*Bulletin of TICMI*
Vol. 23, No. 1, 2019, 35–66

# On BVPs for Piezoelectric Transversely Isotropic Cusped Bars

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(*Received March 30, 2019; Revised June 14, 2019; Accepted June 26, 2019*)

In $(0, 0)$ approximation of hierarchical models of piezoelectric transversely isotropic cusped bars we consider static and oscilation problems. We analyze peculiarities of nonclassical setting boundary conditions.

**Keywords:** Hierarchical models, Piezoelectrics, Transversely isotropic cusped bars.

**AMS Subject Classification**: 70K10, 74F15, 74D05, 74E10, 35L80, 35L81, 34A36, 34B05.

## Preface

The present paper is devoted to static and oscillation problems of hierarchical models of piezoelectric transversely isotropic cusped bars in $(0, 0)$ approximation. A special attention is given to analysis peculiarities of nonclassical setting boundary conditions (BCs). Namely, the criteria are established when on one end or on both ends of the bar no data need be prescribed. Weighted BCs are set as well On the face surfaces of the bar under consideration stress vectors and outward normal components of the electric displacement vectors are prescribed, while at the ends of the bar all the admissible [in sense of well posedness of boundary value problems(BVPs)] BCs, including mixed ones, with respect to weighted $(0, 0)$ moments of the components of the mechanical displacement vectors and electric potential, and $(0, 0)$ moments of the components of the stress and electric displacement vectors are prescribed.

The paper is organized as follows. In Section 1 we compile auxiliary materials concerning geometry of cusped, in general, prismatic bars with rectangular cross-sections and double mathematical moments of functions. In section 2 we briefly sketch field equations of the transversaly isotropic elastic piezoelectric materials. In Section 3 we derive governing equations of $(0, 0)$ hierarchical model. In Section 4 we deduce governing equations of one-dimensional particular case of three-dimensional model. In Section 5 we study BVPs which are solved in the explicit forms. In Section 6 mechanical interpretation of results of analysis of peculiarities of setting BCs for cusped bars is given. In Section 7 we make some bibliographical remarks.

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ISSN: 1512-0082 print

## 1. Introductory section

Let the closure of a domain of $R^3$, occupied by a piezoelectric elastic bar $\overline{V}$ with rectangular cross-sections (see [1], [2]) be:

$$\overline{V} := \Big\{(x_1, x_2, x_3) \in R^3 : 0 \leq x_3 \leq L;\ \overset{(-)}{h}_\alpha(x_3) \leq x_\alpha \leq \overset{(+)}{h}_\alpha(x_3),$$
$$\alpha = 1, 2;\ L = const\Big\} \qquad (1.1)$$

with

$$2h_\alpha(x_3) := \overset{(+)}{h}_\alpha(x_3) - \overset{(-)}{h}_\alpha(x_3) > 0,\ \ 0 < x_3 < L,$$
$$2h_\alpha(0) \geq 0,\ \ 2h_\alpha(L) \geq 0,\ \ h_\alpha \in C([0, L]) \cap C^1(]0, L[),\ \alpha = 1, 2. \qquad (1.2)$$

$C([0, L])$ and $C^1(]0, L[)$ denote classes of continuous and continuously differentiable functions on the indicated intervals, respectively. Let $2h_1$ and $2h_2$ conventionally be the thickness and the width of the bar and their maxima be significantly less than the length $L$ of the bar (see Figure 1.1).

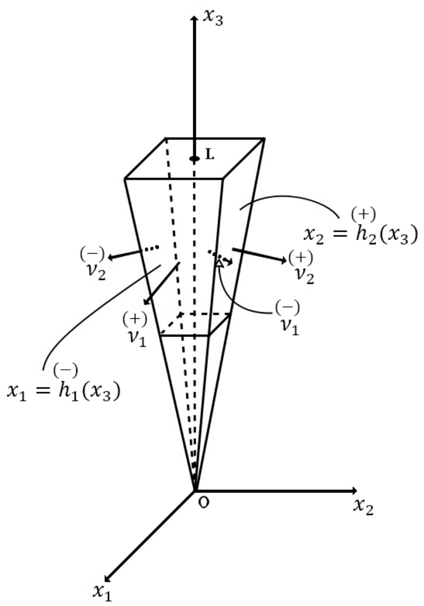


Figure 1.1. An example of a cusped prismatic bar with rectangular cross-sections

Since

$$x_1 = \overset{(\pm)}{h}_1(x_3) \ \text{ and } \ x_2 = \overset{(\pm)}{h}_2(x_3) \qquad (1.3)$$

are the face surfaces, clearly, $\Big[(\overset{(\pm)}{\nu}_\alpha)_i$, $\alpha = 1, 2$, $i = 1, 2$, mean the projections of the normals $\overset{(\pm)}{\nu}_\alpha$ to the face surfaces $\overset{(\pm)}{h}_\alpha$ on $x_i$-axis]

$$\Big(\overset{(\pm)}{\nu}_1\Big)_1 = \frac{\pm 1}{\sqrt{1+(\overset{(\pm)}{h}_{1,3})^2}} = \cos(\overset{(\pm)}{\nu}_1, x_1), \tag{1.4}$$

$$\begin{aligned} \Big(\overset{(\pm)}{\nu}_1\Big)_2 &= \frac{\mp \overset{(\pm)}{h}_{1,2}}{\sqrt{1+(\overset{(\pm)}{h}_{1,3})^2}} = \cos(\overset{(\pm)}{\nu}_1, x_2) = 0 \ \text{ since } \ \overset{(\pm)}{h}_{1,2} = 0, \\ \Big(\overset{(\pm)}{\nu}_1\Big)_3 &= \frac{\mp \overset{(\pm)}{h}_{1,3}}{\sqrt{1+(\overset{(\pm)}{h}_{1,3})^2}} = \cos(\overset{(\pm)}{\nu}_1, x_3), \end{aligned} \tag{1.5}$$

whence, in view of (1.2)-(1.5),

$$\begin{aligned} &\pm 1 = \sqrt{1+(\overset{(\pm)}{h}_{1,3})^2}\Big(\overset{(\pm)}{\nu}_1\Big)_1, \\ &0 = \mp \overset{(\pm)}{h}_{1,2} = \sqrt{1+(\overset{(\pm)}{h}_{1,3})^2}\Big(\overset{(\pm)}{\nu}_1\Big)_2, \\ &\mp \overset{(\pm)}{h}_{1,3} = \sqrt{1+(\overset{(\pm)}{h}_{1,3})^2}\Big(\overset{(\pm)}{\nu}_1\Big)_3; \end{aligned} \tag{1.6}$$

$$\Big(\overset{(\pm)}{\nu}_2\Big)_1 = \frac{\mp \overset{(\pm)}{h}_{2,1}}{\sqrt{1+(\overset{(\pm)}{h}_{2,3})^2}} = 0 \ \text{ since } \ \overset{(\pm)}{h}_{2,1} = 0, \tag{1.7}$$

$$\Big(\overset{(\pm)}{\nu}_2\Big)_2 = \frac{\pm 1}{\sqrt{1+(\overset{(\pm)}{h}_{2,3})^2}}, \quad \Big(\overset{(\pm)}{\nu}_2\Big)_3 = \frac{\mp \overset{(\pm)}{h}_{2,3}}{\sqrt{1+(\overset{(\pm)}{h}_{2,3})^2}}, \tag{1.8}$$

hence, by virtue of (1.2), (1.3), (1.7), (1.8),

$$
\begin{aligned}
0 = \mp \overset{(\pm)}{h}_{2,1} &= \sqrt{1+(\overset{(\pm)}{h}_{2,3})^2}\Big(\overset{(\pm)}{\nu_2}\Big)_1,\\
\pm 1 &= \sqrt{1+(\overset{(\pm)}{h}_{2,3})^2}\Big(\overset{(\pm)}{\nu_2}\Big)_2,\\
\mp \overset{(\pm)}{h}_{2,3} &= \sqrt{1+(\overset{(\pm)}{h}_{2,3})^2}\Big(\overset{(\pm)}{\nu_2}\Big)_3.
\end{aligned}
\tag{1.9}
$$

Note, that also geometrically $(\overset{(\pm)}{\nu_\alpha})_\beta = 0$, $\alpha \neq \beta$, $\alpha, \beta = 1, 2$.

Let further

$$f(x_1, x_2, x_3) \in C^1(\overline{V}),$$

and at points $x_3$, where both the thickness and the width of the bar do not vanish, i.e. the area of the cross-section does not vanish, we define $(0,0)$ double moment of the function $f$ and its first derivatives as follows:

$$f_{00}(x_3) := \int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_1(x_3)} \int\limits_{\overset{(-)}{h}_2(x_3)}^{\overset{(+)}{h}_2(x_3)} f(x_1, x_2, x_3)dx_1dx_2, \tag{1.10}$$

$$
\begin{aligned}
{}_3f_{00}(x_3) :=& \int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_1(x_3)} \int\limits_{\overset{(-)}{h}_2(x_3)}^{\overset{(+)}{h}_2(x_3)} f_{,3}(x_1, x_2, x_3)dx_1dx_2 = \int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_1(x_3)} dx_1 \Big[\frac{\partial}{\partial x_3} \int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_2(x_3)} f(x_1, x_2, x_3)dx_2 \\
& -f(x_1, \overset{(+)}{h}_2, x_3)\overset{(+)}{h}_{2,3} + f(x_1, \overset{(-)}{h}_2, x_3)\overset{(-)}{h}_{2,3}\Big] \\
=& \; f_{00,3}(x_3) - \int\limits_{\overset{(-)}{h}_2(x_3)}^{\overset{(+)}{h}_2(x_3)} f(\overset{(+)}{h}_1, x_2, x_3)dx_2\overset{(+)}{h}_{1,3} + \int\limits_{\overset{(-)}{h}_2(x_3)}^{\overset{(+)}{h}_2(x_3)} f(\overset{(-)}{h}_1, x_2, x_3)dx_2\overset{(-)}{h}_{1,3} \\
& - \int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_1(x_3)} f(x_1 \overset{(+)}{h}_2, x_3)dx_1\overset{(+)}{h}_{2,3} + \int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_1(x_3)} f(x_1, \overset{(-)}{h}_2, x_3)dx_1\overset{(-)}{h}_{2,3} \\
=& \; f_{00,3}(x_3) + \overset{(3)}{f}(x_3),
\end{aligned}
\tag{1.11}
$$

where

$$\overset{(3)}{f}(x_3) := -\int\limits_{\overset{(-)}{h}_2(x_3)}^{\overset{(+)}{h}_2(x_3)} f(\overset{(+)}{h}_1, x_2, x_3)dx_2\overset{(+)}{h}_{1,3} + \int\limits_{\overset{(-)}{h}_2(x_3)}^{\overset{(+)}{h}_2(x_3)} f(\overset{(-)}{h}_1, x_2, x_3)dx_2\overset{(-)}{h}_{1,3}$$

$$-\int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_1(x_3)} f(x_1\overset{(+)}{h}_2, x_3)dx_1\overset{(+)}{h}_{2,3} + \int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_1(x_3)} f(x_1, \overset{(-)}{h}_2, x_3)dx_1\overset{(-)}{h}_{2,3}, \quad (1.12)$$

$${}_1f_{00}(x_3) := \int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_1(x_3)} \int\limits_{\overset{(-)}{h}_2(x_3)}^{\overset{(+)}{h}_2(x_3)} f_{,1}(x_1, x_2, x_3)dx_1dx_2$$

$$= \int\limits_{\overset{(-)}{h}_2(x_3)}^{\overset{(+)}{h}_2(x_3)} \left[f\left(\overset{(+)}{h}_1, x_2, x_3\right) - f\left(\overset{(-)}{h}_1, x_2, x_3\right)\right]dx_2 =: \overset{(1)}{f}(x_3), \quad (1.13)$$

$${}_2f_{00}(x_3) := \int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_1(x_3)} \int\limits_{\overset{(-)}{h}_2(x_3)}^{\overset{(+)}{h}_2(x_3)} f_{,2}(x_1, x_2, x_3)dx_1dx_2$$

$$= \int\limits_{\overset{(-)}{h}_1(x_3)}^{\overset{(+)}{h}_1(x_3)} \left[f\left(x_1, \overset{(+)}{h}_2, x_3\right) - f\left(x_2, \overset{(-)}{h}_2, x_3\right)\right]dx_1 =: \overset{(2)}{f}(x_3). \quad (1.14)$$

At the point $x_3$, where the area of the cross-section vanishes, the $(0,0)$ double moment we define as limit of the $(0,0)$ double moments from points, where the area of cross-sections do not vanish.

## 2. Field equations

Let us now consider the transversely isotropic elastic piezoelectric material in the case when the poling axis coincides with one of the material symmetry axes [3]. Let it be $x_3$-axis. A material behavior is said to be transversely isotropic if it is invariant with respect to an arbitrary rotation about a given axis. This material behavior is of special importance in the modelling of fibre-reinforced composite materials with a coordinate axis in the fibre direction and assumed isotropic in cross-sections orthogonal to fibre direction [4] (in our case to poling axis as well, since in the case under consideration they coincide, see Figures 2.1, 2.2). The transverse isotropic model is also suitable for biological applications because it adequately describes the elastic properties of bundled fibers aligned in one direction [5] (see also [6]).

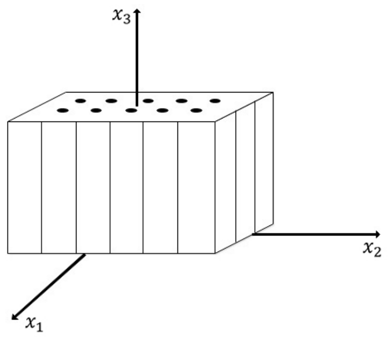


Figure 2.1. Transversely isotropic material with the fibers parallel to $x_3$-axis

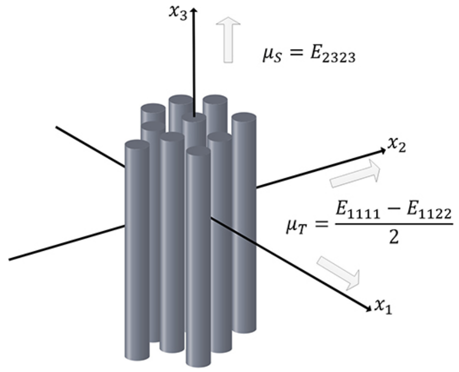


Figure 2.2. A bundle of fiber aligned parallel to $x_3$-axis

It is well-known [3] that the electric field that develops in piezoelectrics can be assumed to be quasi-static because the velocity of the elastic waves is much smaller than the velocity of electromagnetic waves. Therefore, the magnetic field due to the elastic waves is negligible and the vector of magnetic induction $\mathbf{B} \approx 0$. This fact implies that

$$\frac{\partial \mathbf{B}}{\partial t} \approx 0.$$

So one of Maxwell's equations of electrodynamics becomes

$$rot\, \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \approx 0,$$

hence the vector of the electric field

$$\mathbf{E} = -grad\, \mathcal{X},$$

where $\mathcal{X}$ is the electric potential.

Consequently, considering transversely isotropic piezoelectric continuum, it will be based on the governing equations of elastodynamics in the case of small deformations and quasi-electrostatic fields. Note that piezoelectric materials show in most cases a crystal structure with a symmetry of hexagonal 6 mm class. In the case when the poling axis coincides with one of the material symmetry axes these materials become transversely isotropic. Restricting to the case of time-harmonic motion with frequency $o$, i.e., all the sought quantities, s.c. free members of governing equations, and boundary data are represented as the products of $e^{iot}$ and of the same quantities (to avoid redundant indices and symbols we leave the same notation) depending only on the space variables, we get the following governing equations (see [7], [3], and also [8], [9])

$$X_{ij,j} + \rho o^2 u_i = -\Phi_i, \quad i = \overline{1,3} \tag{2.1}$$

(note that the motion equation in the general dynamical case has the form

$$\left. X_{ij,j} - \rho \frac{\partial^2 u_i}{\partial t^2} = -\Phi_i, \;\; i = \overline{1,3} \right);$$

$$D_{j,j} = f_e; \tag{2.2}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad i,j = \overline{1,3}; \tag{2.3}$$

$$E_i = -\chi_{,i}, \quad i = \overline{1,3}; \tag{2.4}$$

$$\begin{pmatrix} X_{11} \\ X_{22} \\ X_{33} \\ X_{23} \\ X_{31} \\ X_{12} \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = C \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{2.5}$$

with (see [3])

$$C := \begin{pmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 & 0 & 0 & p_{311} \\ E_{1122} & E_{1111} & E_{1133} & 0 & 0 & 0 & 0 & 0 & p_{311} \\ E_{1133} & E_{1133} & E_{3333} & 0 & 0 & 0 & 0 & 0 & p_{333} \\ 0 & 0 & 0 & E_{2323} & 0 & 0 & 0 & p_{113} & 0 \\ 0 & 0 & 0 & 0 & E_{2323} & 0 & p_{113} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(E_{1111} - E_{1122}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{113} & 0 & -\varsigma_{11} & 0 & 0 \\ 0 & 0 & 0 & p_{113} & 0 & 0 & 0 & -\varsigma_{11} & 0 \\ p_{311} & p_{311} & p_{333} & 0 & 0 & 0 & 0 & 0 & -\varsigma_{33} \end{pmatrix}, \tag{2.6}$$

where $X_{ij}$ and $e_{ij}$ are the mechanical stress and strain second rank tensors, $u_i$ are the mechanical displacements, $\Phi_i$ are components of the volume force, $D_j$ are components of the vector of electric displacement, $\rho$ is the mass density, $f_e$ is the free electric volume charge, $E_{ijkl}$ are the elastic stiffness constants, $p_{ijk}$ are the piezoelectric constants, $\varsigma_{ij}$ are the dielectric permittivity constants.

From (2.5), (2.6) we have

$$X_{11} = E_{1111}e_{11} + E_{1122}e_{22} + E_{1133}e_{33} - p_{311}E_3, \tag{2.7}$$

$$X_{22} = E_{1122}e_{11} + E_{1111}e_{22} + E_{1133}e_{33} - p_{311}E_3, \tag{2.8}$$

$$X_{33} = E_{1133}e_{11} + E_{1133}e_{22} + E_{3333}e_{33} - p_{333}E_3, \tag{2.9}$$

$$X_{32} = X_{23} = 2E_{2323}e_{23} - p_{113}E_2, \;\; X_{13} = X_{31} = 2E_{2323}e_{31} - p_{113}E_1, \tag{2.10}$$

$$X_{21} = X_{12} = (E_{1111} - E_{1122})e_{12}, \tag{2.11}$$

$$D_1 = 2p_{113}e_{13} + \varsigma_{11}E_1, \;\; D_2 = 2p_{113}e_{23} + \varsigma_{11}E_2, \tag{2.12}$$

$$D_3 = p_{311}e_{11} + p_{311}e_{22} + p_{333}e_{33} + \varsigma_{33}E_3. \tag{2.13}$$

The following reciprocal symmetries hold

$$E_{ijkl} = E_{jikl} = E_{klij}, \;\; p_{ijk} = p_{ikj}, \;\; \varsigma_{ij} = \varsigma_{ji}.$$

In the case of transversely isotropic solids it is known that [10]

$$E_{1111} > |E_{1122}|, \;\; (E_{1111} + E_{1122})E_{3333} > 2E_{1133}^2, \;\; E_{2323} > 0,$$
$$\varsigma_{11} > 0, \;\; \varsigma_{33} > 0,$$

i.e.,

$$E_{1111} + E_{1122} > 0$$

and

$$E_{3333} > \frac{2E_{1133}^2}{E_{1111} + E_{1122}} > 0.$$

## 3. Construction of $(0, 0)$ hierarchical model

Let

$$(u_i, \chi)(x_1, x_2, x_3, t) \cong \frac{1}{4h_1h_2}(u_{i00}, \chi_{00})(x_3, t) =: \frac{1}{4}(v_{i00}, \tilde{\chi}_{00})(x_3, t). \tag{3.1}$$

Let further constitutive coefficients depend only on $x_3$.

After integrating on $]\overset{(-)}{h}_1, \overset{(+)}{h}_2[\times]\overset{(-)}{h}_1, \overset{(+)}{h}_2[$ for fixed $x_3 \in [0, L]$ [see (1.1)],
(i) from (2.1), taking into account (1.10), (1.11), (1.13), (1.14), we obtain (on the

face surfaces we assume prescribed stress vectors)

$$X_{i300,3} + \rho o^2 u_{i00} = -\Phi_{i00} - \sum_{j=1}^{3} \overset{(j)}{X}_{ij}, \quad i = \overline{1,3}; \tag{3.2}$$

(ii) from (2.2) taking into account (1.10), (1.11), (1.13), (1.14), we obtain (on the face surfaces we assume prescribed electric displacements)

$$D_{300,3} = f_{e00} - \sum_{j=1}^{3} \overset{(j)}{D}_j; \tag{3.3}$$

(iii) from (2.3), taking into account (1.10), (1.11), (1.13), (1.14), because of (3.1) [as values of $u_i(x_1,x_2,x_3,t)$ and $\chi(x_1,x_2,x_3,t)$ on the face surfaces we take the approximate values (3.1)], we have

$$e_{\alpha\underline{\alpha}00} = \overset{(\alpha)}{u}_{\underline{\alpha}} \equiv 0, \quad \alpha = 1,2, \tag{3.4}$$

$$e_{3300} = u_{300,3} + \overset{(3)}{u}_3 = (h_1h_2v_{300})_{,3} - \frac{u_{300}}{4h_1h_2}(2h_22h_{1,3} + 2h_12h_{2,3}) = h_1h_2v_{300,3}, \tag{3.5}$$

$$\begin{aligned} e_{3200} = e_{2300} &= \frac{1}{2}(u_{200,3} + \overset{(3)}{u}_2 + \overset{(2)}{u}_3) \\ &= \frac{1}{2}\Big[(h_1h_2v_{200})_{,3} - \frac{u_{200}}{4h_1h_2}(h_1h_2)_{,3} + 0\Big] = \frac{1}{2}h_1h_2v_{200,3}, \end{aligned} \tag{3.6}$$

$$e_{1300} = e_{3100} = \frac{1}{2}(u_{100,3} + \overset{(3)}{u}_1 + \overset{(1)}{u}_3) = \frac{1}{2}h_1h_2v_{100,3}, \tag{3.7}$$

$$e_{2100} = e_{1200} = \frac{1}{2}(\overset{(2)}{u}_1 + \overset{(1)}{u}_2) \equiv 0, \tag{3.8}$$

(iv) from (2.4), taking into account (1.10), (1.11), (1.13), (1.14), we have

$$E_{\alpha 00} = 0, \quad \alpha = 1,2. \tag{3.9}$$

$$E_{300} = -h_1h_2\tilde{\chi}_{300,3}. \tag{3.10}$$

Substituting (3.4)-(3.10) into (2.7)-(2.13), we get

$$X_{1100} = E_{1133}h_1h_2v_{300,3} + p_{311}h_1h_2\tilde{\chi}_{00,3}, \tag{3.11}$$

$$X_{2200} = E_{1133}h_1h_2v_{300,3} + p_{311}h_1h_2\tilde{\chi}_{00,3}, \tag{3.12}$$

$$X_{3300} = E_{3333}h_1h_2v_{300,3} + p_{333}h_1h_2\tilde{\chi}_{00,3}, \tag{3.13}$$

$$X_{3200} = X_{2300} = E_{2323}h_1h_2v_{200,3}, \tag{3.14}$$

$$X_{1300} = X_{3100} = E_{2323} h_1 h_2 v_{100,3}, \tag{3.15}$$

$$X_{2100} = X_{1200} = 0, \tag{3.16}$$

$$D_{100} = p_{113} h_1 h_2 v_{100,3}, \tag{3.17}$$

$$D_{200} = p_{113} h_1 h_2 v_{200,3}, \tag{3.18}$$

$$D_{300} = p_{333} h_1 h_2 v_{300,3} - \varsigma_{33} h_1 h_2 \tilde{\chi}_{00,3}. \tag{3.19}$$

If we substitute (3.13)-(3.15) and (3.19) into (3.2) and (3.3), respectively, we derive

$$(E_{2323} h_1 h_2 v_{\alpha 00,3})_{,3} + \rho o^2 h_1 h_2 v_{\alpha 00} = -\Phi_{\alpha 00} - \sum_{j=1}^{3} \overset{(j)}{X}_{\alpha j}, \quad \alpha = 1, 2, \tag{3.20}$$

$$(E_{3333} h_1 h_2 v_{300,3})_{,3} + (p_{333} h_1 h_2 \tilde{\chi}_{00,3})_{,3} + \rho o^2 h_1 h_2 v_{300} = -\Phi_{300} - \sum_{j=1}^{3} \overset{(j)}{X}_{3j}, \tag{3.21}$$

$$(p_{333} h_1 h_2 v_{300,3})_{,3} - (\varsigma_{33} h_1 h_2 \tilde{\chi}_{00,3})_{,3} = f_{e00} - \sum_{j=1}^{3} \overset{(j)}{D}_j, \tag{3.22}$$

where

$$\sum_{j=1}^{3} \overset{(j)}{X}_{ij} \tag{3.23}$$

$$= \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \Big[\sqrt{1 + (\overset{(+)}{h}_{1,3})^2} X_{\overset{(+)}{\nu}_1 i}(\overset{(+)}{h}_1, x_2, x_3, t) + \sqrt{1 + (\overset{(-)}{h}_{1,3})^2} X_{\overset{(-)}{\nu}_1 i}(\overset{(-)}{h}_1, x_2, x_3, t)\Big] dx_2$$

$$+ \int_{\overset{(-)}{h}_1}^{\overset{(+)}{h}_1} \Big[\sqrt{1 + (\overset{(+)}{h}_{2,3})^2} X_{\overset{(+)}{\nu}_2 i}(x_1, \overset{(+)}{h}_2, x_3, t) + \sqrt{1 + (\overset{(-)}{h}_{2,3})^2} X_{\overset{(-)}{\nu}_2 i}(x_1, \overset{(-)}{h}_2, x_3, t)\Big] dx_1,$$

$$\sum_{j=1}^{3} \overset{(j)}{D}_j = \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \Big[\sqrt{1 + (\overset{(+)}{h}_{1,3})^2} D_{\overset{(+)}{\nu}_1}(\overset{(+)}{h}_1, x_2, x_3, t) \tag{3.24}$$

$$+ \sqrt{1 + (\overset{(-)}{h}_{1,3})^2} D_{\overset{(-)}{\nu}_1}(\overset{(-)}{h}_1, x_2, x_3, t)\Big] dx_2$$

$$+ \int_{\overset{(-)}{h}_1}^{\overset{(+)}{h}_1} \Big[\sqrt{1 + (\overset{(+)}{h}_{2,3})^2} D_{\overset{(+)}{\nu}_2}(x_1, \overset{(+)}{h}_2, x_3, t) + \sqrt{1 + (\overset{(-)}{h}_{2,3})^2} D_{\overset{(-)}{\nu}_2}(x_1, \overset{(-)}{h}_1, x_3, t)\Big] dx_1,$$

where $D_{\overset{(\pm)}{\nu}_\alpha} := D_i(\overset{(\pm)}{\nu}_\alpha)_i$, $\alpha = 1, 2$, mean the projections of the vector of electric displacement $\mathbf{D} := (D_1, D_2, D_3)$ on the directions $\overset{(\pm)}{\nu}_\alpha$.

Indeed,

$$\overset{(1)}{X}_{i1} = \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \Big[ X_{i1}(\overset{(+)}{h}_1, x_2, x_3, t) - X_{i1}(\overset{(-)}{h}_1, x_2, x_3, t) \Big] dx_2,$$

$$\overset{(2)}{X}_{i2} = \int\limits_{\overset{(-)}{h}_1}^{\overset{(+)}{h}_1} \Big[ X_{i2}(x_1, \overset{(+)}{h}_2, x_3, t) - X_{i2}(x_1, \overset{(-)}{h}_2, x_3, t) \Big] dx_1,$$

$$\overset{(3)}{X}_{i3} = -\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} X_{i3}(\overset{(+)}{h}_1, x_2, x_3, t) dx_2 \overset{(+)}{h}_{1,3} + \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} X_{i3}(\overset{(-)}{h}_1, x_2, x_3, t) dx_2 \overset{(-)}{h}_{1,3}$$

$$-\int\limits_{\overset{(-)}{h}_1}^{\overset{(+)}{h}_1} X_{i3}(x_1, \overset{(+)}{h}_2, x_3, t) dx_1 \overset{(+)}{h}_{2,3} + \int\limits_{\overset{(-)}{h}_1}^{\overset{(+)}{h}_1} X_{i3}(x_1, \overset{(-)}{h}_2, x_3, t) dx_1 \overset{(-)}{h}_{2,3}.$$

Therefore, by virtue of (1.6), (1.9),

$$\sum_{j=1}^{3} \overset{(j)}{X}_{ij} = \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \Big[ X_{i1}(\overset{(+)}{h}_1, x_2, x_3, t)(\overset{(+)}{\nu}_1)_1 \sqrt{1 + (\overset{(+)}{h}_{1,3})^2}$$

$$+X_{i1}(\overset{(-)}{h}_1, x_2, x_3, t)(\overset{(-)}{\nu}_1)_1 \sqrt{1 + (\overset{(-)}{h}_{1,3})^2} + X_{i2}(\overset{(+)}{h}_1, x_2, x_3, t)(\overset{(+)}{\nu}_1)_2 \sqrt{1 + (\overset{(+)}{h}_{1,3})^2}$$

$$+X_{i2}(\overset{(-)}{h}_1, x_2, x_3, t)(\overset{(-)}{\nu}_1)_2 \sqrt{1 + (\overset{(-)}{h}_{1,3})^2} + X_{i3}(\overset{(+)}{h}_1, x_2, x_3, t)(\overset{(+)}{\nu}_1)_3 \sqrt{1 + (\overset{(+)}{h}_{1,3})^2}$$

$$+X_{i3}(\overset{(-)}{h}_1, x_2, x_3, t)(\overset{(-)}{\nu}_1)_3 \sqrt{1 + (\overset{(-)}{h}_{1,3})^2}] dx_2$$

$$+ \int\limits_{\overset{(-)}{h}_1}^{\overset{(+)}{h}_1} \Big[ X_{i1}(x_1, \overset{(+)}{h}_2, x_3, t)(\overset{(+)}{\nu}_2)_1 \sqrt{1 + (\overset{(+)}{h}_{2,3})^2}$$

$$+X_{i1}(x_1, \overset{(-)}{h}_2, x_3, t)(\overset{(-)}{\nu}_2)_1 \sqrt{1 + (\overset{(-)}{h}_{2,3})^2} + X_{i2}(x_1, \overset{(+)}{h}_2, x_3, t)(\overset{(+)}{\nu}_2)_2 \sqrt{1 + (\overset{(+)}{h}_{2,3})^2}$$

$$+X_{i2}(x_1,\overset{(-)}{h}_2,x_3,t)(\overset{(-)}{\nu}_2)_2\sqrt{1+(\overset{(-)}{h}_{2,3})^2}$$

$$+X_{i3}(x_1,\overset{(+)}{h}_2,x_3,t)(\overset{(+)}{\nu}_2)_3\sqrt{1+(\overset{(+)}{h}_{2,3})^2}$$

$$+X_{i3}(x_1,\overset{(-)}{h}_2,x_3,t)(\overset{(-)}{\nu}_2)_3\sqrt{1+(\overset{(-)}{h}_{2,3})^2}]dx_1$$

$$=\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\Big[\sqrt{1+(\overset{(+)}{h}_{1,3})^2}X_{\overset{(+)}{\nu}_1 i}(\overset{(+)}{h}_1,x_2,x_3,t)+\sqrt{1+(\overset{(-)}{h}_{1,3})^2}X_{\overset{(-)}{\nu}_1 i}(\overset{(-)}{h}_1,x_2,x_3,t)\Big]dx_2$$

$$+\int\limits_{\overset{(-)}{h}_1}^{\overset{(+)}{h}_1}\Big[\sqrt{1+(\overset{(+)}{h}_{2,3})^2}X_{\overset{(+)}{\nu}_2 i}(x_1,\overset{(+)}{h}_2,x_3,t)+\sqrt{1+(\overset{(-)}{h}_{2,3})^2}X_{\overset{(-)}{\nu}_2 i}(x_1,\overset{(-)}{h}_2,x_3,t)\Big]dx_1,$$

$$i=\overline{1,3},$$

since

$$X_{\overset{(\pm)}{\nu}_2 i}(x_1,\overset{(\pm)}{h}_2,x_3,t)=X_{ji}(\overset{(\pm)}{\nu}_2)_j,\quad i=\overline{1,3}, \tag{3.25}$$

$$X_{\overset{(\pm)}{\nu}_1 i}(\overset{(\pm)}{h}_1,x_2,x_3,t)=X_{ji}(\overset{(\pm)}{\nu}_1)_j,\quad i=\overline{1,3}. \tag{3.26}$$

We similarly derive (3.24).

**Remark 1:** Note that in the general dynamical case the governing system has the form

$$(E_{2323}h_1h_2v_{\alpha00,3})_{,3}-\rho h_1h_2\frac{\partial^2 v_{\alpha00}}{\partial t^2}=-\Phi_{\alpha00}-\sum_{j=1}^{3}\overset{(j)}{X}_{\alpha j},\ \ \alpha=1,2,$$

$$(E_{3333}h_1h_2v_{300,3})_{,3}+(p_{333}h_1h_2\tilde{\chi}_{00,3})_{,3}-\rho h_1h_2\frac{\partial^2 v_{300}}{\partial t^2}=-\Phi_{300}-\sum_{j=1}^{3}\overset{(j)}{X}_{3j},$$

$$(p_{333}h_1h_2v_{300,3})_{,3}-(\varsigma_{33}h_1h_2\tilde{\chi}_{00,3})_{,3}=f_{e00}-\sum_{j=1}^{3}\overset{(j)}{D}_j.$$

## 4. One-dimensional case

In the present section we consider a piezoelectric elastic infinite layer (see Figures 2.1, 2.2)

$$\widetilde{V} := \{(x_1, x_2, x_3) \in R^3 : 0 \leq x_3 \leq L,\ -\infty < x_\alpha < +\infty,\ \alpha = 1, 2\}$$

with the thickness $L = \text{const}$.

Let all the quantities of three-dimensional model under consideration depend only on $x_3 \in ]0, L[$, then from (2.1)-(2.4), (2.7)-(2.13) it follows that

$$X_{i3,3} + \rho o^2 u_i = \Phi_i, \quad i = \overline{1,3}, \tag{4.1}$$

$$D_{3,3} = f_e, \tag{4.2}$$

$$e_{\alpha\beta} = 0, \quad \alpha, \beta = 1, 2; \quad e_{33} = u_{3,3}, \quad e_{23} = \frac{1}{2} u_{2,3}, \quad e_{31} = \frac{1}{2} u_{1,3}, \tag{4.3}$$

$$E_\alpha = 0, \quad E_3 = -\chi_{,3}, \tag{4.4}$$

and, with regard to (4.3), (4.4),

$$X_{11} = E_{1133} u_{3,3} - p_{311} E_3, \tag{4.5}$$

$$X_{22} = E_{1133} u_{3,3} - p_{311} E_3, \tag{4.6}$$

$$X_{33} = E_{3333} u_{3,3} - p_{333} E_3, \tag{4.7}$$

$$X_{23} = E_{2323} u_{2,3}, \tag{4.8}$$

$$X_{31} = E_{2323} u_{1,3}, \tag{4.9}$$

$$X_{12} = 0, \tag{4.10}$$

$$D_1 = p_{113} u_{1,3}, \tag{4.11}$$

$$D_2 = p_{113} u_{2,3}, \tag{4.12}$$

$$D_3 = p_{333} u_{3,3} + \varsigma_{33} E_3. \tag{4.13}$$

Substituting (4.5)-(4.10) into (4.1) we have

$$(E_{2323} u_{1,3})_{,3} + \rho o^2 u_1 = -\Phi_1, \tag{4.14}$$

$$(E_{2323} u_{2,3})_{,3} + \rho o^2 u_1 = -\Phi_2, \tag{4.15}$$

taking into account (4.4),

$$(E_{3333}u_{3,3})_{,3} - (p_{333}E_3)_{,3} + \rho o^2 u_3 = -\Phi_3, \tag{4.16}$$

If we substitute (4.13) into (4.2) we get

$$(p_{333}u_{3,3})_{,3} + (\varsigma_{33}E_3)_{,3} = f_e. \tag{4.17}$$

We rewrite (4.16) and (4.17), by virtue of (4.4), as follows

$$(E_{3333}u_{3,3})_{,3} + (p_{333}\chi_{,3})_{,3} + \rho o^2 u_3 = -\Phi_3 \tag{4.18}$$

and

$$(p_{333}u_{3,3})_{,3} - (\varsigma_{33}\chi_{,3})_{,3} = f_e, \tag{4.19}$$

respectively.

## 5. Boundary value problems

Let us consider the static case. Then from (3.20)-(3.22), assuming $o = 0$, we get the following governing system

$$(E_{2323}h_1h_2v_{\alpha 00,3})_{,3} = -\Phi_{\alpha 00} - \sum_{j=1}^{3} \overset{(j)}{X}_{\alpha j}, \quad \alpha = 1,2, \tag{5.1}$$

$$(E_{3333}h_1h_2v_{300,3})_{,3} + (p_{333}h_1h_2\tilde{\chi}_{00,3})_{,3} = -\Phi_{300} - \sum_{j=1}^{3} \overset{(j)}{X}_{3j}, \tag{5.2}$$

$$(p_{333}h_1h_2v_{300,3})_{,3} - (\varsigma_{33}h_1h_2\tilde{\chi}_{00,3})_{,3} = f_{e00} - \sum_{j=1}^{3} \overset{(j)}{D}_j. \tag{5.3}$$

Equations (5.1) are independent of each other and of system (5.2), (5.3). Their general solutions have the form

$$v_{\alpha 00}(x_3) = -\int_{L^*}^{x_3} (E_{2323}h_1h_2)^{-1}(\tau)d\tau \int_{L^*}^{\tau} [\Phi_{\alpha 00}(t) + \sum_{j=1}^{3} \overset{(j)}{X}_{\alpha j}(t)]dt$$

$$+c_\alpha^2 + c_\alpha^1 \int_{L^*}^{x_3} (E_{2323}h_1h_2)^{-1}(\tau)d\tau, \quad \alpha = 1,2, \quad x_3 \in ]0, L[, \tag{5.4}$$

$$L^*, c_\alpha^1, c_\alpha^2 = const, \quad L^* \in ]0, L[,$$

provided the integrals exist; e.g., they exist if integrands are continuous on $[0, L]$.

Since we allow

$$E_{2323}h_1h_2\Big|_{x_3=0,L} = 0,$$

the integrals may exist as improper ones for $x_3 \to 0+, L-$, in general.

**Statement 5.1.** *If*

$$\int_0^{L^*} (E_{2323}h_1h_2)^{-1}(\tau)d\tau < +\infty \tag{5.5}$$

*and*

$$\int_{L^*}^{L} (E_{2323}h_1h_2)^{-1}(\tau)d\tau < +\infty, \tag{5.6}$$

*then the BVP (Problem D) under BCs*

$$v_{\alpha 00}(0) = c_0^\alpha, \quad c_0^\alpha = const, \quad \alpha = 1, 2, \tag{5.7}$$

*and*

$$v_{\alpha 00}(L) = c_L^\alpha, \quad c_L^\alpha = const, \quad \alpha = 1, 2, \tag{5.8}$$

*for solutions* $v_{\alpha 00} \in C^0[0, L],\ E_{2323}h_1h_2v_{\alpha 00,3} \in C^1(]0, L[)$ *of equations* (5.1) *is well posed. Unique explicit solutions have the form* (5.4)*. Constants* $c_\alpha^\beta,\ \alpha, \beta = 1, 2,$ *we easily calculate from BCs* (5.7)*,* (5.8)*.*

**Statement 5.2.** *If* $E_{2323}h_1h_2v_{\alpha 00,3} \in C^1([0, L])$*, then a mixed weighted BVP is well-posed when either BCs* (5.7) *or BCs* (5.8) *are replaced by BCs*

$$\lim_{x_3 \to 0+} X_{3\alpha 00}(x_3) = \lim_{x_3 \to 0+} (E_{2323}h_1h_2v_{\alpha 00,3})(x_3) = d_0^\alpha, \ \ d_0^\alpha = const, \ \ \alpha = 1, 2, \tag{5.9}$$

*or*

$$\lim_{x_3 \to L-} X_{3\alpha 00}(x_3) = \lim_{x_3 \to L-} (E_{2323}h_1h_2v_{\alpha 00,3})(x_3) = d_L^\alpha, \ \ d_L^\alpha = const, \ \ \alpha = 1, 2, \tag{5.10}$$

*respectively.*

Constants $c_\alpha^1$ we easily find from the weighted Neumann BCs (5.9), (5.10), while constants $c_\alpha^2$ we easily calculate from the Dirichlet conditions (5.7), (5.8). Namely, under BCs (5.7), (5.10),

$$c_\alpha^1 = d_L^\alpha + \int_{L^*}^{L} [\Phi_\alpha(t) + \sum_{i=1}^{3} \overset{(j)}{X}_{\alpha j}(t)]dt, \ \ \alpha = 1, 2, \tag{5.11}$$

$$c_\alpha^2 = c_0^\alpha + \int\limits_{L^*}^{0} (E_{2323}h_1h_2)^{-1}(\tau)d\tau \int\limits_{L^*}^{\tau} [\Phi_\alpha(t) + \sum_{j=1}^{3} \overset{(j)}{X}_{\alpha j}(t)]dt$$

$$-\Big\{d_L^\alpha + \int\limits_{L^*}^{L} [\Phi_\alpha(t) + \sum_{j=1}^{3} \overset{(j)}{X}_{\alpha j}(t)]dt\Big\} \int\limits_{L^*}^{0} (E_{2323}h_1h_2)^{-1}(\tau)d\tau, \quad \alpha = 1, 2; \quad (5.12)$$

under BCs (5.8), (5.9) we similarly calculate $c_\alpha^\beta, \quad \alpha, \beta = 1, 2$.

It is easily seen from (5.4) that on both the ends of the bar we can not set the weighted Neumann conditions, since after differentiation disappears $c_\alpha^2, \quad \alpha = 1, 2$, and by two constants $c_\alpha^1, \quad \alpha = 1, 2$, we are not able to satisfy four BCs.

The above mixed BVP will be called the Dirichlet – Weighted Neumann Problem.

**Statement 5.3.** *If* (5.5) *is fulfilled but*

$$\int\limits_{L^*}^{L} (E_{2323}h_1h_2)^{-1}(\tau)d\tau = +\infty, \quad (5.13)$$

*then the BVP (Problem E) under BCs* (5.7) *and condition ("O" is the Landau symbol)*

$$v_{\alpha 00}(x_3) = O(1); \quad x_3 \to L-, \quad \alpha = 1, 2, \quad (5.14)$$

*for bounded, solutions* $v_{\alpha 00} \in C([0, L[), \ E_{2323}h_1h_2v_{\alpha 00,3} \in C^1(]0, L[$ *of equations* (5.1) *is well posed. Unique explicit solutions have the form* (5.4), *where* $c_\alpha^1 = 0$, $\alpha = 1, 2$ *(otherwise solutions will be unbounded and some conditions of vanishing of* $\Phi_\alpha$, $X_{\overset{(\pm)}{\nu_\alpha}}$ *as* $x_2 \to L-$ *are required as well), while* $c_\alpha^2, \quad \alpha = 1, 2$, *we easily calculate from BCs* (5.7). *If* $E_{2323}h_1h_2v_{\alpha 00,3} \in C(]0, L])$, (5.14) *we can replace by the weighted Neumann BCs* (5.10).

We analogously prove the following

**Statement 5.4** *If* (5.6) *is fulfilled but*

$$\int\limits_{0}^{L^*} (E_{2323}h_1h_2)^{-1}(\tau)d\tau = +\infty, \quad (5.15)$$

*then the BVP (Problem E) under BCs* (5.8) *and*

$$v_{\alpha 00}(x_3) = O(1); \quad x_3 \to 0+, \quad \alpha = 1, 2, \quad (5.16)$$

*for bounded solutions* $v_{\alpha 00} \in C([0, L]), \ E_{2323}h_1h_2v_{300,3} \in C^1(]0, L[)$ *of equations* (5.1) *is well posed. If* $E_{2323}h_1h_2v_{\alpha 00,3} \in C([0, L[)$, (5.16) *we can replace by the* *weighted Neumann BCs* (5.9).

We solve all the BVPs in the explicit forms (5.4). Here we omit simple calculations of constants $c_{\alpha}^{\beta},\ \alpha\beta = 1,2$.

Now we proceed to studing of system (5.2), (5.3). Let

$$p_{333}(x_3) = K\varsigma_{33}(x_3), \quad K = const. \tag{5.17}$$

If $p_{333}$ and $\varsigma_{33}$ are constants, such a constant $K$ always exists.

Multiplying (5.3) by $K$ and summing the obtained equation and equation (5.2), by virtue of (5.17), we get

$$[(E_{3333} + Kp_{333})h_1h_2v_{300,3}]_{,3} = -\Phi_3 - \sum_{j=1}^{3} \overset{(j)}{X}_{3j} + K(f_{e00} - \sum_{j=1}^{3} \overset{(j)}{D}_j), \tag{5.18}$$

which contains only one unknown function $v_{300}$ and we investigate BVPs for equation (5.18) similarly to equations (5.1). After having the solutions $v_{300}(x_3)$ of BVPs in the explicit forms, we substitute them into (5.3) and obtain the following equation

$$(\varsigma_{33}h_1h_2\tilde{\chi}_{00,3})_{,3} = (p_{333}h_1h_2v_{300,3})_{,3} - f_{e00} + \sum_{j=1}^{3} \overset{(j)}{D}_j \tag{5.19}$$

with respect to $\tilde{\chi}_{00}(x_3)$ with the known right-hand side. We investigate BVPs for equation (5.19) similarly to equations (5.1) and construct their solutions in the explicit form.

Here we confine ourselves to formulation of statements concerning well posed BVPs for system (5.2), (5.3), which we have reduced to successive consideration of equation (5.18) and equation (5.19).

Peculiarities of setting BCs for $v_{300}$ and $\tilde{\chi}_{00}$ depend on the conditions:

$$\int_0^{L^*} [(E_{3333} + Kp_{333})h_1h_2]^{-1}(\tau)d\tau < +\infty, \tag{5.20}$$

$$\int_0^{L^*} [(E_{3333} + Kp_{333})h_1h_2]^{-1}(\tau)d\tau = +\infty, \tag{5.21}$$

$$\int_{L^*}^{L} [(E_{3333} + Kp_{333})h_1h_2]^{-1}(\tau)d\tau < +\infty, \tag{5.22}$$

$$\int\limits_{L^*}^{L} [(E_{3333} + Kp_{333})h_1h_2]^{-1}(\tau)d\tau = +\infty, \tag{5.23}$$

$$\int\limits_{0}^{L^*} (\varsigma_{33}h_1h_2)^{-1}(\tau)d\tau < +\infty, \tag{5.24}$$

$$\int\limits_{0}^{L^*} (\varsigma_{33}h_1h_2)^{-1}(\tau)d\tau = +\infty, \tag{5.25}$$

$$\int\limits_{L^*}^{L} (\varsigma_{33}h_1h_2)^{-1}(\tau)d\tau < +\infty, \tag{5.26}$$

$$\int\limits_{L^*}^{L} (\varsigma_{33}h_1h_2)^{-1}(\tau)d\tau = +\infty. \tag{5.27}$$

**Statement 5.5.** *If* (5.20), (5.22), (5.24), *and* (5.26) *are fulfilled, then Problem D for system* (5.2), (5.3) *under BCs*

$$v_{300}(0) = c_0^3, \quad c_0^3 = const, \tag{5.28}$$

$$\tilde{\chi}_{00}(0) = c_0^4, \quad c_0^4 = const, \tag{5.29}$$

$$v_{300}(L) = c_L^3, \quad c_L^3 = const, \tag{5.30}$$

$$\tilde{\chi}_{00}(L) = c_L^4, \quad c_L^4 = const, \tag{5.31}$$

*in the class of functions* $v_{300}, \tilde{\chi}_{00} \in C[0, L]$,

$$(E_{3333} + Kp_{333})h_1h_2v_{300,3}, \ \ \varsigma_{33}h_1h_2\chi_{00,3} \in C^1(]0, L[), \tag{5.32}$$

*is well posed.*

**Statement 5.6.** *If either*

$$(E_{3333}+Kp_{333})h_1h_2v_{300,3}),\ \ \varsigma_{33}h_1h_2\chi_{00,3}\in C^1([0,L[)$$

*or*

$$(E_{3333}+Kp_{333})h_1h_2v_{300,3},\ \ \varsigma_{33}h_1h_2\chi_{00,3}\in C^1(]0,L]),$$

*then the Dirichlet – weighted Neumann problem for system* (5.2), (5.3) *is well posed when either BCs* (5.28), (5.29) {*provided* (5.22), (5.26) *are fulfilled and* $v_{300}$, $\chi_{00}\in C(]0,L])$} *or* (5.30), (5.31) {*provided* (5.20), (5.24) *are fulfilled and* $v_{300}$, $\chi_{00}\in C([0,L[)$} *are replaced by BCs*

$$\lim_{x_3\to 0+}[(E_{3333}+Kp_{333})h_1h_2v_{300,3}](x_3)=d_0^3,\ \ d_0^3=const, \tag{5.33}$$

$$\lim_{x_3\to 0+}(\varsigma_{33}h_1h_2\chi_{00,3}(x_3)=d_0^4,\ \ d_0^4=const, \tag{5.34}$$

*and*

$$\lim_{x_3\to L-}[(E_{3333}+Kp_{333})h_1h_2v_{300,3}](x_3)=d_L^3,\ \ d_L^3=const, \tag{5.35}$$

$$\lim_{x_3\to L-}(\varsigma_{33}h_1h_2\chi_{00,3}(x_3)=d_L^4,\ \ d_L^4=const, \tag{5.36}$$

*respectively.*

**Statement 5.7.** *If* (5.20), (5.24), (5.23), (5.27) *are fulfilled, then Problem E for system* (5.2), (5.3) *under BCs* (5.28), (5.29) *and conditions*

$$v_{300}(x_3)=O(1),\ \ x_3\to L-, \tag{5.37}$$

$$\tilde{\chi}_{00}(x_3)=O(1),\ \ x_3\to L-, \tag{5.38}$$

*in the class of bounded functions*

$$v_{300},\tilde{\chi}_{00}\in C^0([0,L[)$$

*with* (5.32) *are well posed.*

*If*

$$(E_{3333}+Kp_{333})h_1h_2v_{300,3},\ \ \varsigma_{33}h_1h_2\tilde{\chi}_{00,3}\in C(]0,L]),$$

*conditions* (5.37), (5.38) *we can replace by the weighted Neumann BCs* (5.35), (5.36).

**Statement 5.8.** *If* (5.21), (5.25), (5.22), (5.26) *are fulfilled, then Problem E for system* (5.2), (5.3) *under BCs* (5.30), (5.31) *and*

$$v_{300}(x_3) = O(1), \quad x_3 \to 0+, \tag{5.39}$$

$$\tilde{\chi}_{00}(x_3) = O(1), \quad x_3 \to 0+, \tag{5.40}$$

*in the class of bounded functions*

$$v_{300}, \tilde{\chi}_{00} \in C(]0, L])$$

*with* (5.32) *are well posed.*

*If*

$$(E_{3333} + Kp_{333})h_1 h_2 v_{300,3}; \ \ \varsigma_{33} h_1 h_2 \tilde{\chi}_{00,3} \in C([0, L[),$$

*we can replace conditions* (5.39), (5.40) *by the weighted Neumann BCs* (5.33), (5.34).

In the last case from (5.18), clearly

$$v_{300}(x_3) = -\int\limits_{L^*}^{x_3} [(E_{3333} + Kp_{333})h_1 h_2]^{-1}(\tau) d\tau \int\limits_{L^*}^{\tau} \Big\{ \Phi_3(t) + \sum_{j=1}^{3} \overset{(j)}{X}_j(t)$$

$$-K\Big[ f_{eoo}(t) - \sum_{j=1}^{3} \overset{(j)}{D}_j(t) \Big] \Big\} dt + c_3^1 \int\limits_{L^*}^{x_3} [(E_{3333} + Kp_{333})h_1 h_2]^{-1}(\tau) d\tau + c_3^2 \tag{5.41}$$

After integration from (5.19) we obtain

$$\varsigma_{33} h_1 h_2 \tilde{\chi}_{00,3} = p_{333} h_1 h_2 v_{300,3} + \int\limits_{L^*}^{x_3} \Big( \sum_{j=1}^{3} \overset{(j)}{D}_j - f_{eoo} \Big)(t) dt + c_4^1. \tag{5.42}$$

Hence

$$\tilde{\chi}_{00,3} = \varsigma_{33}^{-1} p_{333} v_{300,3} + (\varsigma_{33} h_1 h_2)^{-1}(x_3) \int\limits_{L^*}^{x_3} \Big( \sum_{j=1}^{3} \overset{(j)}{D}_j - f_{eoo} \Big)(t) dt$$

$$+ c_4^1 (\varsigma_{33} h_1 h_2)^{-1}(x_3)$$

and

$$\tilde{\chi}_{00} = \int\limits_{L^*}^{x_3} \varsigma_{33}^{-1}(\tau) p_{333}(\tau) \frac{dv_{300}}{d\tau} d\tau$$

$$+ \int\limits_{L^*}^{x_3} (\varsigma_{33} h_1 h_2)^{-1}(\tau) \int\limits_{L^*}^{\tau} \Big( \sum_{j=1}^{3} \overset{(j)}{D}_j - f_{eoo} \Big)(t) dt + c_4^1 \int\limits_{L^*}^{x_3} (\varsigma_{33} h_1 h_2)^{-1}(\tau) d\tau + c_4^2. \quad (5.43)$$

After differentiation of to (5.41), we have

$$v_{300,3} = -[(E_{3333} + Kp_{333}) h_1 h_2]^{-1}(x_3) \int\limits_{L^*}^{x_3} \Big\{ \Phi_3(t) + \sum_{j=1}^{3} \overset{(j)}{X}_j$$

$$-K \Big[ f_{eoo}(t) - \sum_{j=1}^{3} \overset{(j)}{D}_j(t) \Big] \Big\} dt + c_3^1 [(E_{3333} + Kp_{333}) h_1 h_2]^{-1}(x_3). \quad (5.44)$$

Substituting (5.44) into (5.43), we get

$$\tilde{\chi}_{00} = -\int\limits_{L^*}^{x_3} \Big\langle \varsigma_{33}^{-1}(\tau) p_{333}(\tau) [(E_{3333} + Kp_{333}) h_1 h_2]^{-1}(\tau) \int\limits_{L^*}^{\tau} \{ \Phi_3(t)$$

$$+ \sum_{j=1}^{3} \overset{(j)}{X}_j(t) - K \Big[ f_{eoo}(t) - \sum_{j=1}^{3} \overset{(j)}{D}_j(t) \Big] \Big\} dt$$

$$-c_3^1 \varsigma_{33}^{-1}(\tau) p_{333}(\tau) [(E_{3333} + Kp_{333}) h_1 h_2]^{-1}(\tau) \Big\rangle d\tau$$

$$+ \int\limits_{L^*}^{x_3} (\varsigma_{33} h_1 h_2)^{-1}(\tau) d\tau \int\limits_{L^*}^{\tau} \Big( \sum_{j=1}^{3} \overset{(j)}{D}_j - f_{eoo} \Big)(t) dt + c_4^1 \int\limits_{L^*}^{x_3} (\varsigma_{33} h_1 h_2)^{-1}(\tau) d\tau + c_4^2. \quad (5.45)$$

We omit simple but long calculation of constants $c_3^\beta, c_4^\beta,\ \beta = 1, 2,$, from the corresponding BCs.

Clearly, we easily formulate statements similar to above statements 5.5-5.8 under the following BCs:

(i) (5.28), (5.40), (5.30), (5.31);
(ii) (5.28), (5.29), (5.37), (5.31);
(iii) (5.28), (5.29), (5.30), (5.38);

(iv) (5.39), (5.29), (5.30), (5.31);
(v) (5.39), (5.29), (5.37), (5.31);
(vi) (5.39), (5.29), (5.30), (5.38);
(vii) (5.39), (5.40), (5.30), (5.38);
(viii) (5.33), (5.29), (5.30), (5.36), etc.
All the above BVPs we solve in the explicit forms.

**Remark 2:** If we compare equations (3.20) for $\alpha = 1, 2$ with (4.14) and (4.15), respectively, and equations (3.21) and (3.22) with equations (4.18) and (4.19), respectively, we easily see that system (4.14), (4.15), (4.18), (4.19) and system (3.20)-(3.22) coincide if we remove $h_1 h_2$ in the left-hand sides and sums in the right-hand sides of equations (3.20)-(3.22) and replace $v_{i00}$ and $\Phi_{i00}$, $i = \overline{1,3}$, by $u_i$ and $\Phi_i$, $i = \overline{1,3}$, respectively. Thus all results obtained in the present section concerning equations (3.20)-(3.22) with $o = 0$, i.e. in the static case, we easily reformulate for equations (4.14), (4.15), (4.18), (4.19), with $o = 0$.

Let us now consider the particular case of cusped, in general, prismatic piezoelectric prismatic bars with

$$E_{2323} h_1 h_2 = E_0 x_3^{\kappa_1} (L - x_3)^{\delta_1}, \quad E_0 = const > 0, \quad \kappa_1, \delta_1 = const \geq 0; \tag{5.46}$$

$$E_{3333} h_1 h_2 = \widehat{E}_0 x_3^{\widehat{\kappa}_1} (L - x_3)^{\widehat{\delta}_1}, \quad \widehat{E}_0 = const > 0, \quad \widehat{\kappa}_1, \widehat{\delta}_1 = const \geq 0; \tag{5.47}$$

$$p_{333} h_1 h_2 = p_0 x_3^{\kappa_2} (L - x_3)^{\delta_2}, \quad p_0 = const > 0, \quad \kappa_2, \delta_2 = const \geq 0; \tag{5.48}$$

$$\varsigma_{33} h_1 h_2 = \varsigma_0 x_3^{\kappa_3} (L - x_3)^{\delta_3}, \quad \varsigma_0 = const > 0, \quad \kappa_3, \delta_3 = const \geq 0; \tag{5.49}$$

$$E_{2323} = \widetilde{E}_0 x_3^{\widetilde{\kappa}_1} (L - x_3)^{\widetilde{\delta}_1}, \quad \widetilde{E}_0 = const > 0, \quad \widetilde{\kappa}_1, \widetilde{\delta}_1 = const \geq 0; \tag{5.50}$$

$$E_{3333} = \widetilde{\widetilde{E}}_0 x_3^{\widetilde{\widetilde{\kappa}}_1} (L - x_3)^{\widetilde{\widetilde{\delta}}_1}, \quad \widetilde{\widetilde{E}}_0 = const > 0, \quad \widetilde{\widetilde{\kappa}}_1, \widetilde{\widetilde{\delta}}_1 = const \geq 0; \tag{5.51}$$

$$p_{333} = \widetilde{p}_0 x_3^{\widetilde{\kappa}_2} (L - x_3)^{\widetilde{\delta}_2}, \quad \widetilde{p}_0 = const > 0, \quad \widetilde{\kappa}_2, \widetilde{\delta}_2 = const \geq 0; \tag{5.52}$$

$$\varsigma_{33} = \widetilde{\varsigma}_0 x_3^{\widetilde{\kappa}_3} (L - x_3)^{\widetilde{\delta}_3}, \quad \widetilde{\varsigma}_0 = const > 0, \quad \widetilde{\kappa}_3, \widetilde{\delta}_3 = const \geq 0; \tag{5.53}$$

$$h_\alpha = h_\alpha^0 x_3^{\kappa_4^\alpha} (L - x_3)^{\delta_4^\alpha}, \quad h_\alpha^0 = const > 0, \quad \kappa_4^\alpha, \delta_4^\alpha = const \geq 0, \quad \alpha = 1, 2. \tag{5.54}$$

Then

$$E_0 = \widetilde{E}_0 h_1^0 h_2^0, \quad \kappa_1 = \widetilde{\kappa}_1 + \kappa_4^1 + \kappa_4^2, \quad \delta_1 = \widetilde{\delta}_1 + \delta_4^1 + \delta_4^2; \tag{5.55}$$

$$\widehat{E}_0 = \widetilde{\widetilde{E}}_0 h_1^0 h_2^0, \quad \widehat{\kappa}_1 = \widetilde{\widetilde{\kappa}}_1 + \kappa_4^1 + \kappa_4^2, \quad \widehat{\delta}_1 = \widetilde{\widetilde{\delta}}_1 + \delta_4^1 + \delta_4^2; \tag{5.56}$$

$$p_{333} = p_0 h_1^0 h_2^0, \quad \kappa_2 = \widetilde{\kappa}_2 + \kappa_4^1 + \kappa_4^2, \quad \delta_2 = \widetilde{\delta}_2 + \delta_4^1 + \delta_4^2; \tag{5.57}$$

$$\varsigma_{33} = \varsigma_0 h_1^0 h_2^0, \quad \kappa_3 = \widetilde{\kappa}_3 + \kappa_4^1 + \kappa_4^2, \quad \delta_3 = \widetilde{\delta}_3 + \delta_4^1 + \delta_4^2. \tag{5.58}$$

In this case, on account of (5.46)-(5.58), we express criteria (5.5), (5.6), (5.13), (5.15), (5.20)-(5.27) of the well posedness of BVPs by means of exponents $\kappa_m^\alpha$, $\delta_m^\alpha$, $\alpha = 1, 2$, $m = \overline{1, 4}$. Namely, it follows that

$$\kappa_1 = \widetilde{\kappa}_1 + \kappa_4^1 + \kappa_4^2 < 1 \tag{5.59}$$

from (5.5), (5.46), (5.55);

$$\delta_1 = \widetilde{\delta}_1 + \delta_4^1 + \delta_4^2 < 1 \tag{5.60}$$

from (5.6), (5.46), (5.55);

$$\delta_1 = \widetilde{\delta}_1 + \delta_4^1 + \delta_4^2 \geq 1 \tag{5.61}$$

from (5.13), (5.46), (5.55);

$$\kappa_1 = \widetilde{\kappa}_1 + \kappa_4^1 + \kappa_4^1 \geq 1 \tag{5.62}$$

from (5.15), (5.46), (5.55);

$$\max\{\widehat{\kappa}_1 = \widetilde{\widetilde{\kappa}}_1 + \kappa_4^1 + \kappa_4^2, \ \ \kappa_2 = \widetilde{\kappa}_2 + \kappa_4^1 + \kappa_4^2\} < 1 \tag{5.63}$$

from (5.20), (5.47), (5.48), (5.56), (5.57);

$$\min\{\widehat{\kappa}_1, \kappa_2\} \geq 1 \ or \ \widehat{\kappa}_1 < 1, \kappa_2 \geq 1 \ or \ \widehat{\kappa}_1 \geq 1, \kappa_2 < 1 \tag{5.64}$$

from (5.21), (5.47), (5.48), (5.56), (5.57);

$$\max\{\widehat{\delta}_1 = \widetilde{\widetilde{\delta}}_1 + \kappa_4^1 + \kappa_4^2, \ \ \delta_2 = \widetilde{\delta}_2 + \kappa_4^1 + \kappa_4^2\} < 1 \tag{5.65}$$

from (5.22), (5.47), (5.48), (5.56), (5.57);

$$\min\{\widehat{\delta}_1, \delta_2\} \geq 1 \ or \ \widehat{\delta}_1 < 1, \delta_2 \geq 1 \ or \ \widehat{\delta}_1 \geq 1, \delta_2 < 1 \tag{5.66}$$

from (5.23), (5.47), (5.48), (5.56), (5.57);

$$\kappa_3 = \widehat{\kappa}_3 + \kappa_4^1 + \kappa_4^2 < 1 \tag{5.67}$$

from (5.24), (5.49), (5.58);

$$\kappa_3 = \widehat{\kappa}_3 + \kappa_4^1 + \kappa_4^2 \geq 1 \tag{5.68}$$

from (5.25), (5.49), (5.58);

$$\delta_3 = \widehat{\delta}_3 + \delta_4^1 + \delta_4^2 < 1 \tag{5.69}$$

from (5.26), (5.49), (5.58);

$$\delta_3 = \widehat{\delta}_3 + \delta_4^1 + \delta_4^2 \geq 1 \tag{5.70}$$

from (5.27), (5.49), (5.58);

So, for the case (5.50)-(5.54) in statements 5.1-5.8 conditions (5.5), (5.6), (5.13), (5.15), (5.20)-(5.27) we replace by (5.59)-(5.70), respectively.

$\kappa_4^\alpha$ and $\delta_4^\alpha$, $\alpha = 1, 2$, specify geometry of tapering at the cusped ends $x_3 = 0$ and $x_3 = L$, respectively (see Figures 5.1-5.3, where some examples of longitudinal sections by the plane $x_2 = 0$ (or $x_1 = 0$) of the cusped bars are shown. $\overset{(\pm)}{T}_{1(2)}$ are tangents at the point $O$ to the curves obtained by longitudinal sections by the above plane of the face surfaces $x_3 = \overset{(\pm)}{h}_{1(2)}(x_3)$; we have the similar picture at the end $x_3 = L$ of the bar depending on $\delta_4^\alpha$, $\alpha = 1, 2$).

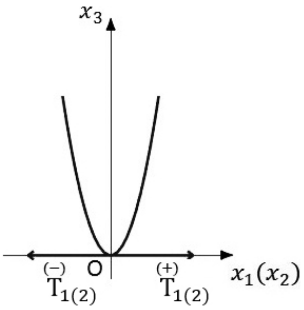


Figure 5.1. $0 < \kappa_4^1(\kappa_4^2) < 1$

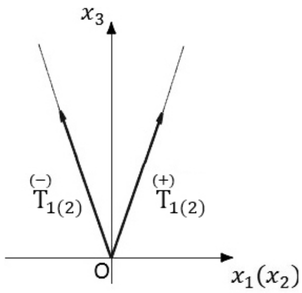


Figure 5.2. $\kappa_4^1(\kappa_4^2) = 1$

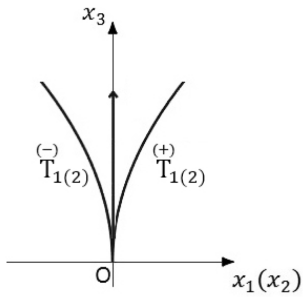


Figure 5.3. $\kappa_4^1(\kappa_4^2) > 1$

**Remark 3:** We now come back to time-harmonic motion. We consider as an example BVPs for equation (3.20). For the sake of simplicity we assume

$$\Phi_{\alpha 00} \equiv 0, \;\; X_{\alpha j}^{(j)} \equiv 0, \;\; \alpha = 1, 2. \;\; j = \overline{1, 3}.$$

Let further

$$\widetilde{\kappa}_1 = 0, \;\; \widetilde{\delta}_1 = 0, \;\; \delta_4^\alpha = 0$$

in (5.50), (5.54) and

$$h_0 := h_1^0 h_2^0, \;\; \kappa_4 := \kappa_4^1 + \kappa_4^2, \;\; \rho = \rho_0 x_3^{\kappa_4}.$$

Then from (3.20) we ontain the following Euler equation

$$x_3^2 v_{\alpha 00,33} + \kappa_4 x_3 v_{\alpha 00,3} + \widetilde{E}_0^{-1} h_0^{-1} \rho_0 o^2 v_{\alpha 00} = 0, \;\; \alpha = 1, 2 \tag{5.71}$$

Introducing new independent variable

$$t = \ln x_3 \quad (x_3 = e^t), \tag{5.72}$$

we rewrite (5.71) as

$$\frac{d^2 v_{\alpha 00}}{dt^2} + (\kappa_4 - 1))\frac{d v_{\alpha 00}}{dt} + \widetilde{E}_0^{-1} h_0^{-1} \rho_0 o^2 v_{\alpha 00} = 0, \;\; \alpha = 1, 2 \tag{5.73}$$

The characteristic algebraic equation of equation (5.73) has the form

$$\lambda^2 + (\kappa_4 - 1)\lambda + \widetilde{E}_0^{-1} h_0^{-1} \rho_0 o^2 = 0.$$

Its solutions are

$$\overset{(\pm)}{\lambda} = \frac{1 - \kappa_4 \pm \sqrt{D}}{2}$$

$$\text{for } \ D > 0, \ \text{ i.e., } \ o^2 < \frac{(\kappa_4 - 1)^2 \widetilde{E}_0 h_0}{4\rho_0}; \tag{5.74}$$

$$\lambda = \overset{(\pm)}{\lambda} = \frac{1 - \kappa_4}{2}$$

$$\text{for } \ D = 0, \ \text{ i.e., } \ o^2 = \frac{(\kappa_4 - 1)^2 \widetilde{E}_0 h_2}{4\rho_0}; \tag{5.75}$$

$$\overset{(\pm)}{\lambda} = \frac{1 - \kappa_4 \pm i\sqrt{-D}}{2} =: \alpha \pm i\beta$$

$$\text{for } \ D < 0, \ \text{ i.e., } \ o^2 > \frac{(\kappa_4 - 1)^2 \widetilde{E}_0 h_0}{4\rho_0}. \tag{5.76}$$

where the discriminant

$$D = (\kappa_4 - 1)^2 - 4\widetilde{E}^{-1} h_0^{-1} \rho_0 o^2 \tag{5.77}$$

and

$$\frac{(\kappa_4 - 1)^2 \widetilde{E}_0 h_0}{4\rho_0} \begin{cases} > 0 \; if \; \kappa_4 \neq 1, \\ = 0 \; if \; \kappa_4 = 1. \end{cases} \tag{5.78}$$

Correspondingly, we have the general solutions of equation (5.73) in the form of

$$v_{\alpha 00} = C_1 e^{\overset{(+)}{\lambda} t} + C_2 e^{\overset{(-)}{\lambda} t} \;\; for \;\; D > 0, \tag{5.79}$$

$$v_{\alpha 00} = C_1 e^{\lambda t} + C_2 t e^{\lambda t} \;\; for \;\; D > 0, \tag{5.80}$$

$$v_{\alpha 00} = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t) \;\; for \;\; D < 0, \tag{5.81}$$

$C_1$ and $C_2$ are arbitrary constants.

Substituting (5.72) into (5.78)-(5.80) we get the general solutions of equation (5.71) in the following forms

$$v_{\alpha 00} = C_1 x_3^{\overset{(+)}{\lambda}} + C_2 x_3^{\overset{(-)}{\lambda}} \quad for \;\; D > 0, \tag{5.82}$$

$$v_{\alpha 00} = C_1 x_3^{\lambda} + C_2 x_3^{\lambda} \ln x_3 \quad for \;\; D > 0, \tag{5.83}$$

$$v_{\alpha 00} = C_1 x_3^{\alpha} \cos(\beta \ln x_3) + C_2 x_3^{\alpha} \sin(\beta \ln x_3) \quad for \;\; D < 0, \tag{5.84}$$

respectively.

Let us now analyze which BCs can be satisfied by means of solutions (5.81)-(5.83).

From (5.81)-(5.83), taking into account (5.74)-(5.76), it is easily seen that:

(i) the Dirichlet type BVP [see BCs (5.7), (5.8)] is well posed

$$\text{if} \;\; 0 \le \kappa_4 = 1 - \sqrt{D} \quad for \;\; D > 0,$$

a unique solution has the form

$$v_{\alpha 00}(x_3) = L^{-\overset{(+)}{\lambda}} (c_L^{\alpha} - c_0^{\alpha}) x_3^{\overset{(+)}{\lambda}} + c_0^{\alpha};$$

(ii) the Keldysh type BVP [see BCs (5.16), (5.8)] in the class of bounded functions is well posed

$$if \;\; 1 - \sqrt{D} < \kappa_4 \le 1 + \sqrt{D} \quad for \;\; D > 0,$$

a unique solution has the form

$$v_{\alpha 00} = L^{-\overset{(+)}{\lambda}} c_L^{\alpha} x_3^{\overset{(+)}{\lambda}}$$

and has a unique solution up to the summand

$$C_1 \Big[ \cos(\beta \ln x_3) - \cot(\beta \ln L) \sin(\beta \ln x_3) \Big]$$

$$\Big\{ \text{or} \;\; C_2 \Big[ \sin(\beta \ln x_3) - \tan(\beta \ln L) \cos(\beta \ln x_3) \Big] \Big\}$$

$C_1$ (or $C_2$) is an arbitrary constant,

$$\text{if} \;\; \kappa_4 = 1 \;\; \text{for} \;\; D < 0;$$

(iii) the Keldysh type BVP [see BC (5.8)] in the class of unbounded as $x_3 \to 0_+$ functions is solvable

$$\text{if} \;\; \kappa_4 > 1 + \sqrt{D} \;\; \text{for} \;\; D > 0$$

and

$$\text{if } \kappa_4 > 1 \text{ for } D = 0 \text{ and } D < 0$$

and a solution is defined up to the summand

$$C_1\Big(x_3^{\overset{(+)}{\lambda}} - L^{\overset{(+)}{\lambda} - \overset{(-)}{\lambda}} x_3^{\overset{(-)}{\lambda}}\Big), \;\; C_1 x_3^{\lambda}\Big(1 - \frac{\ln x_3}{\ln L}\Big), \;\; \text{and}$$

$$C_1 x_3^{\alpha}\Big[\cos(\beta \ln x_3) - \cot(\beta \ln L)\sin(\beta \ln x_3)\Big],$$

$C_1$ is an arbitrary constant [or

$$C_2\Big(x_3^{\overset{(-)}{\lambda}} - L^{\overset{(-)}{\lambda} - \overset{(+)}{\lambda}} x_3^{\overset{(+)}{\lambda}}\Big), \;\; C_2 x_3^{\lambda}\Big(\ln \frac{x_3}{L}\Big), \;\; \text{and}$$

$$C_2 x_3^{\alpha}\Big[\sin(\beta \ln x_3) - \tan(\beta \ln L)\cos(\beta \ln x_3)\Big],$$

$C_2$ is an arbitrary constant], respectively;

(iv) If $0 \le \kappa_4 = 1 - \sqrt{D}$ for $D > 0$, then the Dirichlet-weighted Neumann mixed BVP [see BCs (5.9, 5.8] is well posed. In the case under consideration BC (5.9) has the following form

$$\lim_{x_3 \to 0+} X_{3\alpha 00} = \lim_{x_3 \to 0+} (E_0 h_0 x_3^{\kappa_4} v_{\alpha 00,3})(x_3) = d_0^{\alpha}, \;\; \alpha = 1, 2.$$

A unique solution looks like

$$v_{\alpha 00} = \sqrt{D}^{-1} E_0^{-1} h_0^{-1} d_0^{\alpha} (x_3^{\sqrt{D}} - L^{\sqrt{D}}) + c_L^{\alpha}$$

We investigate BVPs for system (3.21), (3.22) and the system corresponding to the case (5.46)-(5.54) in the same way and construct solutions in the explicit forms.

## 6. Mechanical interpretations

We first note that

$$X_{\underline{\alpha}\alpha 00} = E_{1133} h_1 h_2 v_{300,3} + p_{311} h_1 h_2 \tilde{X}_{00,3}, \;\; \alpha = 1, 2,$$

$$X_{3300} = E_{3333} h_1 h_2 v_{300,3} + p_{333} h_1 h_2 \tilde{X}_{00,3},$$

are the integrated total stress tensor components, depending on electric and mechanical displacements;

$$X_{3200} = X_{2300} = E_{2323} h_1 h_2 v_{200,3},$$

$$X_{1300} = X_{3100} = E_{2323} h_1 h_2 v_{100,3},$$

$$X_{2100} = X_{1200} = 0$$

are the integrated mechanical stress tensor components depending on mechanical displacement and independing of electric displacements;

$$\frac{1}{4h_1h_2}\Big(X_{3200} = X_{2300}, X_{1300} = X_{3100}, X_{1200} = 0\Big)(x_3, t)$$

are the mechanical stresses in $(0,0)$ model, while

$$\frac{1}{4h_1h_2}\Big(X_{\underline{\alpha}\alpha 00}, X_{3300}\Big), \quad \alpha = 1, 2$$

are the total stress tensor components in $(0,0)$ model;

$$u_i = \frac{1}{4} v_{i00}(x_3, t), \quad i = \overline{1,3},$$

are the displacement vector components in $(0,0)$ model;

$$D_i = \frac{1}{4h_1h_2} D_{i00}(x_3, t), \quad i = \overline{1,3},$$

are the electric displacement vector components in $(0,0)$ model;

$$\chi = \frac{1}{4}\widetilde{\chi}_{00}(x_3, t), \quad i = \overline{1,3},$$

is the electric potential in $(0,0)$ model;

$$E_i = \frac{1}{4h_1h_2} E_{i00}(x_3, t), \quad i = \overline{1,3},$$

are the electric field vector components in $(0,0)$ model.

**Problem 6.1** Let at the end $x_3 = 0$ of the bar BCs (5.9) and

$$X_{3300}(0) = \lim_{x_3 \to 0+}\Big(E_{3333} h_1 h_2 v_{300,3} + p_{333} h_1 h_2 \widetilde{\chi}_{00,3}\Big) = d_0^3, \tag{6.1}$$

$$D_{300}(0) = \lim_{x_3 \to 0+}\Big(p_{333} h_1 h_2 v_{300,3} - \varsigma_{33} h_1 h_2 \widetilde{\chi}_{00,3}\Big) = \widetilde{d}_0^3, \tag{6.2}$$

along with BCs (5.8), (5.30), (5.31) at another end $x_3 = L$ of the bar are prescribed, provided (5.6), (5.22), (5.26) hold. Clearly, by virtue of (5.17),

$$X_{3300}(0) + K D_{300}(0) = \lim_{x_3 \to 0+}\Big(E_{3333} h_1 h_2 v_{300,3} + K p_{333} h_1 h_2 v_{300,3}\Big) = d_0^3 + K\widetilde{d}_0^3. \tag{6.3}$$

**Solution of Problem 6.1.** We first find solution $v_{300}$ of equation (5.18) under BC (6.3) (see Statement 5.6). We substitute the solution obtained into the right-hand side of equation (5.19) and into BC (6.2). Therefore, from the last we get the following BC for $\widetilde{\chi}_{00}$:

$$\lim_{x_3 \to 0+} \varsigma_{33} h_1 h_2 \widetilde{\chi}_{00,3} = \lim_{x_3 \to 0+} p_{333} h_1 h_2 v_{300,3} - \widetilde{d}_0^3 \tag{6.4}$$

with known

$$\lim_{x_3 \to 0+} p_{333} h_1 h_2 v_{300,3}.$$

Note that now the right-hand side of equation (5.19) is known as well. Solving the last BVP, taking into account BC (5.31), we thus construct $\widetilde{\chi}_{00}$. Clearly, constructed $v_{300}$ and $\widetilde{\chi}_{00}$ satisfy (6.4). Therefore (6.2) is valid. Substracting from (6.3) multiplied by $K$ (6.2), we conclude that (6.1) is valid.

Adding explicit solutions $v_{\alpha 00}$, constructed in Statement 5.2, we arrive at the explicit solution of the BVP when at the end $x_3 = L$ of the bar the mechanical displacements $u_i = \frac{1}{4} v_{i00}$, $i = \overline{1,3}$ and electric potential $\chi = \frac{1}{4}\widetilde{\chi}_{00}$ are prescribed, while at the end $x_3 = 0$ $(0,0)$ moment $D_{300}$ of the electric displacement $D_3$ and $(0,0)$ moments $X_{3i00}$, $i = \overline{1,3}$, of the stress vector $(X_{31}, X_{32}, X_{33})$ [which in the case of the cusped end is concentrated along the segment (if either $2h_1(0) = 0$, $2h_2(0) \neq 0$ or $2h_1(0) \neq 0$, $2h_2(0) = 0$) or at the point $x_3 = 0$ (if $2h_1(0) = 0$, $2h_2(0) = 0$) force] are prescribed.

**Remark 4 :** Applying Statement 5.1 and Statement 5.5, we construct the explicit solution of BVP for the system (5.1)-(5.3) when at the ends $x_3 = 0$ and $x_3 = L$ of the bar displacements $u_i = \frac{1}{4} v_{i00}$, $i = \overline{1,3}$, and electric potential $\chi = \frac{1}{4}\widetilde{\chi}_{00}$ are prescribed.

We solve similarly other well posed BVPs and give corresponding mechanical interpretation (meaning).

**Remark 5 :** Since equations (5.1)-(5.3) contain the product $h_1 h_2$, characterizing tapering of the ends of the cusped bars with the rectangular cross-sections, and the constitutive coefficients

$$E_{2323}, \ \ E_{3333}, \ \ p_{333}, \ \ \varsigma_{33}$$

only as products

$$E_{2323} h_1 h_2, \ \ E_{3333} h_1 h_2, \ \ p_{333} h_1 h_2, \ \ \varsigma_{33} h_1 h_2,$$

the peculiarities of setting BCs caused by the cusped ends of the bar, provided the constitutive coefficients are constants, we may attain for the bar of the constant cross-section by appropriate choice of the variable constitutive coefficients and vice versa. In other words, in the case of bars of the constant cross-section we may achieve the intrinsic effect of peculiarities of setting BCs for cusped bars by appropriate selection of the non-homogeneous material.

**Remark 6 :** In the case of the infinite layer considered in Section 4 the stress-strain state along every fiber parallel to $x_3$-axis is the same.

## 7. Some bibliographical remarks

**Section 1.** contains well-known materials on geometry of cusped, in general, prismatic bars with rectangular cross-sections and on double mathematical moments of functions. For more details in this connection we refer to [1], [2], [11], [12].

**Section 2.** For the basics of piezoelectricity see, e.g., [13]-[15], [8], [9]; from the physical point of view see, e.g., a review in [3] and the references given there.

**Section 3.** For I. Vekua's dimension reduction method of constructing of differential hierarchical models for elastic shells and plates and its generalization for elastic bars see [16], [1], [2], [17]-[19]; for hierarchical models of piezoelectric viscoelastic Kelvin-Voight, in particular cusped, prismatic shells see [7]; piezoelectric plates are studied in [20]-[35]; $(0,0)$ hierarchical model of similarly constructed differential hierarchical models for piezoelectric bars is presented and investigated in the present paper in the case of piezoelectric transversely isotropic, in particular cusped, bars (see, sections 3, 5, 6); piezoelectric bars and beams are studied in [36]-[42].

## 8. Conclusions

(1) Differential hierarchical $(0,0)$ model for the transversely isotropic elastic piezoelectric bar in the case when the poling axis coincides with one of the material symmetry axis is constructed. In the case of cusped bars peculiarities of non-classical, in general, setting of BCs are fully investigated and all the Dirichlet, the Keldysh, the weighted, and the mixed BVPs are solved in the explicit form. Using a generalization of I. Vekua's dimension reduction method (see [16], [12], [2]), $(N_1, N_2)$ hierarchical models ($N_\alpha = 0, 1, ...,$ $\alpha = 1, 2$) can be constructed and investigation of similar to $(0,0)$ model non-classical BVPs can be carried out.

(2) Depending on the character of tapering of the bar, at the cusped end displacements and electric potential may be prescribed or not. In the last case we have no BCs at such an end of the bar.

(3) Forces applied at cusped ends of the bars are concentrated either at a line or at a point forces depending on that the cross-section of the bar degenerates into a segment of the line or into the point.

(4) Since governing equations contain the product $h_1 h_2$, characterizing tapering of the ends of the cusped bars with the rectangular cross-sections, and the constitutive coefficients

$$E_{2323}, \quad E_{3333}, \quad p_{333}, \quad \varsigma_{33}$$

only as products

$$E_{2323} h_1 h_2, \quad E_{3333} h_1 h_2, \quad p_{333} h_1 h_2, \quad \varsigma_{33} h_1 h_2,$$

the peculiarities of setting BCs caused by the cusped ends of the bar, provided the constitutive coefficients are constants, we may attain for the bar of the constant cross-section by appropriate choice of the variable constitutive coefficients and vice versa. In other words, in the case of bars of the constant cross-section we may achieve the intrinsic effect of peculiarities of setting BCs for cusped bars by appropriate selection of the non-homogeneous material.

(5) As far as the constitutive coefficients are functions of $x_3$, in general (in particular, power functions), the elastic piezo-electric bars under consideration are functionally graded ones (for elastic orthotropic and isotropic functionally graded beams see, e.g., [6]).

**_Acknowledgement_**

This work is supported by Shota Rustaveli National Science Foundation [217596, Construction and investigation of hierarchical models for thermoelastic piezoelectric structures].

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