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Trigonometric Fractional Approximation of Stochastic Processes

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Here we encounter and study very general stochastic positive linear operators induced by general positive linear operators that are acting on continuous functions in the trigonometric sense. These are acting on the space of real fractionally differentiable stochastic processes. Under some very mild, general and natural assumptions on the stochastic processes we produce related trigonometric fractional stochastic Shisha-Mond type inequalities of L^q-type $1 \leq q < \infty$ and corresponding trigonometric fractional stochastic Korovkin type theorems. These are regarding the trigonometric stochastic q-mean fractional convergence of a sequence of stochastic positive linear operators to the stochastic unit operator for various cases. All convergences are produced with rates and are given via the trigonometric fractional stochastic inequalities involving the stochastic modulus of continuity of the α-th fractional derivatives of the engaged stochastic process, $\alpha > 0$, $\alpha \notin \mathbb{N}$. The impressive fact is that only two basic real Korovkin test functions assumptions, one of them trigonometric, are enough for the conclusions of our trigonometric fractional stochastic Korovkin theory. We give applications to stochastic Bernstein operators in the trigonometric sense.

Keywords: Stochastic positive linear operator, trigonometric fractional stochastic Korovkin theory and trigonometric fractional inequalities, trigonometric fractional stochastic Shisha-Mond inequality, stochastic modulus of continuity, stochastic process.

AMS Subject Classification: 26A33, 41A17, 41A25, 41A36, 60E15, 60H25

1. Introduction

Inspiration for this work comes from [2], [3], [14], [15], [16]. This work continues our earlier work [5], now at the stochastic fractional level. First we mention the foundations of Stochastic fractional calculus in the direct analytical sense, see [10], in section 2, this is in the Caputo fractional direction. In the section 3, about background, we talk about the q-mean $(1 \leq q < \infty)$ first modulus of continuity of a stochastic process and its upper bounds. There we describe completely our setting by introducing our stochastic positive linear operator M, see (15), which is based on the positive linear operator $\widetilde{L}$ from $C([-\pi, \pi])$ into itself. The operator M is acting on a wide space of Caputo fractional differentiable real valued stochastic processes X. See there Assumptions 3.6, 3.8, 3.9. We first give the main trigonometric pointwise fractional stochastic Shisha-Mond type inequalities ([14]), see Theorems 4.1, 4.2, and their several corollaries covering important trigonometric special cases.

We continue with trigonometric fractional q-mean uniform Shisha-Mond type inequalities, see Theorems 4.3, 4.4, and their interesting corollaries. All this theory is regarding the trigonometric fractional stochastic convergence of operators

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M to I (stochastic unit operator) given quantitatively with rates. An extensive trigonometric application about the stochastic Bernstein operators follows in full details. Based on our Shisha-Mond type inequalities of our main Theorems 4.1 - 4.4 we derive trigonometric pointwise and uniform Stochastic Korovkin theorems ([12]) on stochastic processes, see Theorems 6.1 - 6.4. The amazing fact here is, that basic conditions on operator $\widetilde{L}$ regarding two simple real valued functions, one of them trigonometric, that are not stochastic, are able to enforce fractional stochastic convergence on all stochastic processes we are dealing with; see Concepts 3.5 and Assumptions 3.6-3.9 on $[-\pi,\pi]$.

2. Foundation of Stochastic Fractional Calculus ([10])

Let $t \in [a,b] \subset \mathbb{R}$, $\omega \in \Omega$, where $(\Omega,\mathcal{F},P)$ is a probability space. Here $X(t,\omega)$ stands for a stochastic process. Case of $X(\cdot,\omega)$ being continuous on $[a,b]$, $\forall\, \omega \in \Omega$. Then by Caratheodory Theorem 20.15, p.156, [1], we get that $X(t,\omega)$ is jointly measurable.

Next we define the left and right respectively, Riemann-Liouville stochastic fractional integrals ([10]), where $\alpha > 0$ is not an integer:

$$I_{a+}^{\alpha} X(x,\omega) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} X(t,\omega)\, dt, \tag{1}$$

and

$$I_{b-}^{\alpha} X(x,\omega) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} X(t,\omega)\, dt, \tag{2}$$

$\forall\, x \in [a,b]$, $\forall\, \omega \in \Omega$, where Γ is the gamma function.

In the following important cases we prove that $I_{a+}^{\alpha}X$, $I_{b-}^{\alpha}X$ are stochastic processes:

i) Assume that $(\Omega,\mathcal{F},P)$ is a complete probability space, and that $(x-t)^{\alpha-1} X(t,\omega)$ is an integrable function on $[a,x]\times\Omega$, $\forall\, x \in [a,b]$, then by Fubini's theorem, [13], p. 269, $I_{a+}^{\alpha}X(x,\cdot)$ is an integrable function on Ω, $\forall\, x \in [a,b]$. Similarly, if $(t-x)^{\alpha-1} X(t,\omega)$ is an integrable function on $[x,b]\times\Omega$, $\forall\, x \in [a,b]$, then again by Fubini's theorem $I_{b-}^{\alpha}X(x,\cdot)$ is an integrable function on Ω, $\forall$ $x \in [a,b]$. That is $I_{a+}^{\alpha}X(x,\omega)$ and $I_{b-}^{\alpha}X(x,\omega)$ are stochastic processes.

ii) Assume a general probability space $(\Omega,\mathcal{F},P)$ and the Lebesgue measure spaces on $[a,x]$, $[x,b]$, $\forall\, x \in [a,b]$. These are clearly σ-finite measure spaces. We assume that the jointly measurable stochastic process $X(t,\omega) \geq 0$ on $[a,b]\times\Omega$, hence $(x-t)^{\alpha-1} X(t,\omega) \geq 0$ on $[a,x]\times\Omega$, and $(t-x)^{\alpha-1} X(t,\omega) \geq 0$ on $[x,b]\times\Omega$, $\forall$ $x \in [a,b]$, and both are jointly measurable. Then by Tonelli's theorem, [13], p. 270, we get that $I_{a+}^{\alpha}X(x,\cdot)$, $I_{b-}^{\alpha}X(x,\cdot)$ are measurable functions on Ω, $\forall\, x \in [a,b]$. That is $I_{a+}^{\alpha}X$, $I_{b-}^{\alpha}X$ are stochastic processes. The above facts provide the foundation of stochastic fractional calculus in the direct analytical sense. So it is not unusual to consider that $I_{a+}^{\alpha}X$, $I_{b-}^{\alpha}X$ are stochastic processes.

iii) Given that $X(\cdot,\omega)$ is in $L_1([a,b])$ then $I_{a+}^{\alpha}X(\cdot,\omega) \in L_1([a,b])$, $\forall\, \omega \in \Omega$, see [11], p. 13, and $I_{b-}^{\alpha}X(\cdot,\omega) \in L_1([a,b])$, $\forall\, \omega \in \Omega$, see [8], p. 334.

And given that $X(\cdot,\omega) \in L_\infty([a,b])$, then $I_{a+}^{\alpha}X(\cdot,\omega) \in C([a,b])$, when $0 < \alpha <$

1, and $I_{a+}^{\alpha}X(\cdot,\omega) \in AC([a,b])$ (absolutely continuous functions), when $\alpha \geq 1$, $\forall\ \omega \in \Omega$, see [6], p. 388. Similarly, if $X(\cdot,\omega) \in L_{\infty}([a,b])$, then $I_{b-}^{\alpha}X(\cdot,\omega) \in C([a,b])$, when $0 < \alpha < 1$, and $I_{b-}^{\alpha}X(\cdot,\omega) \in AC([a,b])$, when $\alpha \geq 1$, $\forall\ \omega \in \Omega$, see [9].

We need

Definition 2.1: ([10]) Let a non-integer $\alpha > 0$, $n = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ be the ceiling of the number), $t \in [a,b] \subset \mathbb{R}$, $\omega \in \Omega$, where $(\Omega, \mathcal{F}, P)$ is a general probability space. Here $X(t,\omega)$ stands for a stochastic process. Assume that $X(\cdot,\omega) \in AC^n([a,b])$ (spaces of functions $X(\cdot,\omega)$ with $X^{(n-1)}(\cdot,\omega) \in AC([a,b])$), $\forall\ \omega \in \Omega$.

We the call stochastic left Caputo fractional derivative

$$D_{*a}^{\alpha}X(x,\omega) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} X^{(n)}(t,\omega)\,dt, \tag{3}$$

$\forall\ x \in [a,b]$, $\forall\ \omega \in \Omega$.

And, we call the stochastic right Caputo fractional derivative

$$D_{b-}^{\alpha}X(x,\omega) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (z-x)^{n-\alpha-1} X^{(n)}(z,\omega)\,dz, \tag{4}$$

$\forall\ x \in [a,b]$, $\forall\ \omega \in \Omega$.

Remark 1: (to Definition 2.1) We further assume here that

$$\left|X^{(n)}(t,\omega)\right| \leq M^*, \ \ \forall\ (t,\omega) \in [a,b] \times \Omega,$$

where $M^* > 0$. Then, by (3), we have

$$|D_{*a}^{\alpha}X(x,\omega)| \leq \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \left|X^{(n)}(t,\omega)\right| dt \leq$$

$$\frac{M^*}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1}\,dt = \frac{M^*(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}.$$

That is

$$|D_{*a}^{\alpha}X(x,\omega)| \leq \frac{M^*(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \ \ \forall\ x \in [a,b]. \tag{5}$$

Also, from (4) we get

$$\left|D_{b-}^{\alpha}X(x,\omega)\right| \leq \frac{1}{\Gamma(n-\alpha)} \int_x^b (z-x)^{n-\alpha-1} \left|X^{(n)}(z,\omega)\right| dz \leq$$

$$\frac{M^*}{\Gamma(n-\alpha)} \int_x^b (z-x)^{n-\alpha-1}\,dz = \frac{M^*(b-x)^{n-\alpha}}{\Gamma(n-\alpha+1)}.$$

That is

$$\left|D_{b-}^{\alpha}X\left(x,\omega\right)\right| \leq \frac{M^{*}\left(b-x\right)^{n-\alpha}}{\Gamma\left(n-\alpha+1\right)}, \quad \forall\, x\in\left[a,b\right]. \tag{6}$$

By (1)-(4), it is not strange to assume that $D_{*a}^{\alpha}X$, $D_{b-}^{\alpha}X$ are stochastic processes.

3. Background (see also [10])

We need

Definition 3.1: We define the relative q-mean first modulus of continuity of stochastic process $X\left(t,\omega\right)$ by

$$\Omega_{1}\left(X,\delta\right)_{L^{q},[c,d]} :=$$

$$\sup\left\{\left(\int_{\Omega}\left|X\left(x,\omega\right)-X\left(y,\omega\right)\right|^{q}P\left(d\omega\right)\right)^{\frac{1}{q}} : x,y\in\left[c,d\right]\subseteq\left[a,b\right],\ \left|x-y\right|\leq\delta\right\}, \tag{7}$$

$\delta>0$, $1\leq q<\infty$.

Definition 3.2: Let $1\leq q<\infty$. Let $X\left(x,\omega\right)$ be a stochastic process. We call X a q-mean uniformly continuous stochastic process over $[a,b]$, iff $\forall\ \varepsilon>0\ \exists\ \delta>0$: whenever $|x-y|\leq\delta$; $x,y\in[a,b]$ implies that

$$\int_{\Omega}\left|X\left(x,s\right)-X\left(y,s\right)\right|^{q}P\left(ds\right)\leq\varepsilon. \tag{8}$$

We denote it as $X\in C_{\mathbb{R}}^{U_{q}}\left(\left[a,b\right]\right)$.

It holds

Proposition 3.3: *([5]) Let $X\in C_{\mathbb{R}}^{U_{q}}\left(\left[a,b\right]\right)$, then $\Omega_{1}\left(X,\delta\right)_{L^{q},[a,b]}<\infty$, any $\delta>0$.*

Besides it holds

Proposition 3.4: *([5]) Let $X\left(t,\omega\right)$ be a stochastic process from $[a,b]\times\left(\Omega,\mathcal{F},P\right)$ into $\mathbb{R}$. Then the following is true ($[c,d]\subseteq[a,b]$):*

(i) $\Omega_{1}\left(X,\delta\right)_{L^{q},[c,d]}$ *is nonnegative and nondecreasing in* $\delta>0$,

(ii) $\lim_{\delta\downarrow0}\Omega_{1}\left(X,\delta\right)_{L^{q},[c,d]}=\Omega_{1}\left(X,0\right)_{L^{q},[c,d]}=0$, *iff* $X\in C_{\mathbb{R}}^{U_{q}}\left(\left[c,d\right]\right)$,

(iii) $\Omega_{1}\left(X,\delta_{1}+\delta_{2}\right)_{L^{q},[c,d]}\leq\Omega_{1}\left(X,\delta_{1}\right)_{L^{q},[c,d]}+\Omega_{1}\left(X,\delta_{2}\right)_{L^{q},[c,d]}$, $\delta_{1},\delta_{2}>0$,

(iv) $\Omega_{1}\left(X,m\delta\right)_{L^{q},[c,d]}\leq m\Omega_{1}\left(X,\delta\right)_{L^{q},[c,d]}$, $\delta>0$, $m\in\mathbb{N}$,

(v) $\Omega_{1}\left(X,\lambda\delta\right)_{L^{q},[c,d]}\leq\lceil\lambda\rceil\,\Omega_{1}\left(X,\delta\right)_{L^{q},[c,d]}\leq\left(\lambda+1\right)\Omega_{1}\left(X,\delta\right)_{L^{q},[c,d]}$, $\lambda>0$, $\delta>0$,

(vi) $\Omega_{1}\left(X+Y,\delta\right)_{L^{q},[c,d]}\leq\Omega_{1}\left(X,\delta\right)_{L^{q},[c,d]}+\Omega_{1}\left(Y,\delta\right)_{L^{q},[c,d]}$, $\delta>0$,

(vii) $\Omega_{1}\left(X,\cdot\right)_{L^{q},[c,d]}$ *is continuous on* $\mathbb{R}_{+}$ *for* $X\in C_{\mathbb{R}}^{U_{q}}\left(\left[c,d\right]\right)$.

We give

Remark 1: (to Proposition 3.4) By Proposition 3.4 (v) we get

$$\Omega_1 (X, |x-y|)_{L^q,[c,d]} \leq \left\lceil \frac{|x-y|}{\delta} \right\rceil \Omega_1 (X, \delta)_{L^q,[c,d]}, \tag{9}$$

$\forall\ x, y \in [c,d]$, any $\delta > 0$.

We give

Remark 2: (continuation of Remark **1**) We assume again that $\left|X^{(n)}(t,\omega)\right| \leq M^*$, $\forall\ (t,\omega) \in [a,b] \times \Omega$, where $M^* > 0$. Let $\delta > 0$, $1 \leq q < \infty$. Then

$$\Omega_1 (D^{\alpha}_{*c} X, \delta)_{L^q,[c,d]} = \sup \left\{ \left(\int_{\Omega} |(D^{\alpha}_{*c} X)(x,\omega) - (D^{\alpha}_{*c} X)(y,\omega)|^q P(d\omega) \right)^{\frac{1}{q}} : \right.$$

$$\left. x, y \in [c,d] \subseteq [a,b],\ |x-y| \leq \delta \right\} \leq$$

$$\sup \left\{ \left(\int_{\Omega} (|(D^{\alpha}_{*c} X)(x,\omega)| + |(D^{\alpha}_{*c} X)(y,\omega)|)^q P(d\omega) \right)^{\frac{1}{q}} : \right.$$

$$\left. x, y \in [c,d] \subseteq [a,b],\ |x-y| \leq \delta \right\} \overset{([5])}{\leq} \tag{10}$$

$$\frac{M^*}{\Gamma(n-\alpha+1)} \sup \left\{ \left(\int_{\Omega} \left((x-c)^{n-\alpha} + (y-c)^{n-\alpha} \right)^q P(d\omega) \right)^{\frac{1}{q}} : \right.$$

$$\left. x, y \in [c,d] \subseteq [a,b],\ |x-y| \leq \delta \right\} =$$

$$\frac{M^*}{\Gamma(n-\alpha+1)} \sup \left\{ \left((x-c)^{n-\alpha} + (y-c)^{n-\alpha} \right) : \right.$$

$$\left. x, y \in [c,d] \subseteq [a,b],\ |x-y| \leq \delta \right\} \leq \frac{2M^*}{\Gamma(n-\alpha+1)} (d-c)^{n-\alpha}.$$

That is

$$\Omega_1 (D^{\alpha}_{*c} X, \delta)_{L^q,[c,d]} \leq \frac{2M^*}{\Gamma(n-\alpha+1)} (d-c)^{n-\alpha}, \tag{11}$$

where $a \leq c < d \leq b$.

Similarly, we have

$$\Omega_1\left(D_{d-}^{\alpha}X,\delta\right)_{L^q,[c,d]}\leq\frac{2M^*}{\Gamma(n-\alpha+1)}(d-c)^{n-\alpha}, \tag{12}$$

where $a\leq c<d\leq b$.

Next let $x_0\in[a,b]$, then

$$\sup_{x_0\in[a,b]}\Omega_1\left(D_{*x_0}^{\alpha}X,\delta\right)_{L^q,[x_0,b]}\leq\frac{2M^*}{\Gamma(n-\alpha+1)}\sup_{x_0\in[a,b]}(b-x_0)^{n-\alpha}$$

$$=\frac{2M^*}{\Gamma(n-\alpha+1)}(b-a)^{n-\alpha}. \tag{13}$$

Similarly, we have

$$\sup_{x_0\in[a,b]}\Omega_1\left(D_{x_0-}^{\alpha}X,\delta\right)_{L^q,[a,x_0]}\leq\frac{2M^*}{\Gamma(n-\alpha+1)}\sup_{x_0\in[a,b]}(x_0-a)^{n-\alpha}$$

$$=\frac{2M^*}{\Gamma(n-\alpha+1)}(b-a)^{n-\alpha}. \tag{14}$$

We need

Concepts 3.5 (see also [10]) Let $\widetilde{L}$ be a positive linear operator from $C([a,b])$ into itself. Let $X(t,\omega)$ be a stochastic process from $[a,b]\times(\Omega,\mathcal{F},P)$ into $\mathbb{R}$, where $(\Omega,\mathcal{F},P)$ is a probability space. Here we assume that for a non-integer $\alpha>0$, $\lceil\alpha\rceil=n$, $X(\cdot,\omega)\in AC^n([a,b])$ with $X^{(n)}(\cdot,\omega)\in L_\infty([a,b])$, $\forall\ \omega\in\Omega$.

We also assume for each $t\in[a,b]$ that $X^{(k)}(t,\cdot)$ is measurable for all $k=1,...,n-1$. Further we assume that $D_{*t}^{\alpha}X(z,\omega)$ is a stochastic process for $z\in[t,b]$, $\omega\in\Omega$, and $D_{t-}^{\alpha}X(z,\omega)$ is a stochastic process for $z\in[a,t]$, $\omega\in\Omega$; $\forall\ t\in[a,b]$.

Define

$$M(X)(t,\omega):=\widetilde{L}(X(\cdot,\omega))(t),\ \ \forall\ \omega\in\Omega,\ \forall\ t\in[a,b], \tag{15}$$

and assume that it is a random variable in ω. Clearly M is a positive linear operator on stochastic processes.

We mention

Assumption 3.6 (as in [10]) Let non-integer $\alpha>0$.

i) For any $t\in[a,b]$ we assume that $D_{*t}^{\alpha}X(z,\omega)$ is continuous in $z\in[t,b]$, uniformly with respect to $\omega\in\Omega$. I.e. $\forall\ \varepsilon>0\ \exists\ \delta>0$: whenever $|z_1-z_2|\leq\delta$; $z_1,z_2\in[t,b]$, then $|D_{*t}^{\alpha}X(z_1,\omega)-D_{*t}^{\alpha}X(z_2,\omega)|\leq\varepsilon$, $\forall\ \omega\in\Omega$.

We denote this by $D_{*t}^{\alpha}X\in C_{\mathbb{R}}^{U}([t,b])$, the space of continuous in x, uniformly with respect to ω, stochastic processes over $[t,b]$.

ii) For any $t\in[a,b]$ we assume that $D_{t-}^{\alpha}X(z,\omega)$ is continuous in $z\in[a,t]$, uniformly with respect to $\omega\in\Omega$. I.e. $\forall\ \varepsilon>0\ \exists\ \delta>0$: whenever $|z_1-z_2|\leq\delta$; $z_1,z_2\in[a,t]$, then $\left|D_{t-}^{\alpha}X(z_1,\omega)-D_{t-}^{\alpha}X(z_2,\omega)\right|\leq\varepsilon$, $\forall\ \omega\in\Omega$.

We denote this by $D_{t-}^{\alpha}X \in C_{\mathbb{R}}^{U}([a,t])$, the space of continuous in x, uniformly with respect to ω, stochastic processes over $[a,t]$.

Remark 3: Assumption 3.6 implies:

i) $D_{*t}^{\alpha}X(\cdot,\omega) \in C([t,b])$, $\forall\ \omega \in \Omega$, and $D_{*t}^{\alpha}X$ is q-mean uniformly continuous in $z \in [t,b]$, that is $D_{*t}^{\alpha}X \in C_{\mathbb{R}}^{U_q}([t,b])$, for any $1 \leq q < \infty$.

ii) $D_{t-}^{\alpha}X(\cdot,\omega) \in C([a,t])$, $\forall\ \omega \in \Omega$, and $D_{t-}^{\alpha}X$ is q-mean uniformly continuous in $z \in [a,t]$, that is $D_{t-}^{\alpha}X \in C_{\mathbb{R}}^{U_q}([a,t])$, for any $1 \leq q < \infty$.

We need

Definition 3.7: Denote by

$$(EX)(t) := \int_{\Omega} X(t,\omega) P(d\omega), \quad \forall\ t \in [a,b], \tag{16}$$

the expectation operator.

We make

Assumption 3.8 (as in [10]) We assume that

$$\left(E\left|X^{(k)}\right|^{q}\right)(t) < \infty, \ \ \forall\ t \in [a,b], \tag{17}$$

$q > 1$, for all $k = 0,1,...,n-1$, $n = \lceil \alpha \rceil$; $\alpha > 0$ non-integer.

We make

Assumption 3.9 (as in [10]) We assume that

$$\left(E\left|X^{(k)}\right|\right)(t) < \infty, \ \ \forall\ t \in [a,b], \tag{18}$$

for all $k = 0,1,...,n-1$, $n = \lceil \alpha \rceil$; $\alpha > 0$ non-integer.

We give

Remark 4: By the Riesz representation theorem ([13]) we can say that there exists μ_t unique, completed Borel measure on $[a,b]$ with

$$m_t := \mu_t([a,b]) = \widetilde{L}(1)(t) > 0, \tag{19}$$

such that

$$\widetilde{L}(f)(t) = \int_{[a,b]} f(x)\, d\mu_t(x), \quad \forall\ t \in [a,b], \forall\ f \in C([a,b]). \tag{20}$$

Consequently we have

$$M(X)(t,\omega) = \int_{[a,b]} X(x,\omega)\, d\mu_t(x), \quad \forall\ (t,\omega) \in [a,b] \times \Omega, \tag{21}$$

and X as in Concepts 3.5.

Here $\chi_{[\gamma,\delta]}(s)$ stands for the characteristic function on $[\gamma,\delta] \subseteq [a,b]$.

Notice that $(r > 0)$

$$\int_{[t,b]} (s-t)^r \mu_t(ds) = \int_{[a,b]} \chi_{[t,b]}(s) |s-t|^r \mu_t(ds) = \widetilde{L}\left(|\cdot - t|^r \chi_{[t,b]}(\cdot)\right)(t), \quad (22)$$

and

$$\int_{[a,t]} (t-s)^r \mu_t(ds) = \int_{[a,b]} \chi_{[a,t]}(s) |s-t|^r \mu_t(ds) = \widetilde{L}\left(|\cdot - t|^r \chi_{[a,t]}(\cdot)\right)(t). \quad (23)$$

Let now $n = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $k = 1, ..., n-1$. Then by Hölder's inequality we obtain

$$\left| \int_{[a,b]} (x-t)^k d\mu_t(x) \right| \leq \int_{[a,b]} |x-t|^k d\mu_t(x) \leq$$

$$\left(\int_{[a,b]} |x-t|^{\alpha+1} d\mu_t(x) \right)^{\left(\frac{k}{\alpha+1}\right)} (\mu_t([a,b]))^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}. \quad (24)$$

Therefore we have

$$\left\| \widetilde{L}\left((\cdot - t)^k\right)(t) \right\|_{\infty,[a,b]} \leq \left\| \widetilde{L}\left(|\cdot - t|^k\right)(t) \right\|_{\infty,[a,b]} \leq$$

$$\left\| \widetilde{L}\left(|\cdot - t|^{\alpha+1}\right)(t) \right\|_{\infty,[a,b]}^{\left(\frac{k}{\alpha+1}\right)} \left\| \widetilde{L}(1) \right\|_{\infty,[a,b]}^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}, \quad (25)$$

all $k = 1, ..., n-1$.

Besides, we observe

$$C([a,b]) \ni |\cdot - t|^{\alpha+1} \chi_{[a,t]}(\cdot) \leq |\cdot - t|^{\alpha+1}, \quad \forall\, t \in [a,b], \quad (26)$$

and

$$C([a,b]) \ni |\cdot - t|^{\alpha+1} \chi_{[t,b]}(\cdot) \leq |\cdot - t|^{\alpha+1}, \quad \forall\, t \in [a,b]. \quad (27)$$

By positivity of $\widetilde{L}$ we obtain

$$\left\| \widetilde{L}\left(|\cdot - t|^{\alpha+1} \chi_{[a,t]}(\cdot)\right)(t) \right\|_{\infty,[a,b]} \leq \left\| \widetilde{L}\left(|\cdot - t|^{\alpha+1}\right)(t) \right\|_{\infty,[a,b]} < \infty, \quad (28)$$

by $\widetilde{L}\left(|\cdot - t|^{\alpha+1}\right)(t)$ being continuous in $t \in [a,b]$, see p. 388 of [4],

and

$$\left\| \widetilde{L}\left(|\cdot - t|^{\alpha+1} \chi_{[t,b]}(\cdot)\right)(t) \right\|_{\infty,[a,b]} \leq \left\| \widetilde{L}\left(|\cdot - t|^{\alpha+1}\right)(t) \right\|_{\infty,[a,b]}. \quad (29)$$

Above (22)-(25) and (28), (29) can be used to derive convergence from (30), (32)-(34) next.

In this work we denote by $\widetilde{L}\left(\chi_{[a,t]}(\cdot)\right)(t) := \mu_t([a,t]) \leq \widetilde{L}(1)(t)$, and by $\widetilde{L}\left(\chi_{[t,b]}(\cdot)\right)(t) := \mu_t([t,b]) \leq \widetilde{L}(1)(t)$.

Next we mention the first main result from [10], on the quantitative stochastic fractional approximation regarding stochastic processes:

Theorem 3.10: *([10]) Suppose Concepts 3.5, Assumptions 3.6 and 3.8. Here the non-integer $\alpha > 0$ is such that $\alpha > \frac{1}{q}$, where $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, $\forall$ $t \in [a,b]$, we have:*

$$
\begin{aligned}
&\left(E\left(|M(X) - X|^q\right)(t)\right)^{\frac{1}{q}} \leq \left(\left(E|X|^q\right)(t)\right)^{\frac{1}{q}}\left|\widetilde{L}(1)(t) - 1\right| \\
&+\sum_{k=1}^{n-1} \frac{\left(\left(E\left|X^{(k)}\right|^q\right)(t)\right)^{\frac{1}{q}}}{k!}\left|\widetilde{L}\left((\cdot - t)^k\right)(t)\right| \\
&+\frac{2^{\frac{1}{p}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}(q+1)^{\frac{\alpha}{q(\alpha+1)}}} \qquad (30) \\
&\times\left\{\left\{\left(\widetilde{L}\left(\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)}\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}\right.\right. \\
&\times\left[\left(\widetilde{L}\left(\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}} + 1\right]^{\frac{1}{q}} \\
&\left.\times\,\Omega_1\left(D_{*t}^{\alpha}X, \left(\frac{1}{(q+1)}\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)}\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[t,b]}\right\} \\
&+\left\{\left(\widetilde{L}\left(\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)}\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}\right. \\
&\times\left[\left(\widetilde{L}\left(\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}} + 1\right]^{\frac{1}{q}} \\
&\left.\left.\times\,\Omega_1\left(D_{t-}^{\alpha}X, \left(\frac{1}{(q+1)}\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)}\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[a,t]}\right\}\right\}.
\end{aligned}
$$

Here we have

$$D^{\alpha}_{*t}X(t,\omega)=D^{\alpha}_{t-}X(t,\omega)=0, \tag{31}$$

$\forall\ \omega\in\Omega$, see [8], pp. 358-359.

We assume also

$$D^{\alpha}_{*t}X(s,\omega)=0,\text{ for } s<t,$$

and

$$D^{\alpha}_{t-}X(s,\omega)=0,\text{ for } s>t,$$

$\forall\ \omega\in\Omega$.

We mention the second main result from [10], the L_1-quantitative stochastic fractional approximation of the stochastic processes, it is the $q=1$ analog of Theorem 3.10. Inequality (32) is much simpler than (30).

Theorem 3.11: *([10]) Suppose Concepts 3.5, Assumptions 3.6 and 3.9. Here $\alpha>0$, $\alpha\notin\mathbb{N}$, $n=\lceil\alpha\rceil$. Then, $\forall\ t\in[a,b]$, we have*

$$E(|M(X)-X|)(t)\leq(E|X|)(t)\left|\widetilde{L}(1)(t)-1\right|$$

$$+\sum_{k=1}^{n-1}\frac{\left(E\left|X^{(k)}\right|\right)(t)}{k!}\left|\widetilde{L}\left((\cdot-t)^{k}\right)(t)\right|+\frac{1}{\Gamma(\alpha+1)}$$

$$\times\left\{\left[\left(\widetilde{L}\left(\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(|\cdot-t|^{\alpha+1}\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{\alpha}{\alpha+1}}\right.$$

$$\times\Omega_1\left(D^{\alpha}_{t-}X,\left(\widetilde{L}\left(|\cdot-t|^{\alpha+1}\chi_{[a,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[a,t]}$$

$$+\left[\left(\widetilde{L}\left(\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(|\cdot-t|^{\alpha+1}\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{\alpha}{\alpha+1}} \tag{32}$$

$$\left.\times\ \Omega_1\left(D^{\alpha}_{*t}X,\left(\widetilde{L}\left(|\cdot-t|^{\alpha+1}\chi_{[t,b]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[t,b]}\right\}.$$

We also mentiom a uniform norm result (denote $\|\cdot\|_{\infty,[a,b]}:=\|f\|_{\infty}$). It is based on Theorem 3.10.

Theorem 3.12: *([10]) Suppose Concepts 3.5, Assumptions 3.6 and 3.8. Here the non-integer $\alpha>0$ is such that $\alpha>\frac{1}{q}$, where $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Additionally* *assume that* $\left|X^{(n)}(t,\omega)\right| \leq M^{*}$, $\forall\ (t,\omega) \in [a,b] \times \Omega$, *where* $M^{*} > 0$. *Then*

$$\|E(|M(X) - X|^{q})\|_{\infty}^{\frac{1}{q}} \leq \|E(|X|^{q})\|_{\infty}^{\frac{1}{q}} \left\|\widetilde{L}(1) - 1\right\|_{\infty}$$

$$+\sum_{k=1}^{n-1} \frac{\left\|E\left(\left|X^{(k)}\right|^{q}\right)\right\|_{\infty}^{\frac{1}{q}}}{k!} \left\|\widetilde{L}\left((\cdot - t)^{k}\right)(t)\right\|_{\infty}$$

$$+\frac{2^{\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}(q+1)^{\frac{\alpha}{q(\alpha+1)}}}$$

$$\times\left[\left\|\widetilde{L}(1)\right\|_{\infty}^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}\left\|\widetilde{L}(1)\right\|_{\infty}^{\frac{1}{p}}$$

$$\times\left\{\left\{\left\|\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)}\chi_{[t,b]}(\cdot)\right)(t)\right\|_{\infty}^{\frac{\alpha}{q(\alpha+1)}}\right.\right. \tag{33}$$

$$\left.\times \sup_{t\in[a,b]} \Omega_{1}\left(D_{*t}^{\alpha}X, \left(\frac{1}{(q+1)}\left\|\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)}\chi_{[t,b]}(\cdot)\right)(t)\right\|_{\infty}\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^{q},[t,b]}\right\}$$

$$+\left\{\left\|\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)}\chi_{[a,t]}(\cdot)\right)(t)\right\|_{\infty}^{\frac{\alpha}{q(\alpha+1)}}\right.$$

$$\left.\left.\times \sup_{t\in[a,b]} \Omega_{1}\left(D_{t-}^{\alpha}X, \left(\frac{1}{(q+1)}\left\|\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)}\chi_{[a,t]}(\cdot)\right)(t)\right\|_{\infty}\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^{q},[a,t]}\right\}\right\}.$$

Based on Theorem 3.11 we mention the following uniform estimate:

Theorem 3.13: *([10]) Suppose Concepts 3.5, Assumptions 3.6 and 3.9. Here* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $n = \lceil\alpha\rceil$. *Additionally assume that* $\left|X^{(n)}(t,\omega)\right| \leq M^{*}$, $\forall\ (t,\omega) \in [a,b] \times \Omega$, *where* $M^{*} > 0$. *Then*

$$\|E(|M(X) - X|)\|_{\infty} \leq \|E(|X|)\|_{\infty}\left\|\widetilde{L}(1) - 1\right\|_{\infty}$$

$$+\sum_{k=1}^{n-1}\frac{\left\|E\left(\left|X^{(k)}\right|\right)\right\|_{\infty}}{k!}\left\|\widetilde{L}\left((\cdot - t)^{k}\right)(t)\right\|_{\infty} + \frac{\left[\left\|\widetilde{L}(1)\right\|_{\infty}^{\frac{1}{\alpha+1}} + \frac{1}{\alpha+1}\right]}{\Gamma(\alpha+1)}$$

$$\times \left\{ \left\| \widetilde{L} \left(|\cdot - t|^{\alpha+1} \chi_{[a,t]}(\cdot) \right)(t) \right\|_{\infty}^{\frac{\alpha}{\alpha+1}} \right.$$

$$\times \sup_{t\in[a,b]} \Omega_1 \left(D_{t-}^{\alpha} X, \left\| \widetilde{L} \left(|\cdot - t|^{\alpha+1} \chi_{[a,t]}(\cdot) \right)(t) \right\|_{\infty}^{\frac{1}{\alpha+1}} \right)_{L^1,[a,t]} \tag{34}$$

$$+ \left\| \widetilde{L} \left(|\cdot - t|^{\alpha+1} \chi_{[t,b]}(\cdot) \right)(t) \right\|_{\infty}^{\frac{\alpha}{\alpha+1}}$$

$$\left. \times \sup_{t\in[a,b]} \Omega_1 \left(D_{*t}^{\alpha} X, \left\| \widetilde{L} \left(|\cdot - t|^{\alpha+1} \chi_{[t,b]}(\cdot) \right)(t) \right\|_{\infty}^{\frac{1}{\alpha+1}} \right)_{L^1,[t,b]} \right\}.$$

We make

Remark 5: Next we specify $[a,b]$ as $[-\pi,\pi]$. Clearly then $\widetilde{L} : C([-\pi,\pi]) \to C([-\pi,\pi])$ is the positive linear operator on hand.

Here $n = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $k = 1,...,n-1$. Next we use Hölder's inequality. We notice that

$$\widetilde{L} \left(\left(\sin \left(\frac{|\cdot - t|}{4} \right) \right)^k \right)(t) = \int_{[-\pi,\pi]} \left(\sin \left(\frac{|x - t|}{4} \right)^k \right) d\mu_t(x)$$

$$\leq \left(\int_{[-\pi,\pi]} \left(\sin \left(\frac{|x - t|}{4} \right)^{\alpha+1} \right) d\mu_t(x) \right)^{\frac{k}{\alpha+1}} (\mu_t([-\pi,\pi]))^{\frac{\alpha+1-k}{\alpha+1}} \tag{35}$$

$$= \left(\widetilde{L} \left(\left(\sin \left(\frac{|\cdot - t|}{4} \right) \right)^{\alpha+1} \right)(t) \right)^{\frac{k}{\alpha+1}} \left(\widetilde{L}(1)(t) \right)^{\frac{\alpha+1-k}{\alpha+1}}.$$

That is

$$\widetilde{L} \left(\left(\sin \left(\frac{|\cdot - t|}{4} \right) \right)^k \right)(t) \tag{36}$$

$$\leq \left(\widetilde{L} \left(\left(\sin \left(\frac{|\cdot - t|}{4} \right) \right)^{\alpha+1} \right)(t) \right)^{\frac{k}{\alpha+1}} \left(\widetilde{L}(1)(t) \right)^{\frac{\alpha+1-k}{\alpha+1}},$$

for $k = 1,...,n-1$; true also for $q(\alpha+1)$ instead of $(\alpha+1)$, for any $1 < q < \infty$.

Next $\|\cdot\|_{\infty}$ denotes $\|\cdot\|_{\infty,[-\pi,\pi]}$.

Consequently, we have

$$\left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{k} \right)(t) \right\|_{\infty} \tag{37}$$

$$\leq \left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{\alpha+1} \right)(t) \right\|_{\infty}^{\frac{k}{\alpha+1}} \left\| \widetilde{L}(1)(t) \right\|_{\infty}^{\frac{\alpha+1-k}{\alpha+1}},$$

for $k = 1, ..., n-1$; (37) is true also for $q(\alpha+1)$ instead of $(\alpha+1)$, for any $1 < q < \infty$.

In this work we use a lot the following well known inequality:

$$|z| \leq \pi \sin\left(\frac{|z|}{2} \right), \ \forall\, z \in [-\pi, \pi]. \tag{38}$$

Notice that, for any $t \in [-\pi, \pi]$, we have $C([-\pi, \pi]) \ni |\cdot - t|\, \chi_{[-\pi,t]}(\cdot) \leq |\cdot - t| \in C([-\pi, \pi])$, therefore

$$C([-\pi, \pi]) \ni \left(\sin\left(\frac{|\cdot - t|\, \chi_{[-\pi,t]}(\cdot)}{4} \right) \right)^{\alpha+1} \leq \left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{\alpha+1} \in C([-\pi, \pi]). \tag{39}$$

Consequently, by positivity of $\widetilde{L}$ we obtain

$$\left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|\, \chi_{[-\pi,t]}(\cdot)}{4} \right) \right)^{\alpha+1} \right)(t) \right\|_{\infty} \leq \left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{\alpha+1} \right)(t) \right\|_{\infty}. \tag{40}$$

Similarly, for any $t \in [-\pi, \pi]$, we have $C([-\pi, \pi]) \ni |\cdot - t|\, \chi_{[t,\pi]}(\cdot) \leq |\cdot - t| \in C([-\pi, \pi])$, thus

$$C([-\pi, \pi]) \ni \left(\sin\left(\frac{|\cdot - t|\, \chi_{[t,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1} \leq \left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{\alpha+1} \in C([-\pi, \pi]). \tag{41}$$

Hence

$$\left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|\, \chi_{[t,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1} \right)(t) \right\|_{\infty} \leq \left\| \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{\alpha+1} \right)(t) \right\|_{\infty}. \tag{42}$$

So, if the right hand side of (40), (42) goes to zero, so do their left hand sides. Above in (39)-(42), one can use $q(\alpha+1)$ instead of $(\alpha+1)$, $1 < q < \infty$.

A further detailed analysis reveals:

We have $(1 \leq q < \infty)$

$$\widetilde{L}\left(|\cdot - t|^{q(\alpha+1)}\, \chi_{[t,\pi]}(\cdot) \right)(t) \overset{(20)}{=} \int_{[t,\pi]} (x-t)^{q(\alpha+1)}\, d\mu_t(x)$$

$$= 2^{q(\alpha+1)} \int_{[t,\pi]} \left(\frac{x-t}{2} \right)^{q(\alpha+1)} d\mu_t(x)$$

$$\overset{(38)}{\leq} (2\pi)^{q(\alpha+1)} \int_{[t,\pi]} \left(\sin\left(\frac{x-t}{4} \right) \right)^{q(\alpha+1)} d\mu_t(x) \tag{43}$$

$$= (2\pi)^{q(\alpha+1)} \int_{[t,\pi]} \left(\sin\left(\frac{|x-t|}{4} \right) \right)^{q(\alpha+1)} d\mu_t(x)$$

$$= (2\pi)^{q(\alpha+1)} \int_{[-\pi,\pi]} \left(\sin\left(\frac{|x-t|}{4} \right) \right)^{q(\alpha+1)} \chi_{[t,\pi]}(x)\, d\mu_t(x)$$

$$= (2\pi)^{q(\alpha+1)} \int_{[-\pi,\pi]} \left(\sin\left(\frac{|x-t|}{4} \chi_{[t,\pi]}(x) \right) \right)^{q(\alpha+1)} d\mu_t(x)$$

$$\overset{(20)}{=} (2\pi)^{q(\alpha+1)} \widetilde{L} \left(\left(\sin\left(\frac{|\cdot-t|}{4} \chi_{[t,\pi]}(\cdot) \right) \right)^{q(\alpha+1)} \right)(t).$$

That is, we have obtained

$$\widetilde{L} \left(|\cdot - t|^{q(\alpha+1)} \chi_{[t,\pi]}(\cdot) \right)(t) \leq (2\pi)^{q(\alpha+1)} \widetilde{L} \left(\left(\sin\left(\frac{|\cdot-t|}{4} \chi_{[t,\pi]}(\cdot) \right) \right)^{q(\alpha+1)} \right)(t). \tag{44}$$

Similarly, we get

$$\widetilde{L} \left(|\cdot - t|^{q(\alpha+1)} \chi_{[-\pi,t]}(\cdot) \right)(t) \leq (2\pi)^{q(\alpha+1)} \widetilde{L} \left(\left(\sin\left(\frac{|\cdot-t|}{4} \chi_{[-\pi,t]}(\cdot) \right) \right)^{q(\alpha+1)} \right)(t). \tag{45}$$

Above inequalities (44), (45) are valid for any $1 \leq q < \infty$.

Furthermore, we observe that

$$\left| \widetilde{L} \left((\cdot - t)^k \right)(t) \right| = \left| \int_{[-\pi,\pi]} (x-t)^k \, d\mu_t(x) \right| \leq \int_{[-\pi,\pi]} |x-t|^k \, d\mu_t(x)$$

$$= 2^k \int_{[-\pi,\pi]} \left(\frac{|x-t|}{2} \right)^k d\mu_t(x) \overset{(38)}{\leq} (2\pi)^k \int_{[-\pi,\pi]} \left(\sin\left(\frac{|x-t|}{4} \right) \right)^k d\mu_t(x) \tag{46}$$

$$= (2\pi)^k \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^k\right)(t).$$

That is

$$\left|\widetilde{L}\left((\cdot - t)^k\right)(t)\right| \leq (2\pi)^k \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^k\right)(t), \tag{47}$$

$\forall\ t \in [-\pi, \pi]$, all $k = 1, ..., n-1$.

4. Main Results

Next we present our first main result on the trigonometric quantitative stochastic fractional approximation of stochastic processes.

It is a pointwise result.

Theorem 4.1 : *Here $[a, b] = [-\pi, \pi]$. Suppose Concepts 3.5, Assumptions 3.6 and 3.8. Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, such that $\alpha > \frac{1}{q}$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, $\forall$ $t \in [-\pi, \pi]$, we have*

$$\left(E\left(|M(X) - X|^q\right)(t)\right)^{\frac{1}{q}} \leq \left(\left(E|X|^q\right)(t)\right)^{\frac{1}{q}} \left|\widetilde{L}(1)(t) - 1\right|$$

$$+\sum_{k=1}^{n-1} \frac{\left(\left(E\left|X^{(k)}\right|^q\right)(t)\right)^{\frac{1}{q}}}{k!} (2\pi)^k \widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^k\right)(t)$$

$$+\frac{2^{\alpha+\frac{1}{p}}\pi^\alpha}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}(q+1)^{\frac{\alpha}{q(\alpha+1)}}} \tag{48}$$

$$\times\left\{\left\{\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}\right.\right.$$

$$\times\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}} + 1\right]^{\frac{1}{q}}$$

$$\left.\times\,\Omega_1\left(D_{*t}^\alpha X, 2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[t,\pi]}\right\}$$

$$+\left\{\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}\right.$$

$$\times\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}$$

$$\left.\left.\times\ \Omega_1\left(D^{\alpha}_{t-}X, 2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[-\pi,t]}\right\}\right\}.$$

Proof: By Theorem 3.10 on $[-\pi,\pi]$ we get ($\forall\ t\in[-\pi,\pi]$) that

$$\left(E\left(|M(X)-X|^q\right)(t)\right)^{\frac{1}{q}}\leq\left(\left(E|X|^q\right)(t)\right)^{\frac{1}{q}}\left|\widetilde{L}(1)(t)-1\right|$$

$$+\sum_{k=1}^{n-1}\frac{\left(\left(E\left|X^{(k)}\right|^q\right)(t)\right)^{\frac{1}{q}}}{k!}\left|\widetilde{L}\left((\cdot-t)^k\right)(t)\right|$$

$$+\frac{2^{\frac{1}{p}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}(q+1)^{\frac{\alpha}{q(\alpha+1)}}}$$

$$\times\left\{\left\{\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(|\cdot-t|^{q(\alpha+1)}\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}\right.\right.$$

$$\times\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}$$

$$\left.\times\ \Omega_1\left(D^{\alpha}_{*t}X,\left(\frac{1}{(q+1)}\widetilde{L}\left(|\cdot-t|^{q(\alpha+1)}\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[t,\pi]}\right\}$$

$$+\left\{\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(|\cdot-t|^{q(\alpha+1)}\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}\right. \qquad (49)$$

$$\times\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}$$

$$\times\ \Omega_1\left(D^{\alpha}_{t-}X,\left(\frac{1}{(q+1)}\widetilde{L}\left(|\cdot-t|^{q(\alpha+1)}\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[-\pi,t]}\Bigg\}\Bigg\}=:(*).$$

Using (44), (45) and (47) we obtain:

$$(*)\leq\left((E|X|^q)(t)\right)^{\frac{1}{q}}\left|\widetilde{L}(1)(t)-1\right|$$

$$+\sum_{k=1}^{n-1}\frac{\left(\left(E\left|X^{(k)}\right|^q\right)(t)\right)^{\frac{1}{q}}}{k!}(2\pi)^k\,\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^k\right)(t)$$

$$+\frac{2^{\frac{1}{p}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}(q+1)^{\frac{\alpha}{q(\alpha+1)}}}$$

$$\times\Bigg\{\Bigg\{\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{p}}(2\pi)^{\alpha}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}$$

$$\times\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}} \tag{50}$$

$$\times\ \Omega_1\left(D^{\alpha}_{*t}X,2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[t,\pi]}\Bigg\}$$

$$+\Bigg\{\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{p}}(2\pi)^{\alpha}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}$$

$$\times\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}$$

$$\times\ \Omega_1\left(D^{\alpha}_{t-}X,2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[-\pi,t]}\Bigg\}\Bigg\},$$

proving the claim. □

We continue with the trigonometric L_1-quantitative stochastic fractional pointwise approximation of stochastic processes.

Theorem 4.2: *Here $[a,b]=[-\pi,\pi]$. Suppose Concepts 3.5, Assumptions 3.6 and 3.9. Here $\alpha>0$, $\alpha\notin\mathbb{N}$, $n=\lceil\alpha\rceil$. Then, $\forall\ t\in[-\pi,\pi]$, we have*

$$E\left(\left|M\left(X\right)-X\right|\right)(t)\leq\left(E\left|X\right|\right)(t)\left|\widetilde{L}\left(1\right)(t)-1\right|$$

$$+\sum_{k=1}^{n-1}\frac{\left(E\left|X^{(k)}\right|\right)(t)}{k!}\left(2\pi\right)^{k}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{k}\right)(t)+\frac{(2\pi)^{\alpha}}{\Gamma\left(\alpha+1\right)}\times$$

$$\left\{\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}\left(\cdot\right)\right)(t)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}\right.$$

$$\times\Omega_{1}\left(D_{t-}^{\alpha}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^{1},[-\pi,t]}$$

$$+\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}\left(\cdot\right)\right)(t)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left.\times\ \Omega_{1}\left(D_{*t}^{\alpha}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^{1},[t,\pi]}\right\}. \qquad (51)$$

Proof: By Theorem 3.11 for $[a,b]=[-\pi,\pi]$, and $\forall\ t\in[-\pi,\pi]$, we get

$$E\left(\left|M\left(X\right)-X\right|\right)(t)\leq\left(E\left|X\right|\right)(t)\left|\widetilde{L}\left(1\right)(t)-1\right|$$

$$+\sum_{k=1}^{n-1}\frac{\left(E\left|X^{(k)}\right|\right)(t)}{k!}\left|\widetilde{L}\left(\left(\cdot-t\right)^{k}\right)(t)\right|+\frac{1}{\Gamma\left(\alpha+1\right)}$$

$$\times\left\{\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}\left(\cdot\right)\right)(t)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(\left|\cdot-t\right|^{\alpha+1}\chi_{[-\pi,t]}\left(\cdot\right)\right)(t)\right)^{\frac{\alpha}{\alpha+1}}\right.$$

$$\times\Omega_{1}\left(D_{t-}^{\alpha}X,\left(\widetilde{L}\left(\left|\cdot-t\right|^{\alpha+1}\chi_{[-\pi,t]}\left(\cdot\right)\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^{1},[-\pi,t]} \qquad (52)$$

$$+\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}\left(\cdot\right)\right)\left(t\right)\right)^{\frac{1}{\alpha+1}}+\frac{1}{\left(\alpha+1\right)}\right]\left(\widetilde{L}\left(\left|\cdot-t\right|^{\alpha+1}\chi_{[t,\pi]}\left(\cdot\right)\right)\left(t\right)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left.\times\,\Omega_1\left(D^{\alpha}_{*t}X,\left(\widetilde{L}\left(\left|\cdot-t\right|^{\alpha+1}\chi_{[t,\pi]}\left(\cdot\right)\right)\left(t\right)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[t,\pi]}\right\}=:\left(**\right).$$

Using (44), (45) and (47) we obtain:

$$\left(**\right)\leq\left(E\left|X\right|\right)\left(t\right)\left|\widetilde{L}\left(1\right)\left(t\right)-1\right|$$

$$+\sum_{k=1}^{n-1}\frac{\left(E\left|X^{(k)}\right|\right)\left(t\right)}{k!}\left(2\pi\right)^k\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\right)\right)^k\right)\left(t\right)+\frac{\left(2\pi\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\times$$

$$\left\{\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}\left(\cdot\right)\right)\left(t\right)\right)^{\frac{1}{\alpha+1}}+\frac{1}{\left(\alpha+1\right)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\chi_{[-\pi,t]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)\left(t\right)\right)^{\frac{\alpha}{\alpha+1}}\right.$$

$$\times\Omega_1\left(D^{\alpha}_{t-}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\chi_{[-\pi,t]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)\left(t\right)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[-\pi,t]}$$

$$+\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}\left(\cdot\right)\right)\left(t\right)\right)^{\frac{1}{\alpha+1}}+\frac{1}{\left(\alpha+1\right)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\chi_{[t,\pi]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)\left(t\right)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left.\times\,\Omega_1\left(D^{\alpha}_{*t}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{\left|\cdot-t\right|}{4}\chi_{[t,\pi]}\left(\cdot\right)\right)\right)^{\alpha+1}\right)\left(t\right)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[t,\pi]}\right\}. \quad (53)$$

The claim is proved. □

We continue with trigonometric fractional uniform estimates ($\|\cdot\|_{\infty,[-\pi,\pi]} := \|\cdot\|_{\infty}$) in L_q-mean ($1 \leq q < \infty$).

Theorem 4.3 : *Here $[a,b]=[-\pi,\pi]$. Suppose Concepts 3.5, Assumptions 3.6 and 3.8. Let $\alpha>0$, $\alpha\notin\mathbb{N}$, $n=\lceil\alpha\rceil$, such that $\alpha>\frac{1}{q}$, where $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Additionally assume that $\left|X^{(n)}\left(t,\omega\right)\right|\leq M^*$, $\forall\left(t,\omega\right)\in[-\pi,\pi]\times\Omega$, where $M^*>0$. Then*

$$\left\|E\left(\left|M\left(X\right)-X\right|^q\right)\right\|_{\infty}^{\frac{1}{q}}\leq\left\|E\left(\left|X\right|^q\right)\right\|_{\infty}^{\frac{1}{q}}\left\|\widetilde{L}\left(1\right)-1\right\|_{\infty}$$

$$+\sum_{k=1}^{n-1}\frac{\left\|E\left(\left|X^{(k)}\right|^{q}\right)\right\|_{\infty}^{\frac{1}{q}}}{k!}(2\pi)^{k}\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{k}\right)(t)\right\|_{\infty}$$

$$+\frac{\pi^{\alpha}2^{\alpha+\frac{1}{p}}\left[\left\|\widetilde{L}(1)\right\|_{\infty}^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}\left\|\widetilde{L}(1)\right\|_{\infty}^{\frac{1}{p}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}(q+1)^{\frac{\alpha}{q(\alpha+1)}}} \tag{54}$$

$$\times\left\{\left\{\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_{\infty}^{\frac{\alpha}{q(\alpha+1)}}\right.\right.$$

$$\left.\times\sup_{t\in[-\pi,\pi]}\Omega_{1}\left(D_{*t}^{\alpha}X,2\pi\left(\frac{1}{(q+1)}\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_{\infty}\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^{q},[t,\pi]}\right\}$$

$$+\left\{\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_{\infty}^{\frac{\alpha}{q(\alpha+1)}}\right.$$

$$\left.\left.\times\sup_{t\in[-\pi,\pi]}\Omega_{1}\left(D_{t-}^{\alpha}X,2\pi\left(\frac{1}{(q+1)}\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_{\infty}\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^{q},[-\pi,t]}\right\}\right\}.$$

Proof: Based on Theorem 4.1, we take into account (13), (14) and the positivity of $\widetilde{L}$. □

We continue with the L_1-mean uniform stochastic fractional result.

Theorem 4.4: *Here $[a,b]=[-\pi,\pi]$. Suppose Concepts 3.5, Assumptions 3.6 and 3.9. Let $\alpha>0$, $\alpha\notin\mathbb{N}$, $n=\lceil\alpha\rceil$. Additionally assume that $\left|X^{(n)}(t,\omega)\right|\leq M^{*}$, $\forall$ $(t,\omega)\in[-\pi,\pi]\times\Omega$, where $M^{*}>0$. Then*

$$\|E(|M(X)-X|)\|_{\infty}\leq\|E(|X|)\|_{\infty}\left\|\widetilde{L}(1)-1\right\|_{\infty}$$

$$+\sum_{k=1}^{n-1}\frac{\left\|E\left(\left|X^{(k)}\right|\right)\right\|_{\infty}}{k!}(2\pi)^{k}\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{k}\right)(t)\right\|_{\infty}$$

$$+\frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)}\left[\left\|\widetilde{L}(1)\right\|_{\infty}^{\frac{1}{\alpha+1}}+\frac{1}{\alpha+1}\right]$$

$$\times\left\{\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}\right.$$

$$\times \sup_{t\in[-\pi,\pi]} \Omega_1 \left(D_{t-}^{\alpha} X, 2\pi \left\| \widetilde{L} \left(\left(\sin \left(\frac{|\cdot - t|}{4} \chi_{[-\pi,t]}(\cdot) \right) \right)^{\alpha+1} \right)(t) \right\|_{\infty}^{\frac{1}{\alpha+1}} \right)_{L^1,[-\pi,t]}$$

$$+ \left\| \widetilde{L} \left(\left(\sin \left(\frac{|\cdot - t|}{4} \chi_{[t,\pi]}(\cdot) \right) \right)^{\alpha+1} \right)(t) \right\|_{\infty}^{\frac{\alpha}{\alpha+1}}$$

$$\left. \times \sup_{t\in[-\pi,\pi]} \Omega_1 \left(D_{*t}^{\alpha} X, 2\pi \left\| \widetilde{L} \left(\left(\sin \left(\frac{|\cdot - t|}{4} \chi_{[t,\pi]}(\cdot) \right) \right)^{\alpha+1} \right)(t) \right\|_{\infty}^{\frac{1}{\alpha+1}} \right)_{L^1,[t,\pi]} \right\}. \quad (55)$$

Proof: Based on Theorem 4.2. We take into account (13), (14) and the positivity of $\widetilde{L}$. □

We continue with interesting corollaries.

Corollary 4.5: *All as in Theorem 4.1. Further assume $\widetilde{L}(1) = 1$ and $X^{(k)}(t_0, \omega) = 0$, $\forall\ \omega \in \Omega$, all $k = 1, ..., n-1$, for a fixed $t_0 \in [-\pi, \pi]$. Then*

$$\left(E\left(|M(X) - X|^q \right)(t_0) \right)^{\frac{1}{q}} \leq \frac{2^{\alpha+\frac{1}{p}} \pi^{\alpha}}{\Gamma(\alpha)\left(p(\alpha-1)+1 \right)^{\frac{1}{p}} (q+1)^{\frac{\alpha}{q(\alpha+1)}}} \quad (56)$$

$$\times \left\{ \left\{ \left(\widetilde{L}\left(\chi_{[t_0,\pi]}(\cdot) \right)(t_0) \right)^{\frac{1}{p}} \left(\widetilde{L} \left(\left(\sin \left(\frac{|\cdot - t_0|}{4} \chi_{[t_0,\pi]}(\cdot) \right) \right)^{q(\alpha+1)} \right)(t_0) \right)^{\frac{\alpha}{q(\alpha+1)}} \right. \right.$$

$$\times \left[\left(\widetilde{L}\left(\chi_{[t_0,\pi]}(\cdot) \right)(t_0) \right)^{\frac{1}{\alpha+1}} (q+1)^{\frac{\alpha}{(\alpha+1)}} + 1 \right]^{\frac{1}{q}}$$

$$\left. \times \Omega_1 \left(D_{*t_0}^{\alpha} X, 2\pi \left(\tfrac{1}{(q+1)} \widetilde{L} \left(\left(\sin \left(\tfrac{|\cdot - t_0|}{4} \chi_{[t_0,\pi]}(\cdot) \right) \right)^{q(\alpha+1)} \right)(t_0) \right)^{\frac{1}{q(\alpha+1)}} \right)_{L^q,[t_0,\pi]} \right\}$$

$$+ \left\{ \left(\widetilde{L}\left(\chi_{[-\pi,t_0]}(\cdot) \right)(t_0) \right)^{\frac{1}{p}} \left(\widetilde{L} \left(\left(\sin \left(\frac{|\cdot - t_0|}{4} \chi_{[-\pi,t_0]}(\cdot) \right) \right)^{q(\alpha+1)} \right)(t_0) \right)^{\frac{\alpha}{q(\alpha+1)}} \right.$$

$$\times \left[\left(\widetilde{L}\left(\chi_{[-\pi,t_0]}(\cdot) \right)(t_0) \right)^{\frac{1}{\alpha+1}} (q+1)^{\frac{\alpha}{(\alpha+1)}} + 1 \right]^{\frac{1}{q}}$$

$$\times\ \Omega_1\left(D^{\alpha}_{t_0-}X, 2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[-\pi,t_0]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t_0)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[-\pi,t_0]}\Bigg\}\Bigg\}.$$

Corollary 4.6: *All as in Theorem 4.1. Further assume that $\widetilde{L}(1)=1$ and $\frac{1}{q}<\alpha<1$ (i.e. $n=1$). Then, $\forall\ t\in[-\pi,\pi]$, we have*

$$(E(|M(X)-X|^q)(t))^{\frac{1}{q}} \leq \frac{2^{\alpha+\frac{1}{p}}\pi^{\alpha}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}(q+1)^{\frac{\alpha}{q(\alpha+1)}}} \tag{57}$$

$$\times\left\{\left\{\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}\right.\right.$$

$$\times\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}$$

$$\times\ \Omega_1\left(D^{\alpha}_{*t}X, 2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[t,\pi]}\Bigg\}$$

$$+\left\{\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{p}}\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{\alpha}{q(\alpha+1)}}\right.$$

$$\times\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}$$

$$\times\ \Omega_1\left(D^{\alpha}_{t-}X, 2\pi\left(\frac{1}{(q+1)}\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[-\pi,t]}\Bigg\}\Bigg\}.$$

Of great interest in Corollary 4.6 is the case of $\alpha=\frac{1}{2}$. We require that $q>2$. Due to lack of space we omit this statement.

Corollary 4.7: *All as in Theorem 4.2. Further assume that $\widetilde{L}(1)=1$ and $X^{(k)}(t_0,\omega)=0$, $\forall\ \omega\in\Omega$, all $k=1,...,n-1$, for a fixed $t_0\in[-\pi,\pi]$. Then*

$$E(|M(X)-X|)(t_0) \leq \frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)}$$

$$\times\left\{\left[\left(\widetilde{L}\left(\chi_{[-\pi,t_0]}(\cdot)\right)(t_0)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[-\pi,t_0]}(\cdot)\right)\right)^{\alpha+1}\right)(t_0)\right)^{\frac{\alpha}{\alpha+1}}\right.$$

$$\times\Omega_1\left(D^{\alpha}_{t_0-}X, 2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[-\pi,t_0]}(\cdot)\right)\right)^{\alpha+1}\right)(t_0)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[-\pi,t_0]}$$

$$+\left[\left(\widetilde{L}\left(\chi_{[t_0,\pi]}(\cdot)\right)(t_0)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[t_0,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t_0)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left.\times\,\Omega_1\left(D^{\alpha}_{*t_0}X, 2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t_0|}{4}\chi_{[t_0,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t_0)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[t_0,\pi]}\right\}. \quad (58)$$

Corollary 4.8: *All as in Theorem 4.2. Further assume that* $\widetilde{L}(1)=1$ *and* $0<\alpha<1$*. Then,* $\forall\ t\in[-\pi,\pi]$*, we have*

$$E\left(|M(X)-X|\right)(t)\leq\frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)}\times$$

$$\left\{\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}\right.$$

$$\times\Omega_1\left(D^{\alpha}_{t-}X, 2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[-\pi,t]}$$

$$+\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left.\times\,\Omega_1\left(D^{\alpha}_{*t}X, 2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[t,\pi]}\right\}. \quad (59)$$

Corollary 4.9: *All as in Theorem 4.2. Further assume* $\widetilde{L}(1)=1$*. Thenm* $\forall$ $t\in[-\pi,\pi]$*, we have*

$$E\left(|M(X)-X|\right)(t)\leq 2\sqrt{2}$$

$$\times\left\{\left[\left(\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{2}{3}}+\frac{2}{3}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{1}{3}}\right.$$

$$\times\Omega_1\left(D_{t-}^{\frac{1}{2}}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[-\pi,t]}$$

$$+\left[\left(\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{2}{3}}+\frac{2}{3}\right]\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{1}{3}}$$

$$\left.\times\,\Omega_1\left(D_{*t}^{\frac{1}{2}}X,2\pi\left(\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[t,\pi]}\right\}. \quad (60)$$

Next we give uniform norm results. They are based on Theorems 4.3, 4.4.

Corollary 4.10: *All as in Theorem 4.3. Assume further* $\frac{1}{q}<\alpha<1$ *and* $\widetilde{L}(1)=1$. *Then*

$$\|E(|M(X)-X|^q)\|_{\infty}^{\frac{1}{q}}\leq\frac{\pi^{\alpha}2^{\alpha+\frac{1}{p}}\left[(q+1)^{\frac{\alpha}{(\alpha+1)}}+1\right]^{\frac{1}{q}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}(q+1)^{\frac{\alpha}{q(\alpha+1)}}} \quad (61)$$

$$\times\left\{\left\{\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_{\infty}^{\frac{\alpha}{q(\alpha+1)}}\right.\right.$$

$$\left.\times\sup_{t\in[-\pi,\pi]}\Omega_1\left(D_{*t}^{\alpha}X,2\pi\left(\frac{1}{(q+1)}\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_{\infty}\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[t,\pi]}\right\}$$

$$+\left\{\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_{\infty}^{\frac{\alpha}{q(\alpha+1)}}\right.$$

$$\left.\left.\times\sup_{t\in[-\pi,\pi]}\Omega_1\left(D_{t-}^{\alpha}X,2\pi\left(\frac{1}{(q+1)}\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)\right\|_{\infty}\right)^{\frac{1}{q(\alpha+1)}}\right)_{L^q,[-\pi,t]}\right\}\right\}.$$

Corollary 4.11: *All as in Theorem 4.4. Assume further* $0<\alpha<1$ *and* $\widetilde{L}(1)=1$. *Then*

$$\|E(|M(X)-X|)\|_{\infty}\leq\frac{(2\pi)^{\alpha}(\alpha+2)}{\Gamma(\alpha+2)}$$

$$\times\left\{\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}\right.$$

$$\times\sup_{t\in[-\pi,\pi]}\Omega_1\left(D_{t-}^{\alpha}X,2\pi\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right)_{L^1,[-\pi,t]}$$

$$+\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}$$

$$\left.\times\sup_{t\in[-\pi,\pi]}\Omega_1\left(D_{*t}^{\alpha}X,2\pi\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right)_{L^1,[t,\pi]}\right\}. \quad (62)$$

Corollary 4.12: *All as in Theorem 4.4. Here* $\alpha=\frac{1}{2}$ *and* $\widetilde{L}(1)=1$. *Then*

$$\|E(|M(X)-X|)\|_{\infty}\leq\frac{10\sqrt{2}}{3}$$

$$\times\left\{\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right\|_{\infty}^{\frac{1}{3}}\right. \quad (63)$$

$$\times\sup_{t\in[-\pi,\pi]}\Omega_1\left(D_{t-}^{\frac{1}{2}}X,2\pi\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right\|_{\infty}^{\frac{2}{3}}\right)_{L^1,[-\pi,t]}$$

$$+\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right\|_{\infty}^{\frac{1}{3}}$$

$$\left.\times\sup_{t\in[-\pi,\pi]}\Omega_1\left(D_{*t}^{\frac{1}{2}}X,2\pi\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right\|_{\infty}^{\frac{2}{3}}\right)_{L^1,[t,\pi]}\right\}.$$

5. Application

Consider the Bernstein polynomials on $[-\pi,\pi]$ for $f \in C([-\pi,\pi])$:

$$B_N(f)(x) = \sum_{k=0}^{N} \binom{N}{k} f\left(-\pi + \frac{2\pi k}{N}\right)\left(\frac{x+\pi}{2\pi}\right)^k \left(\frac{\pi-x}{2\pi}\right)^{N-k}, \quad (64)$$

$N \in \mathbb{N}$, any $x \in [-\pi,\pi]$. There are positive linear operators from $C([-\pi,\pi])$ into itself.

Setting $g(t) = f(2\pi t - \pi)$, $t \in [0,1]$, we have $g(0) = f(-\pi)$, $g(1) = f(\pi)$, and

$$(B_N g)(t) = \sum_{k=0}^{N} \binom{N}{k} g\left(\frac{k}{N}\right) t^k (1-t)^{N-k} = (B_N f)(x), \quad x \in [-\pi,\pi]. \quad (65)$$

Here $x = \varphi(t) = 2\pi t - \pi$ is an $1-1$ and onto map from $[0,1]$ onto $[-\pi,\pi]$. Clearly here $g \in C([0,1])$.

Notice also that

$$\left(B_N\left((\cdot - x)^2\right)\right)(x) = \left[\left(B_N\left((\cdot - t)^2\right)\right)(t)\right](2\pi)^2 = \frac{(2\pi)^2}{N} t(1-t)$$

$$= \frac{(2\pi)^2}{N}\left(\frac{x+\pi}{2\pi}\right)\left(\frac{\pi-x}{2\pi}\right) = \frac{1}{N}(x+\pi)(\pi-x) \le \frac{\pi^2}{N}, \quad \forall\, x \in [-\pi,\pi].$$

I.e.

$$\left(B_N\left((\cdot - x)^2\right)\right)(x) \le \frac{\pi^2}{N}, \quad \forall\, x \in [-\pi,\pi]. \quad (66)$$

In particular

$$(B_N 1)(x) = 1, \quad \forall\, x \in [-\pi,\pi]. \quad (67)$$

Define the corresponding application of M by

$$\widetilde{B}_N(X)(t,\omega) := B_N(X(\cdot,\omega))(t) =$$

$$\sum_{k=0}^{N} \binom{N}{k} X\left(-\pi + \frac{2\pi k}{N}, \omega\right)\left(\frac{t+\pi}{2\pi}\right)^k \left(\frac{\pi-t}{2\pi}\right)^{N-k}, \quad (68)$$

$\forall\, N \in \mathbb{N}$, $\forall\, t \in [-\pi,\pi]$, $\forall\, \omega \in \Omega$, where X is a stochastic process. Clearly $\widetilde{B}_N$ is a stochastic process.

We give

Proposition 5.1: *Let $X(t,\omega)$ be a stochastic process from $[-\pi,\pi] \times (\Omega, \mathcal{F}, P)$ into $\mathbb{R}$, where $(\Omega, \mathcal{F}, P)$ is a probability space. Here $0 < \alpha < 1$ (i.e. $n = 1$) and*

$X(\cdot,\omega) \in AC([-\pi,\pi])$ with $X^{(1)}(\cdot,\omega) \in L_\infty([-\pi,\pi])$, $\forall\, \omega \in \Omega$. Further we assume that $D_{*t}^{\alpha} X(z,\omega)$ is a stochastic process for $z \in [t,\pi]$, $\omega \in \Omega$, and $D_{t-}^{\alpha} X(z,\omega)$ is a stochastic process for $z \in [-\pi,t]$, $\omega \in \Omega$; $\forall\ t \in [-\pi,\pi]$. For any $t \in [-\pi,\pi]$ we assume that $D_{*t}^{\alpha} X(z,\omega)$ is continuous in $z \in [t,\pi]$, uniformly with respect to $\omega \in \Omega$. And for any $t \in [-\pi,\pi]$ we assume that $D_{t-}^{\alpha} X(z,\omega)$ is continuous in $z \in [-\pi,t]$, uniformly with respect to $\omega \in \Omega$. Finally, we assume that $(E|X|)(t) < \infty$, $\forall$ $t \in [-\pi,\pi]$. Then, for any $t \in [-\pi,\pi]$, we have:

$$E\left(\left|\widetilde{B}_N(X) - X\right|\right)(t) \leq \frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)}$$

$$\times\left\{\left[\left(B_N\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}} + \frac{1}{(\alpha+1)}\right]\left(B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}\right.$$

$$\times\Omega_1\left(D_{t-}^{\alpha}X, 2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[-\pi,t]}$$

$$+\left[\left(B_N\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{1}{\alpha+1}} + \frac{1}{(\alpha+1)}\right]\left(B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left.\times\,\Omega_1\left(D_{*t}^{\alpha}X, 2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right)_{L^1,[t,\pi]}\right\}, \quad (69)$$

$\forall\ N \in \mathbb{N}$.

Proof: By Corollary 4.8. □

We give

Proposition 5.2: *All as in Proposition 5.1 with $\alpha = \frac{1}{2}$. Then, for any $t \in [-\pi,\pi]$, we have:*

$$E\left(\left|\widetilde{B}_N(X) - X\right|\right)(t) \leq 2\sqrt{2}$$

$$\times\left\{\left[\left(B_N\left(\chi_{[-\pi,t]}(\cdot)\right)(t)\right)^{\frac{2}{3}} + \frac{2}{3}\right]\left(B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{1}{3}}\right.$$

$$\times\Omega_1\left(D_{t-}^{\frac{1}{2}}X, 2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[-\pi,t]}$$

$$+\left[\left(B_N\left(\chi_{[t,\pi]}(\cdot)\right)(t)\right)^{\frac{2}{3}}+\frac{2}{3}\right]\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{1}{3}}$$

$$\times\,\Omega_1\left(D_{*t}^{\frac{1}{2}}X,2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[t,\pi]}\Bigg\}, \quad (70)$$

$\forall\ N\in\mathbb{N}.$

Proof: By Proposition 5.1. □

We continue with

Proposition 5.3: *All as in Proposition 5.1 with* $\alpha=\frac{1}{2}$. *Then, for any* $t\in[-\pi,\pi]$, *we have:*

$$E\left(\left|\widetilde{B}_N(X)-X\right|\right)(t)\le\frac{10\sqrt{2}}{3}\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{1}{3}}$$

$$\times\left[\Omega_1\left(D_{t-}^{\frac{1}{2}}X,2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[-\pi,t]}\right.$$

$$\left.+\,\Omega_1\left(D_{*t}^{\frac{1}{2}}X,2\pi\left(B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{\frac{3}{2}}\right)(t)\right)^{\frac{2}{3}}\right)_{L^1,[t,\pi]}\right], \quad (71)$$

$\forall\ N\in\mathbb{N}.$

Proof: By (70) and the positivity of B_N, see also (39) and (41). □

We make

Remark 1: By $|\sin x|<|x|$, $\forall\ x\in\mathbb{R}-\{0\}$, in particular, $\sin x\le x$, for $x\ge 0$, we get

$$\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{\frac{3}{2}}\le\left(\frac{|\cdot-t|}{4}\right)^{\frac{3}{2}}=\frac{1}{8}\,|\cdot-t|^{\frac{3}{2}}.$$

Hence

$$\left\|B_N\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{\frac{3}{2}}\right)(t)\right\|_\infty\le\frac{1}{8}\left\|B_N\left(|\cdot-t|^{\frac{3}{2}}\right)(t)\right\|_\infty. \quad (72)$$

We observe that

$$B_N\left(|\cdot - t|^{\frac{3}{2}}\right)(t) = \sum_{k=0}^{N}\left|t+\pi-\frac{2\pi k}{N}\right|^{\frac{3}{2}}\binom{N}{k}\left(\frac{t+\pi}{2\pi}\right)^{k}\left(\frac{\pi-t}{2\pi}\right)^{N-k}$$

(by discrete Hölder's inequality)

$$\leq \left[\sum_{k=0}^{N}\left(t+\pi-\frac{2\pi k}{N}\right)^{2}\binom{N}{k}\left(\frac{t+\pi}{2\pi}\right)^{k}\left(\frac{\pi-t}{2\pi}\right)^{N-k}\right]^{\frac{3}{4}}$$

$$= \left(B_N\left((\cdot - t)^{2}\right)(t)\right)^{\frac{3}{4}} \overset{(66)}{\leq} \frac{\pi^{\frac{3}{2}}}{N^{\frac{3}{4}}},\ \ \forall\ t\in[-\pi,\pi]\,. \tag{73}$$

Consequently, we have

$$\left\|B_N\left(|\cdot - t|^{\frac{3}{2}}\right)(t)\right\|_{\infty} \leq \frac{\pi^{\frac{3}{2}}}{N^{\frac{3}{4}}}, \tag{74}$$

and

$$\left\|B_N\left(\left(\sin\left(\frac{|\cdot - t|}{4}\right)\right)^{\frac{3}{2}}\right)(t)\right\|_{\infty} \leq \frac{\pi^{\frac{3}{2}}}{8N^{\frac{3}{4}}},\ \ \forall\ N\in\mathbb{N}. \tag{75}$$

We further have

Proposition 5.4: *All as in Proposition 5.1 with $\alpha=\frac{1}{2}$. Then, for any $t\in[-\pi,\pi]$ we have:*

$$E\left(\left|\widetilde{B}_N(X)-X\right|\right)(t) \leq \frac{5\sqrt{2\pi}}{\sqrt[4]{N}}$$

$$\times\left[\Omega_1\left(D_{t-}^{\frac{1}{2}}X,\frac{\pi^2}{2\sqrt{N}}\right)_{L^1,[-\pi,t]}+\Omega_1\left(D_{*t}^{\frac{1}{2}}X,\frac{\pi^2}{2\sqrt{N}}\right)_{L^1,[t,\pi]}\right], \tag{76}$$

$\forall\ N\in\mathbb{N}$.

As $N\to+\infty$, we get $E\left(\left|\widetilde{B}_N(X)-X\right|\right)(t)\to 0$.

Proof: By (71) and (75) and positivity of B_N. See also Proposition 3.3. □

Consequently we obtain

Proposition 5.5: *All as in Proposition 5.1 with $\alpha=\frac{1}{2}$. Assume further that $\left|X^{(1)}(t,\omega)\right|\leq M^{*}$, $\forall\ (t,\omega)\in[-\pi,\pi]\times\Omega$, where $M^{*}>0$. Then*

$$\left\|E\left(\left|\widetilde{B}_N(X)-X\right|\right)\right\|_{\infty} \leq \frac{5\sqrt{2\pi}}{\sqrt[4]{N}}$$

$$\times \left[\sup_{t\in[-\pi,\pi]} \Omega_1 \left(D_{t-}^{\frac{1}{2}} X, \frac{\pi^2}{2\sqrt{N}} \right)_{L^1,[-\pi,t]} + \sup_{t\in[-\pi,\pi]} \Omega_1 \left(D_{*t}^{\frac{1}{2}} X, \frac{\pi^2}{2\sqrt{N}} \right)_{L^1,[t,\pi]} \right], \tag{77}$$

$\forall\ N \in \mathbb{N}$.

As $N \to +\infty$, *then* $\left\| E\left(\left| \widetilde{B}_N(X) - X \right| \right) \right\|_\infty \to 0$, *i.e.* $\widetilde{B}_N \to I$ *(stochastic unit operator) in* 1*-mean.*

Proof: By (76) and Remark **2**, see (13), (14). □

Remark 2: (to Proposition 5.5) Assume that

$$\Omega_1 \left(D_{t-}^{\frac{1}{2}} X, \frac{\pi^2}{2\sqrt{N}} \right)_{L^1,[-\pi,t]} \leq \frac{K_1 \pi^2}{2\sqrt{N}},\ \ \forall\ t \in [-\pi,\pi],\ \forall\ N \in \mathbb{N}, \tag{78}$$

where $K_1 > 0$.

And assume that

$$\Omega_1 \left(D_{*t}^{\frac{1}{2}} X, \frac{\pi^2}{2\sqrt{N}} \right)_{L^1,[t,\pi]} \leq \frac{K_2 \pi^2}{2\sqrt{N}},\ \ \forall\ t \in [-\pi,\pi],\ \forall\ N \in \mathbb{N}, \tag{79}$$

where $K_2 > 0$.

Conditions (78), (79) are of Lipschitz type of order 1.

By (77) and (78), (79), we easily derive that

$$\left\| E\left(\left| \widetilde{B}_N(X) - X \right| \right) \right\|_\infty \leq \frac{\theta}{N^{\frac{3}{4}}}, \tag{80}$$

$\forall\ N \in \mathbb{N}$, where $\theta > 0$.

Thus, at smoothness of order $\frac{1}{2}$ we achieve speed of convergence $\frac{1}{N^{\frac{3}{4}}}$, $N \in \mathbb{N}$. Without any smoothness, in [5], we proved that the speed of convergence was $\frac{1}{\sqrt{N}}$, $N \in \mathbb{N}$.

In the presence of the ordinary first derivative, see [5], in deterministic approximation the rate of convergence was $\frac{1}{N}$, $N \in \mathbb{N}$.

Naturally, we have as expected:

$$\frac{1}{N} < \frac{1}{N^{\frac{3}{4}}} < \frac{1}{\sqrt{N}},\ \ \forall\ N \in \mathbb{N} - \{1\}. \tag{81}$$

6. Trigonometric Stochastic Korovkin Results

In this section $\widetilde{L}$, M are meant as sequences of operators.

We give first pointwise results:

Theorem 6.1: *Here all as in Theorem 4.1. Assume further that* $\widetilde{L}(1)(t) \to 1$ *and* $\widetilde{L}\left(\left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{q(\alpha+1)} \right)(t) \to 0$*, pointwise in* $t \in [-\pi,\pi]$.

Then $E(|M(X) - X|^q)(t) \to 0$*, pointwise in* $t \in [-\pi,\pi]$*, that is* $M \to I$ *(stochastic unit operator) in q-mean-pointwise with rates, quantitatively.*

Proof: We use (48), we take into account $\widetilde{L}(1)(t) \to 1$, (36); and $\widetilde{L}\left(\chi_{[t,\pi]}(\cdot)\right)(t)$, $\widetilde{L}\left(\chi_{[-\pi,t]}(\cdot)\right)(t) \leq \widetilde{L}(1)(t)$, which $\widetilde{L}(1)(t)$ is bounded, and by (39), (41) and positivity of $\widetilde{L}$ we get that
$\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[t,\pi]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t)$, $\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\chi_{[-\pi,t]}(\cdot)\right)\right)^{q(\alpha+1)}\right)(t) \leq \widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{q(\alpha+1)}\right)(t) \to 0.$

Finally, we use Proposition 3.4 (ii) for the $\Omega_1(\cdot,\cdot)$'s to go to zero. □

We continue with

Theorem 6.2: *Here all as in Theorem 4.2. Assume further that* $\widetilde{L}(1)(t) \to 1$ *and* $\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{(\alpha+1)}\right)(t) \to 0$, *pointwise in* $t \in [-\pi,\pi]$.

Then $E(|M(X)-X|)(t) \to 0$, *pointwise in* $t \in [-\pi,\pi]$, *that is* $M \to I$ *in* 1*-mean-pointwise with rates, quantitatively.*

Proof: Based on (51), similar to the proof of Theorem 6.1, just take $q = 1$ there. □

Next we give uniform results:

Theorem 6.3: *Here all as in Theorem 4.3. Assume further that* $\widetilde{L}(1) \to 1$, *uniformly, and* $\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{q(\alpha+1)}\right)(t)\right\|_\infty \to 0$.

Then $\|E(|M(X)-X|^q)\|_\infty \to 0$ *over* $[-\pi,\pi]$, *that is* $M \to I$ *in the q-mean, quantitatively with rates.*

Proof: We use (54), we take into account $\widetilde{L}(1) \to 1$, uniformly, (37); $\left\|\widetilde{L}(1)\right\|_\infty$ is bounded, use of (40), (42) and Remark **2**, see there (13), (14). □

Next we give the L_1-mean uniform result

Theorem 6.4: *Here all as in Theorem 4.4. Assume further that* $\widetilde{L}(1) \to 1$, *uniformly, and* $\left\|\widetilde{L}\left(\left(\sin\left(\frac{|\cdot-t|}{4}\right)\right)^{(\alpha+1)}\right)(t)\right\|_\infty \to 0$.

Then $\|E(|M(X)-X|)\|_\infty \to 0$ *over* $[-\pi,\pi]$, *that is* $M \to I$ *in the* 1*-mean, quantitatively with rates.*

Proof: Use of (55), similar to the proof of Theorem 6.3, just take $q = 1$ there. □

Remark 1: An amazing fact/observation follows: In all trigonometric convergence results here, see Theorems 6.1-6.4, the forcing conditions for convergences are based only on $\widetilde{L}$ and basic real valued continuous functions on $[-\pi,\pi]$ and are not related to stochastic processes, but they are giving trigonometric convergence results on stochastic processes!

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