Bulletin of TICMI
Vol. 23, No. 1, 2019, 19–24

Negafibonacci Numbers via Matrices

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(*Received November 01, 2018; Revised June 08, 2019; Accepted June 24, 2019*)

In this paper, negafibonacci numbers are generated by means of matrix methods. A 2×2 matrix is used to obtain some properties of negafibonacci numbers; on the other hand, families of tridiagonal matrices are introduced to generate negafibonacci numbers through determinants.

Keywords: Fibonacci numbers, Matrices, Determinants.

AMS Subject Classification: 11B39, 40C05, 11C20.

1. Introduction

The Fibonacci sequence $\{f_n\}$ is defined by the following recurrence relation

$$f_{n+1} = f_n + f_{n-1}, \text{ for } n \geq 1,$$

with $f_0 = 0$, $f_1 = 1$. The Fibonacci numbers have been widely studied, and the different ways to generate those numbers have gained continued interest, among them matrix methods [10], determinants [5], permanents [6], Pascal's triangle [9], binomial coefficients [3], and many others [8].

An interesting connection between Fibonacci numbers and matrices, introduced in [4], is given by the matrix $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, known as Fibonacci Q-matrix [7] or Fibonacci's matrix [11], such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}.$$

In [1] two tridiagonal Toeplitz matrices were presented

$$H_n = \begin{bmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{bmatrix}_{n\times n}, \quad D_n = \begin{bmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & 1 & 1 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & 1 \end{bmatrix}_{n\times n}$$

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ISSN: 1512-0082 print

such that $\det(H_n) = \det(D_n) = f_{n+1}$.

By the relation $f_{-n} = (-1)^{n+1} f_n$, where n is any positive integer, Fibonacci numbers can be extended to negative index [2], terms in this sequence are called negafibonacci numbers. Since $f_{n+1} = f_n + f_{n-1}$, it is easy to check that $f_{-(n+1)} = -f_{-n} + f_{-(n-1)}$; some negafibonacci numbers are $f_{-1} = 1$, $f_{-2} = -1$, $f_{-3} = 2$, $f_{-4} = -3$, $f_{-5} = 5$. In this paper negafibonacci numbers are generated by means of matrices, and some identities are proved by matrix methods.

2. Negafibonacci identities by matrix methods

Motivated by the Fibonacci Q-matrix, the matrix $N = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is presented, the following proposition shows a connection between N and negafibonacci numbers.

Proposition 2.1: $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{-(n+1)} & f_{-n} \\ f_{-n} & f_{-(n-1)} \end{bmatrix}$.

Proof: Since $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{-2} & f_{-1} \\ f_{-1} & f_0 \end{bmatrix}$, the proposition is true for $n = 1$. Assuming that $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{-(n+1)} & f_{-n} \\ f_{-n} & f_{-(n-1)} \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{n+1}$ is calculated as follows:

$$
\begin{aligned}
\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{n+1} &= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^n \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{-(n+1)} & f_{-n} \\ f_{-n} & f_{-(n-1)} \end{bmatrix} \\
&= \begin{bmatrix} -f_{-(n+1)} + f_{-n} & -f_{-n} + f_{-(n-1)} \\ f_{-(n+1)} & f_{-n} \end{bmatrix} \\
&= \begin{bmatrix} f_{-(n+2)} & f_{-(n+1)} \\ f_{-(n+1)} & f_{-n} \end{bmatrix}.
\end{aligned}
$$

□

The above proposition is useful to prove some identities about negafibonacci numbers.

Proposition 2.2: *For all $n, k \geq 0$:*

$$
\begin{aligned}
f_{-(n+k+1)} &= f_{-(n+1)} f_{-(k+1)} + f_{-n} f_{-k} \qquad & f_{-(n+k)} &= f_{-(n+1)} f_{-k} + f_{-n} f_{-(k-1)} \\
f_{-(n+k)} &= f_{-n} f_{-(k+1)} + f_{-(n-1)} f_{-k} \qquad & f_{-(n+k-1)} &= f_{-n} f_{-k} + f_{-(n-1)} f_{-(k-1)}.
\end{aligned}
$$

Proof: Since $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{-(n+1)} & f_{-n} \\ f_{-n} & f_{-(n-1)} \end{bmatrix}$, by Proposition 2.1, then

$$
\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{n+k} = \begin{bmatrix} f_{-(n+k+1)} & f_{-(n+k)} \\ f_{-(n+k)} & f_{-(n+k-1)} \end{bmatrix}.
$$

On the other hand,

$$\begin{aligned}
\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{n+k} &= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{n} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{k} \\
&= \begin{bmatrix} f_{-(n+1)} & f_{-n} \\ f_{-n} & f_{-(n-1)} \end{bmatrix} \begin{bmatrix} f_{-(k+1)} & f_{-k} \\ f_{-k} & f_{-(k-1)} \end{bmatrix} \\
&= \begin{bmatrix} f_{-(n+1)}f_{-(k+1)} + f_{-n}f_{-k} & f_{-(n+1)}f_{-k} + f_{-n}f_{-(k-1)} \\ f_{-n}f_{-(k+1)} + f_{-(n-1)}f_{-k} & f_{-n}f_{-k} + f_{-(n-1)}f_{-(k-1)} \end{bmatrix}.
\end{aligned}$$

Thus obtaining the desired equalities. □

From Proposition 2.2, we may immediately deduce the following identities.

Corollary 2.3: *For all $n \geq 0$:*

(1) $f_{-2n} = f_{-n}f_{-(n+1)} + f_{-(n-1)}f_{-n}$.
(2) $f_{-(n+2)} = 2f_{-n} - f_{-(n-1)}$.

The following result can be called Cassini's formula for negafibonacci numbers, the reader is referred to [12] for more Cassini-like formulas.

Proposition 2.4: $f_{-(n+1)}f_{-(n-1)} - f_{-n}^2 = (-1)^n$

Proof: Let $N = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$, then $\det(N) = -1$. Since $N^n = \begin{bmatrix} f_{-(n+1)} & f_{-n} \\ f_{-n} & f_{-(n-1)} \end{bmatrix}$, then $\det(N^n) = f_{-(n+1)}f_{-(n-1)} - f_{-n}^2$. On the other hand, $\det(N^n) = (\det(N))^n = (-1)^n$; therefore $f_{-(n+1)}f_{-(n-1)} - f_{-n}^2 = (-1)^n$. □

3. Negafibonacci numbers as tridiagonal matrix determinants

In this section, we present the matrices G_n and K_n defined as follows:

$$G_n = \begin{bmatrix} -1 & -1 & & & \\ 1 & -1 & 1 & & \\ & -1 & -1 & \ddots & \\ & & \ddots & \ddots & (-1)^{n-1} \\ & & & (-1)^n & -1 \end{bmatrix}_{n\times n} \qquad K_n = \begin{bmatrix} -1 & -i & & & \\ -i & -1 & i & & \\ & i & -1 & \ddots & \\ & & \ddots & \ddots & (-1)^{n-1}i \\ & & & (-1)^{n-1}i & -1 \end{bmatrix}_{n\times n}$$

The following proposition shows a connection between negafibonacci numbers and the determinants of a family of tridiagonal matrices.

Proposition 3.1: *For all $n > 0$:*

(1) $\det(G_n) = f_{-(n+1)}$.
(2) $\det(K_n) = f_{-(n+1)}$.

Proof: Here we prove (1); (2) can be similarly proved.

We argue by induction on n. Clearly $G_1 = -1 = f_{-2}$ and $\det(G_2) = 2 = f_{-3}$.

Let G_{n+2} be the matrix

$$G_{n+2} = \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{G_n}} & 0_{n-1\times 1} & 0_{n-1\times 1} \\ & & -1 & 0 \\ 0_{1\times n-1} & (-1)^{n+1} & -1 & (-1)^{n+1} \\ 0_{1\times n-1} & 0 & (-1)^{n+2} & -1 \end{bmatrix}.$$

Assuming that the determinant $\det(G_k) = f_{-(k+1)}$ for all $k \leq n$, we aim to show that $\det(G_{n+2}) = f_{-(n+3)}$. Assuming that n is odd

$$\begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{G_n}} & 0_{n-1\times 1} & 0_{n-1\times 1} \\ & & -1 & 0 \\ 0_{1\times n-1} & 1 & -1 & 1 \\ 0_{1\times n-1} & 0 & -1 & -1 \end{bmatrix}_{R_{n+1}+R_{n+2}} \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{G_n}} & 0_{n-1\times 1} & 0_{n-1\times 1} \\ & & -1 & 0 \\ 0_{1\times n-1} & 1 & -2 & 0 \\ 0_{1\times n+1} & 0 & -1 & -1 \end{bmatrix}.$$

Applying the column operation $C_{n+1} - C_{n+2}$, we obtain

$$\begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{G_n}} & 0_{n-1\times 1} & 0_{n-1\times 1} \\ & & -1 & 0 \\ 0_{1\times n-1} & 1 & -2 & 0 \\ 0_{1\times n-1} & 0 & 0 & -1 \end{bmatrix}.$$

Since the above row and column elementary operations do not change the value of the determinant [11], we have

$$\det(G_{n+2}) = [-1]\left[(-1)^{2[n+1]}[-2]\det(G_n) + (-1)^{2n+1}[1]\det(B_n)\right] \tag{3.1}$$

where $B_n = \begin{bmatrix} \multicolumn{2}{c}{G_{n-1}} & 0_{n-1\times 1} \\ 0_{1\times n-2} & -1 & -1 \end{bmatrix}$. Applying the column operation $C_{n-1} - C_n$, we obtain the equivalent matrix

$$\begin{bmatrix} G_{n-1} & 0_{n-1\times 1} \\ 0_{1\times n-1} & -1 \end{bmatrix}.$$

Therefore $\det(B_n) = [-1]\det(G_{n-1}) = -f_{-n}$. By replacing in equation (3.1) we obtain

$$\begin{aligned} \det(G_{n+2}) &= [-1]\left[[-2][f_{-(n+1)}] + (-1)^{2n+1}[1][-f_{-n}]\right] \\ &= 2f_{-(n+1)} - f_{-n}. \end{aligned}$$

Thus, by corollary 2.3, $\det(G_{n+2}) = f_{-(n+3)}$ for n odd. Similarly, if n is even G_{n+2} is given by

$$G_{n+2} = \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{G_n}} & 0_{n-1\times 1} & 0_{n-1\times 1} \\ & & 1 & 0 \\ 0_{1\times n-1} & -1 & -1 & -1 \\ 0_{1\times n-1} & 0 & 1 & -1 \end{bmatrix}.$$

Applying, first the row operation $R_{n+1} - R_{n+2}$, and after the column operation $C_{n+1} + C_{n+2}$, we obtain the equivalent matrix

$$\begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{G_n}} & 0_{n-1\times 1} & 0_{n-1\times 1} \\ & & 1 & 0 \\ 0_{1\times n-1} & -1 & -2 & 0 \\ 0_{1\times n-1} & 0 & 0 & -1 \end{bmatrix}.$$

Therefore,

$$\det(G_{n+2}) = [-1]\left((-1)^{2n+2}[-2]\det(G_n) + (-1)^{2n+1}[-1]\det(C_n)\right), \tag{3.2}$$

where C_n is given by

$$C_n = \begin{bmatrix} G_{n-1} & 0_{n-1\times 1} \\ 0_{1\times n-2}\ 1 & 1 \end{bmatrix}.$$

Hence, $\det(C_n) = \det(G_{n-1}) = f_{-n}$. By replacing in equation (3.2) we obtain

$$\begin{aligned} \det(G_{n+2}) &= [-1]\left([-2](f_{-(n+1)}) + (f_{-n})\right) \\ &= 2f_{-(n+1)} - f_{-n}. \end{aligned}$$

Thus, by Corollary 2.3, $\det(G_{n+2}) = f_{-(n+3)}$ for n even. □

Given a matrix that generates Fibonacci numbers we can obtain a matrix for negafibonacci numbers and vice-versa, as shows the following proposition.

Proposition 3.2: *F_n is a matrix such that $\det(F_n) = f_{n+1}$ if and only if $N_n = -F_n$ is such that $\det(N_n) = f_{-(n+1)}$.*

Proof: Assuming that $\det(F_n) = f_{n+1}$, for the matrix $N_n = -F_n$ we have $\det(N_n) = \det(-F_n) = (-1)^n \det(F_n) = (-1)^n f_{n+1} = (-1)^{n+2} f_{n+1} = f_{-(n+1)}$.

Assuming that $\det(N_n) = f_{-(n+1)}$, for the matrix $F_n = -N_n$ we have $\det(F_n) = \det(-N_n) = (-1)^n \det(N_n) = (-1)^n f_{-(n+1)} = (-1)^n((-1)^{n+2} f_{n+1}) = f_{n+1}$. □

By Proposition 3.2, we have the following result.

Corollary 3.3: *For all $n \geq 0$:*

(1) $\det(-G_n) = \det(-K_n) = f_{n+1}$.
(2) $\det(-D_n) = \det(-H_n) = f_{-(n+1)}$.

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