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Characterization of (n, m)–Jordan Homomorphisms

Abbas Zivari-Kazempour

Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran
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Let $n \in \mathbb{N}$, $m \in \mathbb{Z} \setminus \{0\}$. In this paper among other things, under special hypotheses, we prove that every (n, m)–Jordan homomorphism between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is a (n, m)–homomorphism.

Keywords: (n, m)-homomorphisms, (n, m)-Jordan homomorphisms, Weakly commutative.

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1. Introduction

Let $n \in \mathbb{N}$, $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then φ is called an n–homomorphism if for all $a_1, a_2, ...a_n \in \mathcal{A}$,

$$\varphi(a_1a_2...a_n) = \varphi(a_1)\varphi(a_2)...\varphi(a_n).$$

The concept of an n–homomorphism was studied for complex algebras in [7] and [3].

Herstein in [8] introduced the notion of n–Jordan homomorphisms. A linear map φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called an n–Jordan homomorphism if

$$\varphi(a^n) = \varphi(a)^n, \qquad a \in \mathcal{A}.$$

A 2-homomorphism (2-Jordan homomorphism) is called simply a homomorphism (Jordan homomorphism).

It is clear that every n-homomorphism is an n-Jordan homomorphism, but in general the converse is false. There are some examples of n-Jordan homomorphisms which are not n-homomorphisms. For $n = 2$, it is proved in [9] that some Jordan homomorphism on the polynomial rings can not be homomorphism.

Herstein in [8] proved the following theorem.

Theorem 1.1: *If φ is a Jordan homomorphism of a ring R onto a prime ring R' of characteristic deferent from 2 and 3, then either φ is a homomorphism or an anti-homomorphism.*

Email: zivari@abru.ac.ir, zivari6526@gmail.com

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It is shown in [4] that every n-Jordan homomorphism between two commutative Banach algebras is an n-homomorphism for $n \in \{2, 3, 4\}$, and this result extended to the case $n = 5$ in [5]. For the case that $n \in \mathbb{N}$ is an arbitrary, Lee in [10] and Gselmann in [6] generalized this result. This challenge is solved in [2] by the different methods which are used in [6] and [10]. For the non-commutative case, Zelazko in [12] presented the following result (see also [11]).

Theorem 1.2 : *Suppose that $\mathcal{A}$ is a Banach algebra, which need not be commutative, and suppose that $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism.*

Later, this result was proved in [14] for 3–Jordan homomorphism with the extra condition that the Banach algebra $\mathcal{A}$ is unital, and it is extended for all $n \in \mathbb{N}$ in [1]. Some significant results concerning Jordan homomorphisms and their automatic continuity on Banach algebras are obtained by the author in [13], [15] and [16].

Let $m \in \mathbb{Z} \setminus \{0\}$, let $\mathcal{A}$ and $\mathcal{B}$ be complex algebras and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then φ is called an (n, m)–homomorphism if for all $a_1, a_2, ...a_n \in \mathcal{A}$,

$$\varphi(a_1a_2...a_n) = m\varphi(a_1)\varphi(a_2)...\varphi(a_n),$$

and it is called an (n, m)–Jordan homomorphism if

$$\varphi(a^n) = m\varphi(a)^n, \qquad a \in \mathcal{A}.$$

Clearly $(n, 1)$–homomorphism and $(n, 1)$-Jordan homomorphism coincide with the classical definitions of n–homomorphism and n–Jordan homomorphism, respectively.

Note that every n–Jordan homomorphism is not necessary (n, m)–Jordan homomorphism for $m \neq 1$, for example, consider the identity map. Also every (n, m)-Jordan homomorphism is not necessary n-Jordan homomorphism for $m \neq 1$. For example, define $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ by $\varphi(x) = \frac{1}{2}x$. Then φ is not n-Jordan homomorphism, but for $m = 2^{(n-1)}$ it is (n, m)–Jordan homomorphism.

Example 1.3 Let

$$\mathcal{A} = \{ \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} : \quad X, Y \in M_2(\mathbb{C})\},$$

and define $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$\varphi(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}) = \frac{1}{k} \begin{bmatrix} X & 0 \\ 0 & Y^T \end{bmatrix},$$

for each $k \in \mathbb{N}$. Then for all $U \in \mathcal{A}$, we have

$$\varphi(U^n) - k^{(n-1)}\varphi(U)^n = \frac{1}{k} \begin{bmatrix} X^n & 0 \\ 0 & (Y^n)^T \end{bmatrix} - k^{(n-1)} \frac{1}{k^n} \begin{bmatrix} X^n & 0 \\ 0 & (Y^T)^n \end{bmatrix} = 0.$$

Thus, φ is (n,m)–Jordan homomorphism for $m = k^{(n-1)}$, but φ is not (n,m)–homomorphism.

In this paper, we prove that every $(3,m)$–Jordan homomorphism φ from unital Banach algebra $\mathcal{A}$ into Banach algebra $\mathcal{B}$ is $(3,m)$-homomorphism if either:

(1) $\mathcal{B}$ is semisimple and commutative, or
(2) $\mathcal{A}$ and $\mathcal{B}$ are weakly commutative.

2. Main Results

For $m = 1$, the following result is Theorem 1.2, and for $m = -1$ it is Lemma 2.1 of [14].

Theorem 2.1: *Every $(2,m)$-Jordan homomorphism φ from Banach algebra $\mathcal{A}$ into $\mathbb{C}$ is a $(2,m)$–homomorphism.*

Proof: Suppose that φ is $(2,m)$–Jordan homomorphism. Then $\varphi(a^2) = m\varphi(a)^2$, for all $a \in \mathcal{A}$. Replacing a by $a+b$, we get

$$\varphi(ab+ba) = 2m\varphi(a)\varphi(b), \qquad (a,b \in \mathcal{A}). \tag{1}$$

Replacing a by a^2 in (1), we have

$$\varphi(a^2b+ba^2) = 2m^2\varphi(a)^2\varphi(b), \qquad (a,b \in \mathcal{A}). \tag{2}$$

Taking $b = ab+ba$ in (1), we see that

$$\varphi(a(ab+ba)+(ab+ba)a) = 2m\varphi(a)\varphi(ab+ba),$$

and hence by (1),

$$\varphi(a^2b+2aba+ba^2) = 4m^2\varphi(a)^2\varphi(b). \tag{3}$$

Subtraction (2) from (3), gives

$$\varphi(aba) = m^2\varphi(a)^2\varphi(b). \tag{4}$$

Fix $a \in \mathcal{A}$ and $b \in \mathcal{A}$ arbitrarily, and put

$$2t = \varphi(ab-ba). \tag{5}$$

It follows from (1) and (5) that

$$\varphi(ab)-t = m\varphi(a)\varphi(b), \qquad \varphi(ba)+t = m\varphi(a)\varphi(b). \tag{6}$$

By (4), (5) and (6),

$$\begin{aligned}4t^2 &= \varphi(ab-ba)^2 = \frac{1}{m}\varphi[(ab-ba)^2]\\ &= \frac{1}{m}\varphi[(ab)^2+(ba)^2-ab^2a-ba^2b]\\ &= [\varphi(ab)^2+\varphi(ba)^2]+\frac{-1}{m}[m^2\varphi(a)^2\varphi(b^2)+m^2\varphi(b)^2\varphi(a^2)]\\ &= [t+m\varphi(a)\varphi(b)]^2+[-t+m\varphi(a)\varphi(b)]^2-[2m^2\varphi(a)^2\varphi(b)^2]\\ &= 2t^2.\end{aligned}$$

Hence $t=0$, which proves $\varphi(ab)=\varphi(ba)$. Therefore by (1), $\varphi(ab)=m\varphi(a)\varphi(b)$, and the proof is complete. □

Corollary 2.2: *Suppose that $\mathcal{A}$ is a Banach algebra and $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each $(2,m)$–Jordan homomorphism $\varphi:\mathcal{A}\longrightarrow\mathcal{B}$ is a $(2,m)$–homomorphism.*

Lemma 2.3: *Let $\mathcal{A}$ be a unital Banach algebra with unit e and let $\varphi:\mathcal{A}\longrightarrow\mathbb{C}$ be a non-zero $(3,m)$–Jordan homomorphism. Then $\varphi(e)\neq 0$.*

Proof: Let φ be non-zero $(3,m)$–Jordan homomorphism, then $\varphi(a^3)=m\varphi(a)^3$, for all $a\in\mathcal{A}$. Replacing a by $a+b$, we get

$$\varphi(ab^2+b^2a+a^2b+ba^2+aba+bab)=m(3\varphi(a)^2\varphi(b)+3\varphi(a)\varphi(b)^2), \tag{7}$$

and replacing b by $-b$ in (7), we obtain

$$\varphi(ab^2+b^2a-a^2b-ba^2-aba+bab)=m(-3\varphi(a)^2\varphi(b)+3\varphi(a)\varphi(b)^2). \tag{8}$$

By (7) and (8) we obtain

$$\varphi(ab^2+b^2a+bab)=3m\varphi(a)\varphi(b)^2, \quad (a,b\in\mathcal{A}). \tag{9}$$

Now assume that $\varphi(e)=0$ and take $b=e$ in (9), then it follows that $\varphi(a)=0$, for all $a\in\mathcal{A}$, which is a contradiction. □

Lemma 2.4: *Let φ be a non-zero $(3,m^2)$-Jordan homomorphism from unital Banach algebra $\mathcal{A}$ into $\mathbb{C}$. Then either φ is $(2,m)$–Jordan or $(2,-m)$–Jordan homomorphism.*

Proof: By assumption for all $a\in\mathcal{A}$,

$$\varphi(a^3)=m^2\varphi(a)^3. \tag{10}$$

Replacing a by $a+e$ in (10), to obtain

$$\varphi(a^2+a)=m^2(\varphi(e)^2\varphi(a)+\varphi(e)\varphi(a)^2).$$

Replacing a by e in (10), we get $\varphi(e) = m^2\varphi(e)^3$. By above Lemma $\varphi(e) \neq 0$, therefore $\varphi(e) = \frac{1}{m}$ or $\varphi(e) = \frac{-1}{m}$. If $\varphi(e) = \frac{1}{m}$, then by the above equation we get

$$\varphi(a^2) = m\varphi(a)^2,$$

hence φ is $(2, m)$–Jordan. Similarly, we have

$$\varphi(a^2) = -m\varphi(a)^2,$$

if $\varphi(e) = \frac{-1}{m}$. Thus, φ is $(2, -m)$–Jordan. □

The next result, which is the main one in the paper, characterizes $(3, m^2)$–Jordan homomorphisms.

Theorem 2.5 : *Suppose that $\mathcal{A}$ is a unital Banach algebra and $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each $(3, m^2)$-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a $(3, m^2)$–homomorphism.*

Proof : We first assume that $\mathcal{B} = \mathbb{C}$ and let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be $(3, m^2)$–Jordan homomorphism, then by Lemma 2.4, φ is either $(2, m)$–Jordan or $(2, -m)$–Jordan homomorphism. If φ is $(2, m)$–Jordan, then by Theorem 2.1 it is $(2, m)$–homomorphism and so it is $(3, m^2)$–homomorphism. If φ is $(2, -m)$–Jordan, then by Theorem 2.1 it is $(2, -m)$–homomorphism. That is, for all $a, b \in \mathcal{A}$,

$$\varphi(ab) = -m\varphi(a)\varphi(b).$$

Therefore

$$\varphi(abc) = -m\varphi(a)\varphi(bc) = -m\varphi(a)[-m\varphi(b)\varphi(c)] = m^2\varphi(a)\varphi(b)\varphi(c),$$

for all $a, b, c \in \mathcal{A}$. Hence, φ is $(3, m^2)$–homomorphism.
Now suppose $\mathcal{B}$ is arbitrary semisimple and commutative. Let $\mathfrak{M}(\mathcal{B})$ be the maximal ideal space of $\mathcal{B}$. We associate with each $f \in \mathfrak{M}(\mathcal{B})$ a function $\varphi_f : \mathcal{A} \longrightarrow \mathbb{C}$ defined by

$$\varphi_f(a) := f(\varphi(a)), \qquad (a \in \mathcal{A}).$$

Pick $f \in \mathfrak{M}(\mathcal{B})$ arbitrary. It is easy to see that φ_f is a $(3, m^2)$–Jordan homomorphism, so by the above argument it is a $(3, m^2)$–homomorphism. Thus by the definition of φ_f we have

$$f(\varphi(abc)) = m^2 f(\varphi(a))f(\varphi(b))f(\varphi(c)) = f(m^2\varphi(a)\varphi(b)\varphi(c)).$$

Since $f \in \mathfrak{M}(\mathcal{B})$ was arbitrary and $\mathcal{B}$ is assumed to be semisimple, we obtain

$$\varphi(abc) = m^2\varphi(a)\varphi(b)\varphi(c),$$

for all $a, b, c \in \mathcal{A}$. This completes the proof. □

Theorem 2.6 : *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras, where $\mathcal{A}$ has a unit element e and $char(\mathcal{B}) > 3$. If every Jordan homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is* *a homomorphism, then every $(3,m)$–Jordan homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is a $(3,m)$–homomorphism.*

Proof: Let φ be a $(3,m)$–Jordan homomorphism, then for all $a \in \mathcal{A}$,

$$\varphi((a+2)^3 - 2(a+e)^3 + a^3) = m(\varphi(a+2)^3 - 2\varphi(a+e)^3 + \varphi(a)^3).$$

Hence,

$$6\varphi(a) + 6\varphi(e) = m(2\varphi(e)^2\varphi(a) + 2\varphi(a)\varphi(e)^2 + 2\varphi(e)\varphi(a)\varphi(e) + 6\varphi(e)^3). \quad (11)$$

By assumption $\varphi(e) = m\varphi(e)^3$, so by (11) we get

$$3\varphi(a) = m(\varphi(e)^2\varphi(a) + \varphi(a)\varphi(e)^2 + \varphi(e)\varphi(a)\varphi(e)). \quad (12)$$

Multiplying $\varphi(e)$ from the right in (12), we get

$$2\varphi(a)\varphi(e) = m(\varphi(e)^2\varphi(a)\varphi(e) + \varphi(e)\varphi(a)\varphi(e)^2). \quad (13)$$

Similarly,

$$2\varphi(e)\varphi(a) = m(\varphi(e)\varphi(a)\varphi(e)^2 + \varphi(e)^2\varphi(a)\varphi(e)). \quad (14)$$

By (13) and (14) we have

$$\varphi(a)\varphi(e) = \varphi(e)\varphi(a), \qquad (a \in \mathcal{A}). \quad (15)$$

It follows from (12) and (15) that

$$\varphi(a) = m\varphi(e)^2\varphi(a) = m\varphi(a)\varphi(e)^2. \quad (16)$$

By assumption

$$\varphi((a+e)^3 - a^3) = m(\varphi(a+e)^3 - \varphi(a)^3). \quad (17)$$

So by (15) and (17) we have

$$3\varphi(a^2) + 3\varphi(a) + \varphi(e) = m(3\varphi(a)^2\varphi(e) + 3\varphi(a)\varphi(e)^2 + \varphi(e)^3). \quad (18)$$

By (16) and (18) we get

$$\varphi(a^2) = m\varphi(a)^2\varphi(e), \qquad (a \in \mathcal{A}). \quad (19)$$

Now define a mapping $f : \mathcal{A} \longrightarrow: \mathcal{B}$ by

$$f(a) := m\varphi(a)\varphi(e),$$

for all $a \in \mathcal{A}$. Then by (19), f is Jordan homomorphism, so it is a homomorphism. By the definition of f and (16) we have

$$f(a)\varphi(e) = \varphi(a). \tag{20}$$

It follows from (16) and (20) that

$$\begin{aligned}\varphi(abc) &= f(abc)\varphi(e)\\ &= f(a)f(b)f(c)\varphi(e)\\ &= (m\varphi(a)\varphi(e))(m\varphi(b)\varphi(e))(m\varphi(c)\varphi(e))\varphi(e)\\ &= m\varphi(a)(m\varphi(b)\varphi(e)^2)(m\varphi(c)\varphi(e)^2)\\ &= m\varphi(a)\varphi(b)\varphi(c).\end{aligned}$$

Thus, φ is $(3, m)$–homomorphism. □

As a consequence of Theorem 1.2 and Theorem 2.6 we deduce the next result.

Corollary 2.7: *Suppose that $\mathcal{A}$ is a unital Banach algebra and $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each $(3, m)$–Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a $(3, m)$–homomorphism.*

3. Weakly commutative Case

We say that the Banach algebra $\mathcal{A}$ is weakly commutative if

$$(ax)^2 = a^2x^2 \quad \text{and} \quad ax^2a = x^2a^2,$$

for all $a, x, \in \mathcal{A}$. Clearly, every commutative Banach algebra is weakly commutative, but in general, the converse is false. For example, let

$$\mathcal{A} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : \quad a, b \in \mathbb{R} \right\}.$$

Then it is obvious to check that with the usual matrix product for all $x, y \in \mathcal{A}$,

$$(xy)^2 = x^2y^2 \quad \text{and} \quad xy^2x = y^2x^2.$$

Thus, $\mathcal{A}$ is weakly commutative, but it is neither unital nor commutative.

Theorem 3.1: *Let $\mathcal{A}$ and $\mathcal{B}$ be two weakly commutative Banach algebras. If $\mathcal{A}$ is unital, then every $(2, m)$–Jordan homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is a $(2, m)$–homomorphism*

Proof: By a similar argument which has been used in the proof of theorem 2.1, for all $a, b \in \mathcal{A}$ we have

$$\varphi(aba) = m^2\varphi(a)\varphi(b)\varphi(a). \tag{21}$$

Replacing b by b^2 in (21), we obtain

$$\varphi(b^2a^2) = \varphi(ab^2a) = m^2\varphi(a)\varphi(b^2)\varphi(a) = m^3\varphi(a)^2\varphi(b)^2 = m\varphi(b^2)\varphi(a^2). \quad (22)$$

Replacing b by $x + y$ in (22), gives

$$\varphi(xya^2 + yxa^2) = m\varphi(xy + yx)\varphi(a^2). \quad (23)$$

Replacing a by $a + b$ in (23), gives

$$\varphi((xy + yx)(ab + ba)) = m\varphi(xy + yx)\varphi(ab + ba), \quad (24)$$

for all $a, b, x, y \in \mathcal{A}$. Replacing y and b with unit the element of $\mathcal{A}$ in (24), we get

$$\varphi(xa) = m\varphi(x)\varphi(a), \quad (25)$$

for all $a, x \in \mathcal{A}$, as claimed. □

Theorem 3.2: *With the hypotheses of Theorem 3.1, every* $(3, m^2)$*–Jordan homomorphism from* $\mathcal{A}$ *into* $\mathcal{B}$ *is a* $(3, m^2)$*–homomorphism.*

Proof: Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be $(3, m^2)$–Jordan homomorphism. Then by Lemma 2.4, φ is $(2, m)$–Jordan or $(2, -m)$–Jordan homomorphism. If φ is $(2, m)$–Jordan, then by Theorem 3.1 it is $(2, m)$–homomorphism and so it is $(3, m^2)$-homomorphism. If φ is $(2, -m)$–Jordan homomorphism, then by Theorem 3.1 it is $(2, -m)$–homomorphism. That is, $\varphi(ab) = -m\varphi(a)\varphi(b)$, for all $a, b \in \mathcal{A}$. Therefore

$$\varphi(abc) = -m\varphi(a)\varphi(bc) = -m\varphi(a)[-m\varphi(b)\varphi(c)] = m^2\varphi(a)\varphi(b)\varphi(c),$$

for all $a, b, c \in \mathcal{A}$. Hence, φ is $(3, m^2)$–homomorphism. □

The following theorem follows from Theorem 3.1 and Theorem 2.6.

Theorem 3.3: *With the hypotheses of Theorem 3.1, every* $(3, m)$*–Jordan homomorphism from* $\mathcal{A}$ *into* $\mathcal{B}$ *is a* $(3, m)$*–homomorphism.*

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