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# Finite Dimensional Applications of the Dunford-Taylor Integral

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The Dunford-Taylor integral is used in order to compute the inverse of a non-singular complex matrix. Then the obtained result is applied to derive the solution of some basic analytic problems as the solution of linear algebraic equations, the solution of matrix equations and of initial value problems for a linear system of ordinary differential equations

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## 1. Introduction

A classic equation of functional analysis, which takes place in the framework of Cauchy's integral formula is known as the Dunford-Taylor integral.

Actually this equation traces back to Luigi Fantappiè [5], creating the theory of analytic functionals and Frigyes Riesz [16], who made fundamental contributions to functional analysis. For this reason the corresponding equation is also called the Riesz-Fantappiè integral.

This equation has been recently used in order to compute the $n$th roots of a non-singular complex matrix [3]. The same methodology is applied in what follows to basic problems of matrix analysis such as the solution of algebraic systems of equations, the solution of matrix equations and of initial value problem for a linear system of ordinary differential equations.

We strongly emphasize that the method proposed here does not claim to replace the results of numerical analysis that solve the same problems in the case of very large or ill conditioned matrices. We only believe that this procedure can sometimes be more convenient, because it avoids having to determine the eigenvalues and sometimes also the eigenvectors of the involved matrices.

As the tools we have used are basic elements of Complex Analysis, we are confident that the considered method become part of the undergraduate teaching.

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Some of the computations, obtained by the first author, by using the computer program Mathematica© are reported in the last section.

The described technique could also be applied to higher order matrices, but we would find it difficult to report results due to the size of the Journal page.

## 2. The Dunford-Taylor integral

The Dunford-Taylor integral [9, 16] in functional analysis is the analog of the Cauchy integral in function theory. It works for holomorphic functions of an operator. In the finite dimensional case, the operator is represented by a matrix $\mathcal{A}$.

**Theorem 2.1 :** *Suppose that $f(\lambda)$ is a holomorphic function in a domain $\Delta \subset \mathbf{C}$, containing all the eigenvalues $\lambda_h$ of $\mathcal{A}$, and let $\gamma \subset \Delta$ be a simple closed smooth curve with positive direction enclosing all the $\lambda_h$ in its interior.*

*Then the matrix function $f(\mathcal{A})$ is defined by the Dunford-Taylor integral*

$$f(\mathcal{A}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma f(\lambda)\,(\lambda\mathcal{I} - \mathcal{A})^{-1} d\lambda\,, \tag{1}$$

where $(\lambda\mathcal{I} - \mathcal{A})^{-1}$ denotes the *resolvent* of $\mathcal{A}$.

An example is the computation of all the square roots of a non-singular complex matrix $\mathcal{A}$, which are given by the formula:

$$\mathcal{A}^{1/n} = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \lambda^{1/n}\,(\lambda\mathcal{I} - \mathcal{A})^{-1} d\lambda\,, \tag{2}$$

as it has been shown in [3].

It is worth to note that there exists in literature another method for computing matrix powers using the Cayley-Hamilton theorem and the so called $F_{k,n}$ functions, which are solutions of linear recursions [14]. This method is purely algebraic, can be used for computing the matrix exponential [11] and does not require quadrature rules, which are necessary for avoiding Cauchy's residue theorem.

If $\mathcal{A}$ is non-singular, both equation (3) and that reported in [14] still work for negative values of the integer $n$, as the FKN functions [1] are defined there even if $n < 0$.

It is worth noting that the application of the Dunford-Taylor's integral requires only the knowledge of matrix entries (and the relevant invariants), whereas for using the Cauchy's residue theorem it is necessary to know the eigenvalues and their multiplicity. Therefore, the first method is computationally more convenient.

## 3. Recalling the resolvent of a matrix

The resolvent of an operator is an important tool for using methods of complex analysis in the theory of operators on Banach spaces [9]. The holomorphic functional calculus gives a formal justification of the used procedure.

In the present case, we are in the finite dimensional case, so that the operator under consideration is a $r \times r$ complex matrix $\mathcal{A}$.

**Definition 3.1:** - Given the $r \times r$, matrix $\mathcal{A} = (a_{ij})$, whose invariants are

$$u_1 := tr\,\mathcal{A}, \quad u_2 := \sum_{i<j}^{1,r} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}, \quad \cdots, \quad u_r := det\,\mathcal{A}, \tag{3}$$

putting for shortness $\mathbf{u} := (u_1, u_2, \ldots, u_r)$, and $(u_0 := 1)$, its *characteristic polynomial* is given by

$$P(\mathbf{u}; \lambda) := det(\lambda\mathcal{I} - \mathcal{A}) = \lambda^r - u_1\lambda^{r-1} + u_2\lambda^{r-2} + \ldots + (-1)^r u_r, \tag{4}$$

and the relative characteristic equation writes:

$$P(\lambda) := P(\mathbf{u}; \lambda) = 0. \tag{5}$$

The coefficients of the characteristic polynomial are called the *invariants* of $\mathcal{A}$, because they are invariant under similarity transformations [10]. The roots of $P(\lambda)$,

$$\lambda_1, \lambda_2, \ldots, \lambda_r, \tag{6}$$

are the eigenvalues of $\mathcal{A}$.

In [4], pp. 93–95, the following representation equation for the resolvent $(\lambda\mathcal{I} - \mathcal{A})^{-1}$, in terms of the invariants of $\mathcal{A}$ is proved:

$$(\lambda\mathcal{I} - \mathcal{A})^{-1} = \frac{1}{P(\lambda)} \sum_{k=0}^{r-1} \left[ \sum_{h=0}^{r-k-1} (-1)^h \, u_h \, \lambda^{r-k-h-1} \right] \mathcal{A}^k. \tag{7}$$

By using equations (4) and (6), we find a representation formula for matrix functions [7], reported in [2].

**Theorem 3.2:** *Let $f(\lambda)$ be a holomorphic function in a domain $\Delta \subset \mathbf{C}$, containing the spectrum of $\mathcal{A}$, and denote by $\gamma \subset \Delta$ a simple contour enclosing all the zeros of $P(\lambda)$. Then the Dunford-Taylor integral writes:*

$$f(\mathcal{A}) = \frac{1}{2\,\pi\,\mathrm{i}} \left[ \sum_{k=1}^{r} \oint_\gamma \frac{f(\lambda)\, \sum_{h=0}^{k-1} (-1)^h u_h \, \lambda^{k-h-1}}{P(\lambda)} \, d\lambda \; \mathcal{A}^{r-k} \right]. \tag{8}$$

Note that, if $\Delta$ does not contain the origin, a simple choice of $\gamma$ is a circle centered at the origin and radius greater then the spectral radius of $\mathcal{A}$. This radius can be determined, only using the entries of $\mathcal{A}$, as a consequence of the Gershgorin circle theorem.

The most useful application deals with the computation of the matrix exponential, which writes:

$$\exp(\mathcal{A}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma e^\lambda \, (\lambda\mathcal{I} - \mathcal{A})^{-1} d\lambda. \tag{9}$$

**Remark 1:** It is worth noting that the use of series expansion for defining a holomorphic matrix function $f(\mathcal{A})$ is useless, since, according to the general theory in the Gantmacher book [6], any holomorphic matrix function is a polynomial of the same matrix, that is: $f(\mathcal{A}) \equiv P(\mathcal{A})$, where $P$ is the polynomial interpolating the function $f$ on the eigenvalues of $\mathcal{A}$) (see e.g. [2, 12] and the references therein).

Consider now the function $f(\lambda) = \lambda^{-1}$. As this function is holomorphic in the open set $\mathbf{C}-\{0\}$, i.e. in the whole plane excluding the origin, the preceding theorem becomes

**Theorem 3.3:** *If $\mathcal{A}$, is a non-singular complex matrix and $\gamma = \gamma_1 \cup \gamma_2$ is a simple contour enclosing all the zeros of $P(\lambda)$ (were $\gamma_1$, oriented counter-clockwise, encircles all the eigenvalues of $\mathcal{A}$ and $\gamma_2$, oriented clockwise, is a circle enclosing the origin and no eigenvalues of $\mathcal{A}$), then the inverse of $\mathcal{A}$ is represented by*

$$\mathcal{A}^{-1} = \frac{1}{2\,\pi\,\mathrm{i}} \left[ \sum_{k=1}^{r} \oint_{\gamma_1 \cup \gamma_2} \frac{\sum_{h=0}^{k-1} (-1)^h u_h \, \lambda^{k-h-1}}{\lambda\, P(\lambda)} \; d\lambda \; \mathcal{A}^{r-k} \right] . \tag{10}$$

Recalling Cauchy's residue theorem [17], and denoting by $\Phi_k = \Phi_k(\lambda)$, $(k = 1, 2, \ldots, r)$ the integrand in equation (8), the contour integral is given by:

$$\oint_{\gamma_1 \cup \gamma_2} \frac{\sum_{h=0}^{k-1} (-1)^h u_h \, \lambda^{k-h-1}}{\lambda\, P(\lambda)} \; d\lambda = (2\,\pi\,\mathrm{i}) \left[ \sum_{\ell=1}^{r} Res_{\Phi_k}(\lambda_\ell) - Res_{\Phi_k}(0) \right] . \tag{11}$$

Supposing, for simplicity, the eigenvalues are all distinct, and putting

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_r),$$

we find:

$$\begin{aligned} \sum_{\ell=1}^{r} Res_{\Phi_k}(\lambda_\ell) &= \sum_{\ell=1}^{r} \lim_{\lambda \to \lambda_\ell} (\lambda - \lambda_\ell) \; \frac{\sum_{h=0}^{k-1} (-1)^h u_h \, \lambda^{k-h-1}}{\lambda\, P(\lambda)} \\ &= \sum_{\ell=1}^{r} \; \frac{\sum_{h=0}^{k-1} (-1)^h u_h \, \lambda_\ell^{k-h-1}}{\lambda_\ell \,(\lambda_\ell - \lambda_1) \cdots (\lambda_\ell - \lambda_{\ell-1})(\lambda_\ell - \lambda_{\ell+1}) \cdots (\lambda_\ell - \lambda_r)} \, , \end{aligned} \tag{12}$$

where we have put, by definition: $(\lambda - \lambda_0) = (\lambda - \lambda_{r+1}) := 1$.

Furthermore,

$$Res_{\Phi_k}(0) = \lim_{\lambda \to 0} \frac{\sum_{h=0}^{k-1} (-1)^h u_h \, \lambda^{k-h-1}}{P(\lambda)} = \frac{(-1)^{k-1} u_{k-1}}{P(0)} \, . \tag{13}$$

Then, equation (8), noting that $P(0) = (-1)^r u_r$, becomes:

$$\mathcal{A}^{-1} = \sum_{k=1}^{r} \left[ \sum_{\ell=1}^{r} \; \frac{\sum_{h=0}^{k-1} (-1)^h u_h \, \lambda_\ell^{k-h-1}}{\lambda_\ell \,(\lambda_\ell - \lambda_1) \cdots (\lambda_\ell - \lambda_{\ell-1})(\lambda_\ell - \lambda_{\ell+1}) \cdots (\lambda_\ell - \lambda_r)} + (-1)^{r-k} \frac{u_{k-1}}{u_r} \right] \mathcal{A}^{r-k} \, . \tag{14}$$

A similar result can be found in case of multiple roots of the characteristic polynomial, by using the more general equation, which holds for a pole of order $m$ at the point $\lambda_\ell$:

$$Res_{\Phi_k}(\lambda_\ell) = \frac{1}{(m-1)!} \lim_{\lambda \to \lambda_\ell} \frac{d^{m-1}}{d\lambda^{m-1}} \left[ (\lambda - \lambda_\ell)^m \, \lambda^{-1} \right] . \tag{15}$$

Examples of computations using Cauchy's residue theorem are given in [15].

**Remark 2 :** Note that the knowledge of eigenvalues is not strictly necessary. It is mandatory if we compute the integral in equation (10) by Cauchy's residue theorem, but actually only the knowledge of the invariants is necessary, since we could compute the contour integral by choosing as $\gamma_1$ a circle centered at the origin with radius greater than the spectral radius of $\mathcal{A}$, and $\gamma_2$ with radius less than the minimum modulus of the eigenvalues.

**Remark 3 :** Recalling Cauchy's bounds for the roots of polynomials [8, 13], it immediately follows that, in case of the characteristic equation (5), the highest (in modulus) eigenvalue is bounded by

$$1 + \max \left\{ \frac{|u_{r-1}|}{|u_r|}, \frac{|u_{r-2}|}{|u_r|}, \cdots, \frac{1}{|u_r|} \right\} . \tag{16}$$

Making the substitution $\mu = \lambda^{-1}$ in the polynomial (4), multiplying by $\mu^r$ and applying the Cauchy bound, we find the upper bound of the zeros of $\mu^r \, P(\mu^{-1})$. Then, returning to the variable $\lambda$, we find the lower bound to the roots of $P(\lambda)$, which is given by

$$[1 + \max \{|u_1|, |u_2|, \ldots, |u_r|\}]^{-1} . \tag{17}$$

## 4. Applications

### 4.1. *Solving linear algebraic systems*

$\mathcal{A}$ is a $r \times r$ non-singular complex matrix, $\mathbf{b}$ and $\mathbf{x}$ are $r \times 1$ column vectors

$$\mathcal{A}\,\mathbf{x} = \mathbf{b} \qquad \Leftrightarrow \qquad \mathbf{x} = \mathcal{A}^{-1}\mathbf{b} . \tag{18}$$

### 4.2. *Solving matrix equations*

$\mathcal{A}$, $\mathcal{B}$ and $\mathbf{X}$ are $r \times r$ non-singular complex matrices and $\mathbf{O}$ is the zero matrix.

**# 1.**

$$\mathcal{A}\,\mathbf{X} + \mathcal{B} = \mathbf{O} \qquad \Leftrightarrow \qquad \mathbf{X} = -\mathcal{A}^{-1}\mathcal{B} . \tag{19}$$

**# 2.**

$$\mathcal{A}\,\mathbf{X}^n + \mathcal{B} = \mathbf{O} \qquad \Leftrightarrow \qquad \mathbf{X} = \left(-\mathcal{A}^{-1}\mathcal{B}\right)^{1/n} . \tag{20}$$

### 4.3. *Solving the Cauchy problem for a linear systems of ODE*

$\mathcal{A}$ is a $r \times r$ non-singular complex matrix, $\mathbf{y}(t)$ is a $r \times 1$ column vector

$$\begin{cases} \mathbf{y}'(t) = \mathcal{A}\,\mathbf{y}(t) \\ \mathbf{y}(0) = \mathbf{y_0} \end{cases} \qquad \Leftrightarrow \qquad \mathbf{y}(t) = \exp(\mathcal{A}\,t)\,\mathbf{y_0}\,. \tag{21}$$

### 4.4. *Numerical examples*

*4.4.1. Computations with non-singular matrix of sixth order*

Let us consider the matrix:

$$\mathcal{A} = \begin{pmatrix} 3 & -2 & 0 & 1 & 5 & 5 \\ 2 & 2 & 0 & 0 & -1 & 0 \\ -18 & 4 & -1 & -2 & -20 & -19 \\ 0 & 1 & -1 & 1 & -2 & 1 \\ 7 & 0 & 0 & 1 & 6 & 7 \\ -8 & 1 & 0 & -1 & -8 & -9 \end{pmatrix}. \tag{22}$$

The relevant invariants are:

$$u_1 = 2\,, \quad u_2 = 2\,, \quad u_3 = 0\,, \quad u_4 = -1\,, \quad u_5 = -2\,, \quad u_6 = -2\,.$$

Therefore, it would not be not difficult to verify that the corresponding eigenvalues are:

$$\lambda_1 = 1 + \mathrm{i}\,, \quad \lambda_2 = 1 - \mathrm{i}\,, \quad \lambda_3 = -1\,, \quad \lambda_4 = \mathrm{i}\,, \quad \lambda_5 = -\mathrm{i}\,, \quad \lambda_6 = 1\,.$$

By using the Dunford-Taylor integral formula (10) in combination with the Gauss-Kronrod integration rule, the following representation of the inverse of $\mathcal{A}$ is obtained:

$$\mathcal{A}^{-1} = \sum_{k=1}^{r} \xi_k\, \mathcal{A}^{r-k}\,,$$

with:

$$\xi_1 = 1/2\,, \quad \xi_2 = -1\,, \quad \xi_3 = 1\,, \quad \xi_4 = 0\,, \quad \xi_5 = -1/2\,, \quad \xi_6 = 1\,,$$

this leading to the conclusion that:

$$\mathcal{A}^{-1} = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} & -1 & 1 & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 1 & -1 & \frac{5}{2} & 0 \\ 9 & 9 & 2 & -3 & -7 & -5 \\ \frac{3}{2} & \frac{1}{2} & 0 & 0 & \frac{3}{2} & 2 \\ -2 & -3 & 0 & 0 & 4 & 2 \\ 3 & 3 & 1 & -1 & -3 & -3 \end{pmatrix}. \tag{23}$$

From here, given the vector of known terms:

$$\mathbf{b} = (-6, 2, 1, 1, 3, -6)^T \,, \tag{24}$$

one can readily find out that the system of linear algebraic equation $\mathcal{A} \cdot \mathbf{x} = \mathbf{b}$ has the solution:

$$\mathbf{x} = \mathcal{A}^{-1} \cdot \mathbf{b} = \left( \frac{1}{2}, \frac{7}{2}, -28, -\frac{31}{2}, 6, -3 \right)^T \,. \tag{25}$$

With reference to the matrix equation (19), under the assumption that:

$$\mathcal{B} = 2\,\mathcal{I}_6 = \begin{pmatrix} 2\,0\,0\,0\,0\,0 \\ 0\,2\,0\,0\,0\,0 \\ 0\,0\,2\,0\,0\,0 \\ 0\,0\,0\,2\,0\,0 \\ 0\,0\,0\,0\,2\,0 \\ 0\,0\,0\,0\,0\,2 \end{pmatrix} , \tag{26}$$

it follows, by trivial algebra, that:

$$\mathbf{X} = -\mathcal{A}^{-1} \cdot \mathcal{B} = \begin{pmatrix} 3 & 1 & 2 & -2 & 1 & -2 \\ -1 & 1 & -2 & 2 & -5 & 0 \\ -18 & -18 & -4 & 6 & 14 & 10 \\ -3 & -1 & 0 & 0 & -3 & -4 \\ 4 & 6 & 0 & 0 & -8 & -4 \\ -6 & -6 & -2 & 2 & 6 & 6 \end{pmatrix} . \tag{27}$$

The invariants of the matrix $\mathbf{X}$ are given:

$$u_{X1} = -2 \,, \quad u_{X2} = 2 \,, \quad u_{X3} = 0 \,, \quad u_{X4} = -16 \,, \quad u_{X5} = 32 \,, \quad u_{X6} = -32 \,,$$

whereas the relevant eigenvalues are:

$$\lambda_{X1} = -2 \,, \quad \lambda_{X2} = 2\mathrm{i} \,, \quad \lambda_{X3} = -2\mathrm{i} \,, \quad \lambda_{X4} = 2 \,, \quad \lambda_{X5} = -1{+}\mathrm{i} \,, \quad \lambda_{X6} = -1{-}\mathrm{i} \,.$$

In this way, upon implementing the technique detailed in [3], and setting:

$$\mathbf{X}^{1/2} = \mathrm{Re}\left\{\mathbf{X}^{1/2}\right\} + \mathrm{i}\;\mathrm{Im}\left\{\mathbf{X}^{1/2}\right\} \,,$$

we easily find that one determination of the square root of $\mathbf{X}$ is such that:

$$\mathrm{Re}\left\{\mathbf{X}^{1/2}\right\} = \frac{1}{20}\,(B_1 \mid B_2) \,,$$

where

$$B_1 = \begin{pmatrix} 40-4\sqrt{2}+6\sqrt{2+10\sqrt{2}} & -2\left(-10+2\sqrt{2}+\sqrt{10\sqrt{2}-2}\right) & 4\left(3+\sqrt{2}-2\sqrt{\sqrt{2}-1}\right) \\ -30+\sqrt{2}+6\sqrt{2+10\sqrt{2}} & 30+\sqrt{2}-2\sqrt{10\sqrt{2}-2} & -18-\sqrt{2}-8\sqrt{\sqrt{2}-1} \\ -50+7\sqrt{2}-12\sqrt{167+145\sqrt{2}} & -190+7\sqrt{2}+4\sqrt{145\sqrt{2}-167} & 18-7\sqrt{2}-8\sqrt{58\sqrt{2}-82} \\ -10-7\sqrt{2}-6\sqrt{10\sqrt{2}-2} & 10-7\sqrt{2}-2\sqrt{2+10\sqrt{2}} & -6+7\sqrt{2}-8\ 2^{3/4}\cos\left(\frac{\pi}{8}\right) \\ 60\ 2^{3/4}\cos\left(\frac{\pi}{8}\right)-5\left(2+\sqrt{2}\right) & -5\left(-14+\sqrt{2}+4\sqrt{\sqrt{2}-1}\right) & -18+5\sqrt{2}-8\sqrt{2\left(5\sqrt{2}-7\right)} \\ 10\left(-2+\sqrt{2}-6\ 2^{3/4}\cos\left(\frac{\pi}{8}\right)\right) & 20\left(-4+\frac{1}{\sqrt{2}}+\sqrt{\sqrt{2}-1}\right) & 8-10\sqrt{2}+8\sqrt{2\left(5\sqrt{2}-7\right)} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 8\sqrt{1+5\sqrt{2}}-28 & 2\left(\sqrt{46+130\sqrt{2}}-6\right) & -4\left(5+3\sqrt{2}-3\sqrt{7+5\sqrt{2}}\right) \\ 12+8\sqrt{1+5\sqrt{2}} & 2\left(\sqrt{46+130\sqrt{2}}-36\right) & 3\left(-10+\sqrt{2}+4\sqrt{7+5\sqrt{2}}\right) \\ 68-8\sqrt{290\sqrt{2}-254} & -4\left(\sqrt{2329+1885\sqrt{2}}-78\right) & 190+21\sqrt{2}-12\sqrt{238+290\sqrt{2}} \\ 4+8\sqrt{5\sqrt{2}-1} & -2\left(12+\sqrt{130\sqrt{2}-46}\right) & -10-21\sqrt{2}+12\sqrt{5\sqrt{2}-7} \\ 8\left(\sqrt{50\sqrt{2}-34}-1\right) & 4\left(\sqrt{359+325\sqrt{2}}-33\right) & 5\left(-14-3\sqrt{2}+6\ 2^{3/4}\csc\left(\frac{\pi}{8}\right)\right) \\ 28-8\sqrt{50\sqrt{2}-34} & -4\left(\sqrt{359+325\sqrt{2}}-33\right) & 10\left(8+3\sqrt{2}-3\ 2^{3/4}\csc\left(\frac{\pi}{8}\right)\right) \end{pmatrix},$$

and:

$$\operatorname{Im}\left\{\mathbf{X}^{1/2}\right\} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -2 & -6 & -2 & 4 & 4 & 6 \\ 1 & 3 & 1 & -2 & -2 & -3 \\ 11 & 33 & 11 & -22 & -22 & -33 \\ 1 & 3 & 1 & -2 & -2 & -3 \\ -3 & -9 & -3 & 6 & 6 & 9 \\ 4 & 12 & 4 & -8 & -8 & -12 \end{pmatrix}.$$

It is not difficult to verify that:

$$\mathcal{A}\cdot\mathbf{X}^2+\mathcal{B}=\mathcal{O}_6\,,$$

where $\mathcal{O}_6$ denotes the zero matrix of order 6.

Let us finally consider the system of linear of ODEs of the first order:

$$\begin{cases} \mathbf{y}'(t)=\mathcal{A}\cdot\mathbf{y}(t)\,, \\ \mathbf{y}(0)=\mathbf{y_0}=(0,-1,0,-1,1,0)^T\ . \end{cases} \tag{28}$$

As reported in (21), the formal solution of (28) can be expressed as:

$$\mathbf{y}(t)=\exp(t\,\mathcal{A})\cdot\mathbf{y_0}\,,$$

where:

$$\exp(t\,\mathcal{A}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma e^{t\,\lambda}\,(\lambda\mathcal{I} - \mathcal{A})^{-1} d\lambda = \sum_{k=1}^{r} \epsilon_k\,(t)\;\; \mathcal{A}^{r-k}\,,$$

with:

$$\epsilon_k\,(t) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{e^{t\,\lambda}\,\sum_{h=0}^{k-1}(-1)^h u_h\,\lambda^{k-h-1}}{P(\lambda)}\; d\lambda\,,$$

for $k = 1, 2, \ldots, r = 6$. In the specific case, using Cauchy's residue theorem, one can determine:

$$\epsilon_1\,(t) = \frac{1}{10}\,[-\sin t - 2\cos t + (3 - 2\sin t)\sinh t - 2(\sin t - 1)\cosh t]\,,$$

$$\begin{aligned}\epsilon_2\,(t) = \frac{1}{10}\,[&4\sin t + 3\cos t - 2\sinh t(-\sin t + \cos t + 2)\\ &+ \cosh t(2\sin t - 2\cos t - 1)]\,,\end{aligned}$$

$$\epsilon_3\,(t) = \frac{1}{2}\,(\sinh t - \sin t)\,,$$

$$\epsilon_4\,(t) = \frac{1}{2}\,(\cosh t - \cos t)\,,$$

$$\epsilon_5\,(t) = \frac{1}{5}\,[\cos t + \sinh t + \sin t(\sinh t + 3) + (\sin t - 1)\cosh t]\,,$$

$$\begin{aligned}\epsilon_6\,(t) = \frac{1}{5}\,[&-2\sin t + \cos t + \sinh t(-\sin t + \cos t + 2)\\ &+ \cosh t(-\sin t + \cos t + 3)]\,.\end{aligned}$$

As a result, it follows that:

$$\mathbf{y}(t) = \frac{1}{10}\begin{pmatrix} 31\sin t - 33\cos t + (16\cos t + 13)\sinh t + (16\cos t + 17)\cosh t \\ -39\sin t - 7\sinh t + 2\cos t(8\sinh t - 9) + 8(2\cos t - 1)\cosh t \\ 19\sin t + 198\cos t - \sinh t(48\sin t + 112\cos t + 79) - 2\cosh t(24\sin t + 56\cos t + 43) \\ -13\sin t - 6\cos t - (16\sin t + 11)\sinh t - 4(4\sin t + 1)\cosh t \\ -34\sin t - 63\cos t + 4\sinh t(4\sin t + 12\cos t + 5) + \cosh t(16\sin t + 48\cos t + 25) \\ 9\sin t + 83\cos t - \sinh t(16\sin t + 48\cos t + 25) - \cosh t(16\sin t + 48\cos t + 35) \end{pmatrix}.$$

*4.4.2. Computations with non-singular matrix of third order*

Let us consider the matrix:

$$\mathcal{A} = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} . \tag{29}$$

The relevant invariants are:

$$u_1 = 3 \,, \quad u_2 = 1 \,, \quad u_3 = -1 \,.$$

Therefore, it would not be not difficult to verify that the corresponding eigenvalues are:

$$\lambda_1 = 1 + \sqrt{2} \,, \quad \lambda_2 = 1 \,, \quad \lambda_3 = 1 - \sqrt{2} \,.$$

By using the Dunford-Taylor integral formula (10) in combination with the Gauss-Kronrod integration rule, the following representation of the inverse of $\mathcal{A}$ is obtained:

$$\mathcal{A}^{-1} = \sum_{k=1}^{r} \xi_k \; \mathcal{A}^{r-k} \,, \tag{30}$$

with:

$$\xi_1 = -1 \,, \quad \xi_2 = 3 \,, \quad \xi_3 = -1 \,,$$

this leading to the conclusion that:

$$\mathcal{A}^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix} .$$

From here, given the vector of known terms:

$$\mathbf{b} = (1, -1, 1)^T \,, \tag{31}$$

one can readily find out that the system of linear algebraic equation $\mathcal{A} \cdot \mathbf{x} = \mathbf{b}$ has the solution:

$$\mathbf{x} = \mathcal{A}^{-1} \cdot \mathbf{b} = (-1, 4, -2)^T \,. \tag{32}$$

With reference to the matrix equation (19), under the assumption that:

$$\mathcal{B} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} , \tag{33}$$

it follows, by trivial algebra, that:

$$\mathbf{X} = -\mathcal{A}^{-1} \cdot \mathcal{B} = \begin{pmatrix} -1 & 0 & -1 \\ 2 & 1 & 4 \\ -2 & -1 & -2 \end{pmatrix} . \tag{34}$$

The invariants of the matrix $\mathbf{X}$ are given:

$$u_{X\,1} = -2\,, \quad u_{X\,2} = 1\,, \quad u_{X\,3} = -2\,,$$

whereas the relevant eigenvalues are:

$$\lambda_{X\,1} = -2\,, \quad \lambda_{X\,2} = \mathrm{i}\,, \quad \lambda_{X\,3} = -\mathrm{i}\,.$$

In this way, upon implementing the technique detailed in [3], and setting:

$$\mathbf{X}^{1/3} = \mathrm{Re}\left\{\mathbf{X}^{1/3}\right\} + \mathrm{i}\;\mathrm{Im}\left\{\mathbf{X}^{1/3}\right\}\,,$$

we easily find that one determination of the cubic root of $\mathbf{X}$ is such that:

$$\mathrm{Re}\left\{\mathbf{X}^{1/3}\right\} = \frac{1}{10}\begin{pmatrix} 3+4\sqrt[3]{2}+\sqrt{3} & 2+\sqrt[3]{2}-\sqrt{3} & 1+3\sqrt[3]{2}-3\sqrt{3} \\ 2\left(-3-4\sqrt[3]{2}+4\sqrt{3}\right) & 1-2\sqrt[3]{2}+7\sqrt{3} & 2\left(4-3\sqrt[3]{2}+3\sqrt{3}\right) \\ 2\left(-1+2\sqrt[3]{2}-2\sqrt{3}\right) & -3+\sqrt[3]{2}-\sqrt{3} & -4+3\sqrt[3]{2}+2\sqrt{3} \end{pmatrix}\,,$$

$$\mathrm{Im}\left\{\mathbf{X}^{1/3}\right\} = \frac{1}{5}\begin{pmatrix} 2\sqrt[3]{2}\sqrt{3} & \frac{\sqrt{3}}{2^{2/3}} & \frac{3\sqrt{3}}{2^{2/3}} \\ -4\sqrt[3]{2}\sqrt{3} & -\sqrt[3]{2}\sqrt{3} & -3\sqrt[3]{2}\sqrt{3} \\ 2\sqrt[3]{2}\sqrt{3} & \frac{\sqrt{3}}{2^{2/3}} & \frac{3\sqrt{3}}{2^{2/3}} \end{pmatrix}\,.$$

It is not difficult to verify that:

$$\mathcal{A} \cdot \mathbf{X}^3 + \mathcal{B} = \mathcal{O}_3\,,$$

where $\mathcal{O}_3$ denotes the zero matrix of order 3.

Let us finally consider the system of linear of ODEs of the first order:

$$\begin{cases} \mathbf{y}'(t) = \mathcal{A} \cdot \mathbf{y}(t)\,, \\ \mathbf{y}(0) = \mathbf{y_0} = (1,1,0)^T\;. \end{cases} \tag{35}$$

As reported in (21), the formal solution of (35) can be expressed as:

$$\mathbf{y}(t) = \exp(t\,\mathcal{A}) \cdot \mathbf{y_0}\,,$$

where:

$$\exp(t\,\mathcal{A}) = \frac{1}{2\,\pi\,\mathrm{i}}\oint_\gamma e^{t\,\lambda}\,(\lambda\mathcal{I}-\mathcal{A})^{-1} d\lambda = \sum_{k=1}^{r} \epsilon_k\,(t)\;\;\mathcal{A}^{r-k}\,,$$

with:

$$\epsilon_k\,(t) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{e^{t\,\lambda}\,\sum_{h=0}^{k-1}(-1)^h u_h\,\lambda^{k-h-1}}{P(\lambda)}\,d\lambda\,,$$

for $k = 1, 2, \ldots, r = 3$. In the specific case, using Cauchy's residue theorem, one can determine:

$$\epsilon_1\,(t) = e^t \sinh^2\left(\frac{t}{\sqrt{2}}\right)\,,$$

$$\epsilon_2\,(t) = \frac{1}{2}e^t\left[\sqrt{2}\sinh\left(\sqrt{2}t\right) - 2\cosh\left(\sqrt{2}t\right) + 2\right]\,,$$

$$\epsilon_3\,(t) = \frac{1}{2}e^t\left[-\sqrt{2}\sinh\left(\sqrt{2}t\right) + \cosh\left(\sqrt{2}t\right) + 1\right]\,.$$

As a result, it follows that:

$$\mathbf{y}(t) = \begin{pmatrix} \frac{1}{2}e^t\left(3\sqrt{2}\sinh\left(\sqrt{2}t\right) + 4\cosh\left(\sqrt{2}t\right) - 2\right) \\ -e^t\left(\sqrt{2}\sinh\left(\sqrt{2}t\right) + \cosh\left(\sqrt{2}t\right) - 2\right) \\ -\frac{1}{4}\left(2+\sqrt{2}\right)e^{t-\sqrt{2}t}\left(e^{\sqrt{2}t} - 1\right)\left(e^{\sqrt{2}t} - 3 + 2\sqrt{2}\right) \end{pmatrix}\,. \tag{36}$$

## 5. Conclusion

The Dunford-Taylor integral is a classical mathematical tool in complex analysis, which is also ascribed to Luigi Fantappiè and Frigyes Riesz.

We have proved that the Dunford-Taylor integral can be used for deriving the solution of some basic problems of matrix theory, avoiding the knowledge of eigenvalues and eigenvectors.

This formula is sometimes ignored in applications of mathematical physics, economics, and engineering. The basic examples, computed in last section show that the same method could be used in every problem in which a fast computation of the inverse of a non singular complex matrix is crucial.

Furthermore, it permits the computation of the matrix exponential in the solution of the Cauchy problem for a linear system of ordinary differential equations with constant coefficients without the use of a series expansion, and the knowledge of the matrix eigenvectors.

Another field is the analysis of linear dynamical systems, in which, according to the most popular texts, the solution using the matrix exponential, requires the knowledge of eigenvectors and eigenvectors, quite a long way to go.

In our opinion J. Hadamard reasoned when he said that “Le plus court chemin entre deux vérités dans le domaine réel passe par le domaine complexe” (The shortest path between two truths in the real domain passes through the complex domain), as quoted in an article on Jacques Hadamard by Jean-Pierre Kahane in *Math. Intelligencer*, **1991**, *13* (1), 26.

An extension of the used technique to more general matrices will be shown in a subsequent article.

## Compliance with ethical standards

**Conflict of interest** The authors declare that they have not received funds from any institution and that they have no conflict of interest.

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