique representation in the form

$$1 = q_1 e_{i_1} + q_2 e_{i_2} + ... + q_n e_{i_n},$$

where $i_1, i_2, ..., i_n$ are some pairwise distinct indices from I and $q_1, q_2, ..., q_n$ are some nonzero rational numbers. Denote

$$H' = H \setminus \{e_{i_1}, e_{i_2}, ..., e_{i_n}\}$$

and consider the finite-dimensional vector space

$$U = \mathrm{lin}_{\mathbf{Q}}(\{e_{i_1}, e_{i_2}, ..., e_{i_n}\}).$$

Using a well-known algorithm, we can replace the basis $\{e_{i_1}, e_{i_2}, ..., e_{i_n}\}$ of U by a basis $\{e'_1, e'_2, ..., e'_n\}$, where $e'_1 = 1$. Further, we put

$$U' = \mathrm{lin}_{\mathbf{Q}}(\{e'_2, e'_3, ..., e'_n\} \cup H').$$

Obviously, we have a representation of $\mathbf{R}$ in the form of a direct sum

$$\mathbf{R} = \mathbf{Q} + U' \qquad (\mathbf{Q} \cap U' = \{0\}),$$

whence it follows that U' is a Vitali set being simultaneously a vector space over $\mathbf{Q}$.

It remains to show that U' is also a projective subset of $\mathbf{R}$. For this purpose, first observe that the set

$$H'' = \{e'_2, e'_3, ..., e'_n\} \cup H'$$

is trivially projective (because H and H' are projective). Further, we may write

$$U' = \cup\{U'_k : k < \omega\},$$

where, for every natural number k, the set U'_k consists of all those elements from U', whose representation via H'' contains at most k nonzero rational coefficients. Clearly, we have a continuous mapping

$$\phi : (\mathbf{Q}H'')^k \to \mathbf{R}$$

defined by the formula

$$\phi(q_1 h_1, q_2 h_2, ..., q_k h_k) = q_1 h_1 + q_2 h_2 + ... + q_k h_k,$$

where

$$(q_1, q_2, ..., q_k) \in \mathbf{Q}^k, \qquad (h_1, h_2, ..., h_k) \in (H'')^k.$$

From this formula we easily infer that each U'_k $(k < \omega)$ is a projective set in $\mathbf{R}$ (belonging to a fixed projective class). Therefore, the entire set U' belongs to the same projective class. This completes the proof. $\square$

Remark 1: Actually, the above argument yields that if there exists a Hamel basis of $\mathbf{R}$ belonging to the projective class $\Pi^1_n(\mathbf{R})$, then there exists a Vitali set which is a vector space over $\mathbf{Q}$ and belongs to the projective class $\Sigma^1_{n+1}(\mathbf{R})$. The same argument shows (within **ZF** theory) that if there exists a Hamel basis of $\mathbf{R}$, then there exists a Vitali set in $\mathbf{R}$ which is a vector space over $\mathbf{Q}$.

Remark 2: According to one old theorem of Sierpiński, no Hamel basis in $\mathbf{R}$ can be an analytic subset of $\mathbf{R}$. On the other hand, by virtue of Miller's result (see [5], [6]), in Gödel's Universe $\mathbf{L}$ there exists a co-analytic Hamel basis. Keeping in mind Theorem 1, we once again come to the statement that in $\mathbf{L}$ there exists a Vitali set which belongs to the projective class $\Delta^1_2(\mathbf{R})$ and is a vector space over $\mathbf{Q}$.

Let $\mathcal{M}(\mathbf{R})$ denote the class of all nonzero σ-finite diffused measures on $\mathbf{R}$ (notice that these measures are defined on various σ-algebras of subsets of $\mathbf{R}$). In [2] the validity of the following statement has been established.

Theorem 2: *Suppose that there exists a well-ordering of $\mathbf{R}$ which is isomorphic to the least uncountable ordinal ω_1 and, simultaneously, is a projective subset of the Euclidean plane $\mathbf{R}^2$. Then there exists a function*

$$f : \mathbf{R} \to \mathbf{R}$$

which is absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$ and whose graph is a projective subset of the plane $\mathbf{R}^2$.

Consequently, there exists a countable family $\{P_i : i \in I\}$ of projective subsets of $\mathbf{R}$ such that, for any measure $\mu \in \mathcal{M}(\mathbf{R})$, infinitely many members of $\{P_i : i \in I\}$ are nonmeasurable with respect to μ.

Moreover, if the above-mentioned well-ordering of $\mathbf{R}$ belongs to the projective class $\Sigma^1_n(\mathbf{R}^2)$, then all members of $\{P_i : i \in I\}$ can be taken from the class $\Sigma^1_{m(n)}(\mathbf{R})$, where the natural number $m(n)$ is completely determined by n.

Notice that a certain analog of Theorem 2 can be proved under Martin's Axiom instead of the Continuum Hypothesis.

By virtue of Example 4 we see that, for every measure $\mu \in \mathcal{M}(\mathbf{R})$ and for any finite subfamily $\{P_j : j \in J\}$ of $\{P_i : i \in I\}$, there exists a measure $\mu' \in \mathcal{M}(\mathbf{R})$ extending μ and such that

$$\{P_j : j \in J\} \subset \mathrm{dom}(\mu'),$$

where $\mathrm{dom}(\mu')$ denotes, as usual, the domain of μ'.

Let $m > 0$ be a natural number and let $\mathcal{L}(\mathbf{R}^m)$ stand for the class of those nonzero σ-finite measures on $\mathbf{R}^m$ which are invariant (quasi-invariant) under the group of all translations of $\mathbf{R}^m$.

We shall say that a set $X \subset \mathbf{R}^m$ is absolutely negligible with respect to $\mathcal{L}(\mathbf{R}^m)$ if, for every measure $\mu \in \mathcal{L}(\mathbf{R}^m)$, there exists a measure $\mu' \in \mathcal{L}(\mathbf{R}^m)$ extending μ and satisfying the relation $\mu'(X) = 0$.

Lemma 1: *For a set $X \subset \mathbf{R}^m$, the following two assertions are equivalent:*

(1) X is absolutely negligible with respect to $\mathcal{L}(\mathbf{R}^m)$;

(2) for any countable family $\{g_i : i \in I\} \subset \mathbf{R}^m$, there exists a countable family $\{h_k : k \in K\} \subset \mathbf{R}^m$ such that

$$\cap\{(h_k + \cup\{g_i(X) : i \in I\}) : k \in K\} = \emptyset.$$

For a proof of Lemma 1, see Chapter 5 in [1].

From Lemma 1 it is not difficult to deduce the following statement.

Lemma 2: *Let H be a Hamel basis of $\mathbf{R}^m$, let k be a natural number, and let H_k denote the set of all those elements of $\mathbf{R}^m$, whose representation via H contains at most k nonzero rational coefficients. Then H_k is absolutely negligible with respect to the class $\mathcal{L}(\mathbf{R}^m)$.*

The next statement may be regarded as an analogue of Theorem 2 for translation invariant (quasi-invariant) measures on $\mathbf{R}$.

Theorem 3: *Suppose that there exists a Hamel basis H of $\mathbf{R}^m$ which is a projective subset of $\mathbf{R}^m$. Then there exists a countable family $\{X_i : i \in I\}$ such that:*

(1) $\cup\{X_i : i \in I\} = \mathbf{R}^m$;

(2) all X_i $(i \in I)$ are projective subsets of $\mathbf{R}^m$;

(3) all X_i $(i \in I)$ are absolutely negligible with respect to the class $\mathcal{L}(\mathbf{R}^m)$.

Consequently, for an arbitrary measure $\mu \in \mathcal{L}(\mathbf{R}^m)$, infinitely many members of $\{X_i : i \in I\}$ are nonmeasurable with respect to μ, and for any finite subfamily $\{X_j : j \in J\}$ of $\{X_i : i \in I\}$, there exists a measure $\mu' \in \mathcal{L}(\mathbf{R}^m)$ extending μ and satisfying the relation

$$\{X_j : j \in J\} \subset \mathrm{dom}(\mu'), \qquad (\forall j \in J)(\mu'(X_j) = 0).$$

Proof: For a natural number k, let H_k denote again the set of all those elements of $\mathbf{R}^m$, whose representation via H contains at most k nonzero rational coefficients. According to Lemma 2, all H_k $(k < \omega)$ are absolutely negligible with respect to the class $\mathcal{L}(\mathbf{R}^m)$. Further, using the same argument as in the proof of Theorem 1, we infer that all H_k are projective subsets of $\mathbf{R}^m$. It remains to observe that the equality

$$\mathbf{R}^m = \cup\{H_k : k < \omega\}$$

holds true. Theorem 3 has thus been proved. □

Remark 3: It is easy to see from the above argument that if a Hamel basis H belongs to the projective class $\Pi^1_n(\mathbf{R}^m)$, then all sets H_k $(n < \omega)$ belong to the projective class $\Sigma^1_{n+1}(\mathbf{R}^m)$.

Remark 4: Recall that a Mazurkiewicz set is any subset of the plane $\mathbf{R}^2$ which meets each straight line contained in $\mathbf{R}^2$ in exactly two points. It can be demonstrated that there exists a measure $\nu \in \mathcal{L}(\mathbf{R}^2)$ extending the standard Lebesgue measure λ_2 on $\mathbf{R}^2$ and such that all Mazurkiewicz sets are of ν-measure zero. In particular, all Mazurkiewicz sets are relatively measurable with respect to the class $\mathcal{L}(\mathbf{R}^2)$. Moreover, every Mazurkiewicz set $Z \subset \mathbf{R}^2$ is small (negligible) with respect to $\mathcal{L}(\mathbf{R}^2)$ in the following sense: for any measure $\mu \in \mathcal{L}(\mathbf{R}^2)$, the relation

$Z \in \mathrm{dom}(\mu)$ implies $\mu(Z) = 0$. In this context, the next two facts are of some interest:

(a) there exists a Mazurkiewicz set which is not absolutely negligible with respect to $\mathcal{L}(\mathbf{R}^2)$;

(b) there exists a Mazurkiewicz set which is a Hamel basis of $\mathbf{R}^2$ and, consequently, is absolutely negligible with respect to $\mathcal{L}(\mathbf{R}^2)$.

For more details about these facts, see [3].

Notice also that in Gödel's Universe $\mathbf{L}$ there are co-analytic Mazurkiewicz sets (see [5], [6]).

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