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On Critical Parameters and Stability of Two-Vortex Interaction in a Continuum

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For the 2-vortex interaction the generalized critical parameter determining qualitative character of interaction of vortices is introduced. For given initial conditions its value divides modes of active interaction ("phase intermixing") and quasi-stationary evolution (regime of stability). The obtained results can have numerous applications on studying the vortex motions in a plasma and fluids.

Keywords: Vortices, Interaction, Stability, Critical parameters, Plasma, Fluids, Theory, Computer simulation

AMS Subject Classification: 0340G, 0270, 4730.

1. Introduction

One of main problems in the investigation of the dynamics of the vortex structures is study of a character of their interaction and, in particular, search of states when a system of the vortices conserves its stability. In this connection, it is interesting from our point of view the question on a possibility of prediction of the result of the evolution of vortices outgoing from the parameters which describe the initial state of the system. In this paper we present the results of numerical simulations and theoretical analysis on study of evolution of the system of the finite area vortex regions (FAVRs) [1]. On the basis of processing of these results we introduce a special generalized parameter which enables us to predict the qualitative character of the interaction of the FAVRs on the reference initial parameters of the vortex system.

2. Basic equations and method of solution

Let us consider the FAVRs which are special class of the solutions of two-dimensional set describing the evolution of such vortices [2]:

$$\begin{cases} \partial_t \zeta + (\mathbf{u}\nabla)\zeta = 0, \\ \triangle \psi = -\zeta, \\ \mathbf{u} = [\nabla, \psi \mathbf{e}_z], \end{cases} \tag{1}$$

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where ξ is the vorticity, ψ is the current function, and $\mathbf{u}$ is the velocity. The regions of positive vorticity (or circulation) correspond to rotation of the convective fluid elements counter-clockwise, therefore let us study the right-handed coordinate system, where the vector of a vorticity is directed along the axis $\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y$.

For the numerical simulation of set (1) we use the contour dynamics (CD) method [3], to some extent modified[1] [4], which enables at considerable saving of computer time to investigate the evolution of both the single FAVRs and the systems of the vortex structures of various configuration consisting of the FAVRs, distinguishing in the parameters. Solving the Poisson equation (1) and integrating the expression for current function $\psi(x, y)$ (see [4]) we obtain the integral for the velocity $\mathbf{u}(x, y)$. Further, solving differential equations

$$\dot{\mathbf{x}} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y = \zeta_0 \int_\Gamma [\ln r][\mathbf{e}_x d\xi + \mathbf{e}_y d\eta], \tag{2}$$

where $r = [(x-\xi)^2 + (y-\eta)^2]^{1/2}$ and $\ln r$ is the Green function of Poisson's equation, we obtain the evolution of the contour coordinates with time. Further transformations associated with the spatial and temporal discretization of equation for current function and Eq. (2) are described in [3] in detail. Whereas the set (1) is conservative, for numerical solution of Eq. (2) we used the centered on time "overstep" scheme approximating initial hyperbolic equation with the accuracy $O(\tau^2)$ [6]. Thus, setting the initial form, size, the vorticity of FAVRs, configuration of a positional relationship of vortices, it is possible to observe the time evolution of such a system [2].

3. Results of numerical simulation

Basing on the results of [2], let us consider in the beginning qualitative differences in a character of interaction of two FAVRs. Our results (see also [1, 6–8]) show that at interaction of a pair of circle vortices some cases can take place:

1. At rather large distance between the centers the vortex regions rotate around of common center, thus there is a deformation of the vortices – they are drawn out, taking the form close to elliptical, but in due course return to an original state [8]. Thus, the interaction is reduced to a cyclic change of the shape of vortices and is called a "quasi-recurrence" phenomenon [1, 2, 4].

2. With decreasing of a distance between the centers the vortices start ever more to be deformed during interaction, that results in formation of the cusps [4, 7]. At further evolution it causes appearance of the filaments of vorticity [7] (see. Fig. 1) and, as a result, the vortices disintegrate.

3. With further decreasing of a distance between the centers of the vortices there is their "phase intermixing" (see [2]), and different configurations are possible too from small coupling of the FAVRs down to full junction of two vortices [9].

In further we shall suppose, that the qualitative change (some kind of a "jump") in a character of interaction of two vortex regions happens with loss of stability of the system and transition in a "phase intermixing" state.

[1] The problem of stability of the CD algorithm was investigated in detail in [4, 5].

The problem is to find some generalized critical parameter describing the interaction of the FAVRs in terms of such a jump, whose value would allow us to predict the qualitative character of the result of vortex interaction.

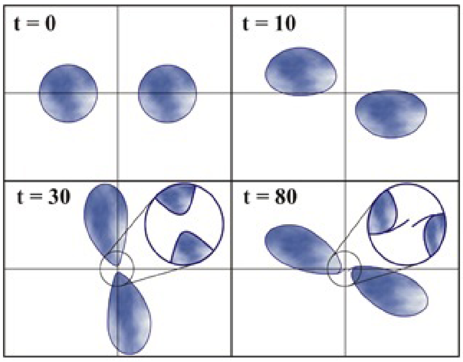

Figure 1. Formation of filaments of vorticity

As such parameter we offer to use the following function of the basic characteristics of interacting vortex structures corresponding their state at $t = 0$:

$$\xi = \frac{S}{l^2}\frac{\zeta_1}{\zeta_2}(1 - e_0)^{-1}(1 + \sin^2\theta), \tag{3}$$

where S is the area of each interacting FAVRs[1], l is the distance between their centers, ζ_1 and ζ_2 are the values of the vorticities (and $\zeta_1 \geq \zeta_2$), $e_0 = (e_1 + e_2)/2$ is the eccentricity averaged on two FAVRs, and $\theta = \theta_1 + \theta_2$ is the sum of angles of inclination of large axes of the FAVRs ellipses concerning a line, connecting their centers [2] (see Fig. 2).

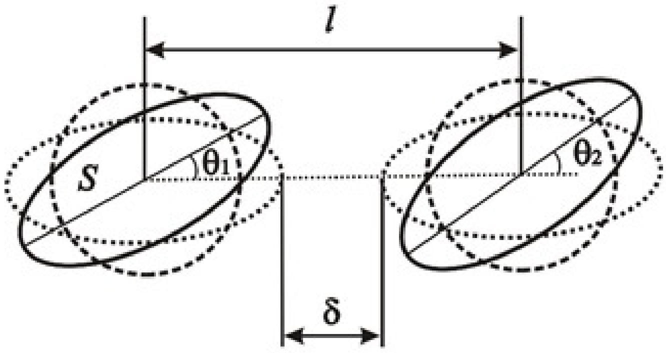

Figure 2. Illustration to definition of the initial state parameters of vortex system

Let us introduce the following denotations for critical parameters corresponding an initial state of a vortex system and determining the transition to the "phase

[1]Suppose, for a determinacy, that the areas of interacting FAVRs $S_1 = S_2 = S$.

intermixing" state with change of the sizes and positional relationship of FAVRs, ratio of their vorticities, eccentricity and angle θ, respectively:

$$\alpha = S/l^2,$$
$$\beta = \zeta_1/\zeta_2,$$
$$\gamma = (1 - e_0)^{-1},$$
$$\theta_0 = 1 + \sin^2\theta$$

and write function ξ in the form

$$\xi = \alpha\beta\gamma\theta_0. \tag{4}$$

To justify the expediency of offered criterion we fulfilled some series of numerical experiments in which the critical values of parameters α, β, γ and θ_0 for vortex regions of the circle and elliptical form, as most often as model meeting in the applications, were calculated.

With the purpose of finding the critical value of the parameter α, the system consisting of two circle FAVRs with equal values of vorticities and radiuses was considered: at a fixed distance between the centers of two FAVRs we increased their radiuses (and, accordingly, the areas) until there was an interaction. Thus the parameters corresponding to the critical state of vortex pair were fixed. The quantities which uniquely determinate initial configuration of the system of two circle vortices are shown in Fig. 2.

In our numerical simulations for the cases corresponding the initial states between the centers of FAVRs $l = 1, 2, \ldots, 5$ we have found that the beginning of interaction in all cases responds the approximately same value of parameter α. The values of parameters, at which there is a qualitative change in the character of interaction – the transition from steadily rotated pair to the "phase intermixing" state, are shown in Table 1. So, the results of numerical simulations enable us to conclude that the critical value of parameter α, at which there is qualitative change in the interaction of the vortices, equals $\alpha_{cr} = 0.267$. For $\alpha < \alpha_{cr}$ the merging of the vortex regions does not happen during interaction, but as soon as parameter α reaches its critical value, there is a qualitative jump in behaviour of the vortex system, and the vortices start to be intermixed.

S	l	δ/l	α
0.267864	1	0.416	0.267864
1.067791	2	0.417	0.266947
2.399785	3	0.417	0.266642
4.271168	4	0.417	0.266947
6.669121	5	0.417	0.266764

Table 1.

The next series of the numerical simulations purposed a calculation of the critical value of the parameter β. Our results showed that the vortices with the greater value of β are exposed to the greater deformation, their filamentation (i.e. formation of the filaments of a vorticity) happens faster, thus the change in character of the interaction happens at the ratio of vorticities $\zeta_1/\zeta_2 > 1.11$ (remind, that $\zeta_1 \geq \zeta_2$), therefore, $\beta_{cr} = 1.11$.

To answer a problem on the critical value of the parameter γ a series of simulations for FAVRs of the elliptical form was conducted (Fig. 2). We fixed the area S, at which the circle vortices still save a stable state, and at $S = \text{const}$ changed the eccentricities e_1 and e_2. Further, we found the critical value e_0, at which the vortex system loses its stability transferring to the "phase intermixing" state. Thus, we considered the cases corresponding the initial states between the centers of FAVRs $l = 1, 2, \ldots, 5$. The values of critical parameters, at which there is a qualitative change in the behaviour of the system of two elliptical FAVRs, are presented in Table 2.

l	δ/l	α	e_0
1	0.180	0.266033	0.863847
2	0.180	0.266033	0.866426
3	0.183	0.266033	0.863834
4	0.180	0.266490	0.863341
5	0.180	0.266764	0.863037

Table 2.

Numerical simulations have shown that there is the same for all cases a critical value of the averaged eccentricity, at which the "phase change" happens – $e_0 \approx 0,864$, that corresponds to $\gamma = 7.143$. Thus, as one can see from Table 2 the ratio δ/l is also a constant in a critical region, however it cannot be used as the critical parameter for the description of interaction, because, at first, it takes different values for elliptical and circle FAVRs (see Table 1), secondly, it is less information as determines only a distance between boundaries of the vortices, nothing speaking about their form. Therefore, for definition of the function ξ we use parameter γ, expressed through the averaged eccentricity.

The further investigations were connected to finding of the critical angle of inclination (see Fig. 2) of the elliptical FAVRs for the initial state of a system, at which the evolution results in qualitative change in character of their interaction. The simulations fulfilled show that increase of the angle of declination of the vortex regions θ at $t = 0$ more than on $4°$ leads to the transition to the unstable state. Thus, we mean as angle of inclination the summing angle $\theta = \theta_1 + \theta_2$, and, for example, the case when $\theta_1 = \theta_2 = 2°$ is analogous to the case $\theta_1 = 4°$, $\theta_2 = 0$. As it follows from processing of the results of this series of simulations, the critical value of corresponding parameter is $\theta_0 = 1.005$.

Summing all presented above results we can define a critical value of the generalized parameter ξ as a multiplication of four parameters α_{cr}, β_{cr}, γ_{cr} and θ_{0cr}:

$$\xi_{cr} = l_{cr}\beta_{cr}\gamma_{cr}\theta_{0cr} = 2.129.$$

The numerical simulations for $|\xi| \geq \xi_{cr}$ with simultaneous variation of critical parameters α, β, γ and θ_0 corresponding to change of the sizes and positional relationship of FAVRs, the ratio of their vorticities, eccentricity and the summing angle of inclination of their major axes θ, respectively, have confirmed a capability and expediency of usage of the parameter ξ for prediction of character of interaction of the vortex structures.

4. Conclusion

The obtained results, obviously, allow to make a conclusion that the character of interaction of the vortex structures is determined by an initial state of the system. Thus, for a pair of the FAVRs we managed to find the function ξ having the sense of critical parameter which uniquely determinates a qualitative character of their interaction. Comparing the value of ξ with its critical value ξ_{cr} we can predict the result of interaction of the vortex regions, namely: if $\xi < \xi_{cr}$ then "phase intermixing" of FAVRs will not be observed with evolution, in the opposite case, when $\xi \geq \xi_{cr}$, the merging of vortices with further formation of the vorticities of more small scales will happen. For the vortices of the circle and elliptical (or close to elliptical) form the value of generalized critical parameter $\xi_{cr} = 2.129$ corresponds to the "phase change" point.

Note, that the obtained results concern only systems which consist of two vortices. The generalization on a case of arbitrary number of vortex regions requires padding investigations.

The obtained results can be useful on studying the stability of vortex structures of different types and origins, including at the solution of the model problems on dynamics of the quasi-geostrophic vortices in atmosphere and ocean [10].

Add also, that the CD method can be effectively used on studying the dynamics of the vortex structures in a plasma [11–14], in particular, in the problems connecting with the vortex movements in the dusty plasma when the dust particles are involved in motion by a vortex region [2, 15]. Thus, the results obtained in the present paper can be useful on studying the mechanisms of process of interaction of the vortex formations with the dust particles.

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