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On Some Applications of Almost Invariant Sets

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Certain properties of almost invariant subsets of uncountable groups are considered and several applications of those subsets are given in measure theory and general topology.

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Dedicated to the 100-th Anniversary of Professor Sh. Pkhakadze.

In this paper we will be concerned with so-called almost invariant sets in infinite (mostly, in uncountable) groups.

Recall that the concept of these sets was first introduced by von Neumann and Halmos. There are various possibilities to define such sets. We will consider below only three variants of their definitions.

Let E be an infinite ground set, let G be a group of transformations of E, and let X be a subset of E.

We shall say that X is almost G-invariant (in the set-theoretical sense) if, for each transformation $g \in G$, one has

$$\mathrm{card}(g(X) \triangle X) < \mathrm{card}(E),$$

where the symbol $\triangle$ denotes the operation of symmetric difference of two sets.

Supposing that E is additionally endowed with some topology, we shall say that a set $Y \subset E$ is almost G-invariant (in the topological sense) if, for each transformation $g \in G$, the set $g(Y) \triangle Y$ is of first category in E.

Analogously, supposing that E is endowed with some measure μ, we shall say that a set $Z \subset E$ is almost G-invariant (in the measure-theoretical sense or, more precisely, with respect to μ) if, for each transformation $g \in G$, the set $g(Z) \triangle Z$ is of μ-measure zero in E.

If E itself is a group, then we may take as G the group of all left translations of E and, in this way, we get the notions of almost E-invariant subsets of E (respectively, in the set-theoretical, topological, and measure-theoretical sense).

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Example 1 Let $\mathbf{R}$ be the real line endowed with the usual Euclidean topology, let $\mathbf{c}$ denote the cardinality of the continuum, and let λ stand for the standard Lebesgue measure on $\mathbf{R}$. We put $E = \mathbf{R}$ and $G = \mathbf{R}$.

(a) Take any bounded λ-measurable set $X \subset \mathbf{R}$ of first category in $\mathbf{R}$ with $\lambda(X) > 0$ (there are many such sets in $\mathbf{R}$). Clearly, X is almost $\mathbf{R}$-invariant in the topological sense, but is not almost $\mathbf{R}$-invariant with respect to λ.

(b) Take any bounded set $Y \subset \mathbf{R}$ of second category and with $\lambda(Y) = 0$ (again, there are many such sets in $\mathbf{R}$). Obviously, Y is almost $\mathbf{R}$-invariant with respect to λ, but is not almost $\mathbf{R}$-invariant in the topological sense.

(c) Take any λ-nonmeasurable set $Z \subset \mathbf{R}$ with $\mathrm{card}(Z) < \mathbf{c}$ (there are models of set theory in which such a set Z does exist). Clearly, Z is almost $\mathbf{R}$-invariant in the set-theoretical sense, but is not almost $\mathbf{R}$-invariant with respect to λ.

(d) Take any second category set $T \subset \mathbf{R}$ with $\mathrm{card}(T) < \mathbf{c}$ (again, there are models of set theory in which a set T does exist). Obviously, T is almost $\mathbf{R}$-invariant in the set-theoretical sense, but is not almost $\mathbf{R}$-invariant in the topological sense.

(e) Assuming that Martin's Axiom holds true, it can easily be seen that every almost $\mathbf{R}$-invariant subset of $\mathbf{R}$ in the set-theoretical sense is simultaneously almost $\mathbf{R}$-invariant with respect to λ and almost $\mathbf{R}$-invariant in the topological sense.

The above example shows, in particular, that the three introduced concepts of almost invariance of subsets of a ground set E (equipped with a group G of its transformations) are mutually independent.

Suppose now that $(G, \cdot)$ is a topological group. The following three basic and well-known facts will be used below.

(i) If A is a second category subset of G having the Baire property, then the set $A \cdot A^{-1}$ is a neighborhood of the neutral element of G.

(ii) If both A and B are second category subsets of G having the Baire property, then there exists an element $g \in G$ such that $gA \cap B$ is also a set of second category in G.

(iii) If G is a σ-compact locally compact group and μ denotes the left Haar measure on G, then for any μ-measurable set $X \subset G$ the relation

$$\lim_{g \to e} \mu(X \triangle gX) = 0$$

holds true, where e denotes the neutral element of G.

Assertion (i) is known as the Banach–Kuratowski–Pettis theorem (cf. [9], [15]).

Assertion (ii) may be considered as the topological transitivity with respect to the family of all second category subsets of G possessing the Baire property.

Assertion (iii) is a standard fact of the theory of Haar measure (see, e.g., [5]).

Notice that (iii) trivially implies the so-called Steinhaus property of μ-measurable sets in G, where μ is again the left Haar measure on G. Namely, if $X \subset G$ is a μ-measurable set with $\mu(X) > 0$, then the set $X \cdot X^{-1}$ is a neighborhood of e. The latter fact may be treated as a measure-theoretical analogue of (i). Various statements closely connected with (iii) are discussed in [1], [2], [10], [11], [16].

In this article we would like to present several applications of almost invariant sets to certain questions which naturally arise in measure theory and general topology. As mentioned in the beginning, the notion of an almost invariant set was first considered by von Neumann and Halmos. Later on, Kakutani and Oxtoby used those sets for constructing non-separable translation invariant extensions of the

Lebesgue measure λ (see their work [8]; cf. [4], and also [5] where some generalization was given to a class of locally compact groups equipped with the Haar measure).

A radically different method for obtaining nonseparable $\mathbf{R}$-invariant extensions of λ was suggested by Kodaira and Kakutani [14] and is based on certain properties of group homomorphisms having thick (massive) graphs. In this connection, see also [11] and [13].

In the fundamental work by Pkhakadze [19] almost invariant sets were applied for giving a relative solution of the general measure extension problem restricted to the class of all nonzero σ-finite $\mathbf{R}$-invariant measures on $\mathbf{R}$. In order to formulate the main results of Pkhakadze, we need two widely known concepts of large cardinal numbers.

A cardinal $\mathbf{a}$ is called measurable in the Ulam sense if there is a probability diffused measure defined on the family of all subsets of $\mathbf{a}$.

A cardinal $\mathbf{b}$ is called two-valued measurable if there is a two-valued probability diffused measure defined on the family of all subsets of $\mathbf{b}$.

It is well known that the existence of cardinals measurable in the Ulam sense (as well as the existence of two-valued measurable cardinals) cannot be established within **ZFC** set theory (see, for instance, [7]).

We also need the concept of ergodicity (metrical transitivity) for invariant measures (see, e.g., [11]).

Let E be a ground set, let G be a group of transformations of E, and let μ be a G-invariant or, more generally, G-quasi-invariant measure on E. This μ is called G-ergodic (or metrically transitive with respect to G) if, for any set $X \in \mathrm{dom}(\mu)$, the relation

$$(\forall g \in G)(\mu(X \triangle g(X)) = 0)$$

implies the relation $\mu(X) = 0 \vee \mu(E \setminus X) = 0$ (cf. (ii) which is a topological version of ergodicity).

Actually, Pkhakadze proved in his extensive work [19] that the following two assertions are valid.

Theorem 1 : *Let μ be an arbitrary nonzero σ-finite $\mathbf{R}$-invariant measure on $\mathbf{R}$. If $\mathbf{c}$ is not measurable in the Ulam sense, then there exists an $\mathbf{R}$-invariant measure μ' on $\mathbf{R}$ which properly extends μ.*

Theorem 2 : *Let μ be an arbitrary nonzero σ-finite $\mathbf{R}$-invariant and $\mathbf{R}$-ergodic measure on $\mathbf{R}$. Then there exists an $\mathbf{R}$-invariant and $\mathbf{R}$-ergodic measure μ' on $\mathbf{R}$ which properly extends μ.*

Remarkably, the method of Pkhakadze is applicable to a much more general situation, where an uncountable group G equipped with a nonzero σ-finite G-invariant measure μ is taken instead of the pair $(\mathbf{R}, \lambda)$ (cf. [6]). So, the above results of Pkhakadze can be reformulated as follows.

1. Let G be an uncountable group endowed with a nonzero σ-finite G-invariant measure μ and let $\mathrm{card}(G)$ be strictly less than the smallest cardinal measurable in the Ulam sense. Then there exists a G-invariant measure μ' on G which properly extends μ.

2. Let G be an uncountable group endowed with a nonzero σ-finite G-invariant G-ergodic measure μ and let $\mathrm{card}(G)$ be strictly less than the smallest two-valued

measurable cardinal. Then there exists a G-invariant G-ergodic measure μ' on G which properly extends μ.

The key role in the process of obtaining these results is played by a certain partition of any uncountable group G.

Let us denote by α the least ordinal number with $\mathrm{card}(\alpha) = \mathrm{card}(G)$. It is not hard to see that there exists a family $\{G_\xi : \xi < \alpha\}$ of subgroups of G such that:

(1) for every $\xi < \alpha$, the group G_ξ properly contains the set $\cup\{G_\zeta : \zeta < \xi\}$;

(2) if $\xi < \alpha$, then $\mathrm{card}(G_\xi) \leq \mathrm{card}(\xi) + \omega$, where ω denotes, as usual, the least infinite cardinal number;

(3) $\cup\{G_\xi : \xi < \alpha\} = G$.

Now, putting $X_\xi = G_\xi \setminus \cup\{G_\zeta : \zeta < \xi\}$ for each $\xi < \alpha$, we come to the decomposition $\{X_\xi : \xi < \alpha\}$ of G which satisfies the relations:

(4) $\mathrm{card}(X_\xi) < \mathrm{card}(G)$ for all $\xi < \alpha$;

(5) for any set $\Xi \subset \alpha$ and for any element $g \in G$, the symmetric difference

$$(g \cdot (\cup\{X_\xi : \xi \in \Xi\}))\triangle(\cup\{X_\xi : \xi \in \Xi\})$$

has cardinality strictly less than $\mathrm{card}(G)$, so the set $\cup\{X_\xi : \xi \in \Xi\}$ is almost G-invariant in the set-theoretical sense.

Let us indicate some works in which the above-mentioned partition of G is applied in constructions of proper G-invariant extensions of nonzero σ-finite G-invariant measures: [4], [6], [8], [10], [11], [18], [19], [20]. We would like to especially notice that [20] is an extensive survey of this topic and some related ones.

To continue, let us recall a few definitions concerning density points (in the measure-theoretical sense) and their natural generalizations.

Let μ be a measure on $\mathbf{R}$ extending the standard Lebesgue measure λ, let $X \in \mathrm{dom}(\mu)$, and let $d \in [0, 1]$.

It is natural to say that X has upper μ-density d at a point $x \in \mathbf{R}$ if

$$\mathrm{limsup}_{\delta\to 0}\mu(X \cap I_\delta(x))/\mu(I_\delta(x)) = d,$$

where $I_\delta(x)$ stands for the interval in $\mathbf{R}$ with center x and diameter δ.

If the stronger condition

$$\lim_{\delta\to 0}\mu(X \cap I_\delta(x))/\mu(I_\delta(x)) = d$$

holds true, then we say that X has μ-density d at x.

In the latter case, for $d = 1$, we get the ordinary μ-density point x of X.

The classical Lebesgue theorem states that if Y is an arbitrary λ-measurable set in $\mathbf{R}$, then λ-almost all points of Y are its λ-density points (see, for instance, [12], [17]).

According to the classical Steinhaus theorem, if Z is a λ-measurable set in $\mathbf{R}$ of strictly positive measure, then the difference set

$$Z - Z = \{x - y : x \in Z,\ y \in Z\}$$

is a neighborhood of zero in $\mathbf{R}$, so $Z - Z$ has a nonempty interior (cf. (i), (iii)).

Various aspects of the Steinhaus theorem and its generalizations are considered in many works (see, e.g., [1], [2], [11], [16] and references therein). As a matter of fact, this theorem can easily be deduced from the above-mentioned Lebesgue theorem. The following statement illustrates a slightly more general fact.

Theorem 3 : *Let μ be an $\mathbf{R}$-invariant measure on $\mathbf{R}$ extending λ and let X be a μ-measurable set having upper μ-density $d > 1/2$ at some point $x \in \mathbf{R}$.*

Then there exists a real $\delta > 0$ such that

$$(\forall h \in \mathbf{R})(|h| < \delta \Rightarrow \mu(X \cap (h + X)) > 0).$$

Consequently, the difference set $X - X$ is a neighborhood of zero of $\mathbf{R}$.

Proof: Since $d > 1/2$, there is a real $\varepsilon > 0$ such that $d > 1/2 + \varepsilon/2$. Since X has upper μ-density d at x, we may choose an interval $I(x)$ with center x which satisfies the inequality

$$\mu(X \cap I(x)) \geq (1/2 + \varepsilon/2)\mu(I(x)).$$

Further, let $\delta > 0$ be so small that

$$\mu(I(x) \cup (I(x) + h)) < (1 + \varepsilon)\mu(I(x))$$

whenever $h \in \mathbf{R}$ and $|h| \leq \delta$. Clearly, for all these h we may write

$$(X \cap I(x)) \cap ((X + h) \cap (I(x) + h)) \subset (X \cap (X + h)).$$

Assuming for a moment that

$$\mu((X \cap I(x)) \cap ((X + h) \cap (I(x) + h))) = 0$$

and keeping in mind the $\mathbf{R}$-invariance of μ, we get

$$\mu(I(x) \cup (I(x) + h)) \geq \mu(X \cap I(x)) + \mu((X + h) \cap (I(x) + h)) \geq$$

$$2(1/2 + \varepsilon/2)\mu(I(x)) = (1 + \varepsilon)\mu(I(x)),$$

which is impossible. Therefore, $\mu(X \cap (X + h)) > 0$, and we conclude that $X - X$ contains the δ-neighborhood of zero of $\mathbf{R}$. This completes the proof. □

Actually, the assertion of Theorem 3 remains valid if there exist a point $x \in \mathbf{R}$ and an interval $I(x)$ such that $\mu(X \cap I(x)) > (1/2)\mu(I(x))$. The proof is essentially the same.

Example 2 It can be demonstrated that there is an $\mathbf{R}$-invariant measure ν on $\mathbf{R}$ satisfying the following conditions:

(a) ν is an extension of the Lebesgue measure λ;

(b) there exists a ν-measurable set Y with $\nu(Y) > 0$ such that $Y - Y$ is not a neighborhood of zero of $\mathbf{R}$ and at ν-almost all points $y \in Y$ the ν-density of Y at y is greater than or equal to $1/2$.

The construction of the measure ν is highly nontrivial and needs a certain modification of the method of Kodaira and Kakutani [14].

Let G be a topological group, let μ be a σ-finite G-invariant measure on G, and let X be a μ-measurable subset of G with $\mu(X) > 0$.

We shall say that this X has the Steinhaus property if there exists a neighborhood U_X of the neutral element of G such that

$$(\forall g \in U_X)(\mu(gX \cap X) > 0).$$

Accordingly, we shall say that a measure μ possesses the Steinhaus property if all μ-measurable sets X with $\mu(X) > 0$ have the Steinhaus property.

In view of Theorem 3 and the remark after it, if μ is an $\mathbf{R}$-invariant measure on $\mathbf{R}$ extending λ and such that, for any μ-measurable set X with $\mu(X) > 0$ there exist a point $x \in \mathbf{R}$ (depending on X) and an interval $I(x)$ satisfying

$$\mu(X \cap I(x)) > (1/2)\mu(I(x)),$$

then μ possesses the Steinhaus property.

Using almost invariant subsets of uncountable groups (in the set-theoretical sense), the following statement can be formulated and proved.

Theorem 4: *Let G be a topological group whose cardinality is strictly less than the first cardinal measurable in the Ulam sense, and let μ be a nonzero σ-finite G-invariant measure on G possessing the Steinhaus property.*

Then there exists a G-invariant measure μ' on G which properly extends μ and also possesses the Steinhaus property.

Proof: Since the argument is essentially the same as that of Pkhakadze [19], we only sketch it. We may suppose, without loss of generality, that the measure μ is complete and all sets in G whose cardinalities are strictly less than $\mathrm{card}(G)$ belong to $\mathrm{dom}(\mu)$. From the assumption of the theorem it follows that there is an almost G-invariant (with respect to μ) set T such that

$$T \notin \mathrm{dom}(\mu), \quad \mu_*(T) = 0,$$

where the symbol μ_* stands for the inner measure canonically associated with μ. Denote by $\mathcal{I}$ the G-invariant σ-ideal of subsets of G generated by $\{T\}$ and consider the G-invariant σ-algebra $\mathcal{A}$ of subsets of G generated by $\mathrm{dom}(\mu) \cup \{T\}$. All elements A of this $\mathcal{A}$ are representable in the form

$$A = (X \cup Z_1) \setminus Z_2,$$

where $X \in \mathrm{dom}(\mu)$ and $\{Z_1, Z_2\} \subset \mathcal{I}$. It can be checked that the functional

$$\mu' : \mathcal{A} \to \mathbf{R}$$

defined by the formula

$$\mu'(A) = \mu'((X \cup Z_1) \setminus Z_2) = \mu(X)$$

gives us a G-invariant measure on G which properly extends μ and also possesses the Steinhaus property. □

A concrete application of almost invariant sets to the Baire property in topological groups can also be given. In this connection, let us formulate the following statement.

Theorem 5: *Let $(G, \cdot)$ be a nondiscrete topological group of second category (not necessarily Hausdorff) and let* $\mathrm{card}(G)$ *be strictly less than the smallest two-valued measurable cardinal.*

Then there exists a subset of G which does not possess the Baire property.

Proof: Suppose to the contrary that all subsets of G possess the Baire property. We may assume, without loss of generality, that $\mathrm{card}(G)$ takes the minimum value. Taking into account fact (i) for topological groups, it is not difficult to verify the validity of these two relations:

(a) $\mathrm{card}(G) > \omega$;

(b) if $X \subset G$ and $\mathrm{card}(X) < \mathrm{card}(G)$, then X is of the first category in G.

Denoting by α the least ordinal number with $\mathrm{card}(\alpha) = \mathrm{card}(G)$ and following the method of Pkhakadze, we construct a partition $\{X_\xi : \xi < \alpha\}$ of G such that:

(c) $\mathrm{card}(X_\xi) < \mathrm{card}(G)$ for all $\xi < \alpha$;

(d) for any set $\Xi \subset \alpha$ and for any $g \in G$, the symmetric difference

$$(g \cdot (\cup\{X_\xi : \xi \in \Xi\})) \triangle (\cup\{X_\xi : \xi \in \Xi\})$$

has cardinality strictly less than $\mathrm{card}(G)$, so is of the first category in view of (b).

Now, for each subset A of α, we put $\mu(A) = 1$ if $\cup\{X_\xi : \xi \in A\}$ is not a set of the first category, and we put $\mu(A) = 0$ if $\cup\{X_\xi : \xi \in A\}$ is a set of the first category. Observe that, by virtue of (ii), we have

$$\mu(A) = 1 \Leftrightarrow \mu(\alpha \setminus A) = 0.$$

A straightforward verification shows that μ is a two-valued diffused probability measure defined on the whole power set of α, which contradicts the assumption that α is strictly less than the smallest two-valued measurable cardinal.

The obtained contradiction finishes the proof. □

Recall that a topological space E is resolvable (in the sense of Hewitt) if there exists a partition of E into two everywhere dense subsets of E.

According to this definition, a topological space E is irresolvable if the above-mentioned partition does not exist for E. Moreover, E is called totally irresolvable if no nonempty open subset of E is resolvable.

It can easily be checked that any set in a totally irresolvable topological space E has the Baire property in E.

If G is a topological group (not necessarily Hausdorff), then the following two assertions are equivalent:

(*) G is an irresolvable topological space;

(**) G is a totally irresolvable topological space.

Using these facts and Theorem 5 proved above, the next statement can be readily obtained.

Theorem 6: *Let $(G, \cdot)$ be a nondiscrete topological group (not necessarily Hausdorff) and let* $\mathrm{card}(G)$ *be strictly less than the smallest two-valued measurable cardinal.*

Then the disjunction of the following two assertions holds true:
(1) G is a resolvable topological space;
(2) G is a first category topological space.

Proof: Suppose that both (1) and (2) are false. This means that G is a second category topological group and, simultaneously, G is a totally irresolvable topological space. Therefore, all subsets of G possess the Baire property. But, keeping in mind the assumption on card(G) and Theorem 5, there must be sets in G which do not have the Baire property. The obtained contradiction gives us the required result. □

In connection with Theorem 5, the following question seems to be of some interest.

Let G be again a nondiscrete topological group of the second category whose cardinality is not a two-valued measurable cardinal and let $\{X_i : i \in I\}$ be a partition of G into sets of the first category in G. Does there exist a subset J of I such that the set $\cup\{X_j : j \in J\}$ lacks the Baire property?

This question remains open. It should be noticed that, without any assumption on the cardinality of G, the answer can be negative. To show this circumstance, we will use the construction of Frankiewicz and Kunen [3].

Example 3 Let κ be an infinite cardinal such that there exists a nontrivial ω_1-complete ultrafilter Φ in the power set of κ (this assumption is logically equivalent to the existence of a two-valued measurable cardinal number). Denote by $D(2^\kappa)$ a discrete commutative group of cardinality 2^κ and consider the topological product group $\Gamma = (D(2^\kappa))^\omega$ which is complete and metrizable. Obviously, we can put

$$\Phi = \{\Phi_\alpha : \alpha < 2^\kappa\},$$

where 2^κ is identified with the least ordinal number of cardinality 2^κ. Also, we may identify the group $D(2^\kappa)$ with the same least ordinal number of cardinality 2^κ. Now, for each ordinal $\alpha < \kappa$, we introduce the set

$$X_\alpha = \{x \in \Gamma : \alpha = \min(\cap\{\Phi_{x(n)} : n < \omega\})\}.$$

It is not hard to verify that:
(a) the family $\{X_\alpha : \alpha < \kappa\}$ is disjoint;
(b) all X_α are the first category subsets of Γ;
(c) $\cup\{X_\alpha : \alpha < \kappa\} = \Gamma$.

Moreover, as demonstrated in [3], if $A \in \Phi$, then the set $\cup\{X_\alpha : \alpha \in A\}$ is co-meager in Γ and if $A \notin \Phi$, then the set $\cup\{X_\alpha : \alpha \in A\}$ is of the first category in Γ. Therefore, for any $A \subset \kappa$, the union $\cup\{X_\alpha : \alpha \in A\}$ has the Baire property in Γ.

Another open problem arising in connection with the preceding results may be formulated as follows.

Let G be again a nondiscrete topological group of second category whose cardinality is not a two-valued measurable cardinal. What additional assumptions should be made on G to guarantee the existence of a subgroup of G lacking the Baire property?

Note that it can be proved in **ZFC** set theory that if G is a commutative nondiscrete locally compact group, then G contains a subgroup without the Baire property.

Let E be a set and let $\mathcal{I}$ be an ideal of subsets of E. We recall that a family $\mathcal{B} \subset \mathcal{I}$ is a base of $\mathcal{I}$ if, for any set $X \in \mathcal{I}$, there exists a set $Y \in \mathcal{B}$ such that $X \subset Y$.

Let E be a topological space. We recall that a family $\mathcal{V}$ of nonempty open subsets of E is a pseudo-base of E if, for any nonempty open set U in E, there exists a set $V \in \mathcal{V}$ such that $V \subset U$.

The following statement is valid.

Theorem 7: *Let E be an infinite topological space of second category and let G be a group of homeomorphisms of E onto itself, acting transitively and freely in E. Suppose also that there is a pseudo-base of E whose cardinality is strictly less than* card(E).

Then it is consistent with **ZFC** *theory that there exists a subset of E which does not possess the Baire property.*

Proof: Assume the Generalized Continuum Hypothesis (**GCH**) and denote $\mathrm{card}(E) = \mathbf{a}$. Without loss of generality, we may assume that $\mathbf{a} > \omega$. Let $\mathcal{V}$ be a pseudo-base of E with minimal cardinality $\mathbf{b}$. According to the condition of the theorem, $\mathbf{b} < \mathbf{a}$, so by virtue of **GCH** we have $2^{\mathbf{b}} \leq \mathbf{a}$. Observe also that E is a Baire topological space (because the group G acts transitively in E).

Let the symbol $\mathcal{K}$ stand for the σ-ideal of all first category subsets of E. It is not difficult to verify that there exists a base $\mathcal{B}$ of $\mathcal{K}$ for which the relation

$$\mathrm{card}(\mathcal{B}) \leq 2^{\mathbf{b}} \leq \mathbf{a}$$

holds true. Denote by $\mathcal{H}$ the family of all those sets $X \subset E$ which admit a representation in the form $X = V \setminus B$, where $V \in \mathcal{V}$ and $B \in \mathcal{B}$. Taking into account the above relation, we may write

$$\mathrm{card}(\mathcal{H}) \leq \mathbf{b} \cdot 2^{\mathbf{b}} \leq \mathbf{a}.$$

In addition, keeping in mind the condition of the theorem that G acts transitively and freely in E, we deduce that

$$(\forall X \in \mathcal{H})(\mathrm{card}(X) = \mathbf{a}).$$

Therefore, we can apply to the family $\mathcal{H}$ a Bernstein type transfinite construction (cf. [12], [15], [17]). In this way we obtain a partition of E into two sets Z_1 and Z_2 such that

$$(\forall X \in \mathcal{H})(Z_1 \cap X \neq \emptyset \ \& \ Z_2 \cap X \neq \emptyset).$$

Finally, it is not hard to see that neither Z_1 nor Z_2 possesses the Baire property in E.

Moreover, slightly modifying the above argument, it can be demonstrated that there are $2^{\mathbf{a}}$ many subsets of E, none of which has the Baire property. □

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