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Stochastic Integral Representation of Past-Dependent Non-Smooth Brownian Functionals

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We study the questions of the stochastic integral representation of stochastically non-smooth Brownian functionals, which are interesting from the point of view of their practical application in the problem of the European option. In particular, we generalize the Clark-Ocone formula to the case when the functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable, and we establish a method for finding its integrand. Moreover, we consider such functionals that do not satisfy even weakened conditions, that is, non-smooth path-dependent functionals whose conditional mathematical expectations are also not stochastically differentiable.

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1. Introduction and preliminaries

The stochastic integral representation theorem, also known as the martingale representation theorem, states that any square integrable Brownian functional is represented as a stochastic integral with respect to a Brownian motion.The first proof of the martingale representation theorem was implicitly provided by Ito himself (see, [1]). Indeed, here it is proved that any square integrable Wiener functional can be expressed as a series of multiple stochastic integrals, further it is shown that a multiple integral can be expressed as an iterated stochastic integral, and, as a result, a stochastic integral representation can be obtained from here.

In particular, Theorem 4.2 [1] states: any L_2-functional F of Wiener process can be expressible in the form: $F = \sum_{n=0}^{\infty} I_n(f_n)$. Further, according to Theorem 5.1 [1], the multtple Wiener integral $I_n(f_n)$ can be expressible as iterated stochastic intgrals. Therefore, we can write

$$F = EF + \sum_{n=1}^{\infty} I_n(f_n) := EF + \sum_{n=1}^{\infty} \int_0^T \widetilde{I}_{n-1}(g_n(\cdot, t))dW_t =$$

$$= EF + \int_0^T \sum_{n=1}^{\infty} \widetilde{I}_{n-1}(g_n(\cdot, t))dW_t := EF + \int_0^T G(t)dW_t,$$

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where

$$I_n(f_n) := \int_0^T \cdots \int_0^T f_n(t_1, ...t_{n-1}, t) dW_{t_1} \cdots dW_{t_{n-1}} dW_t$$

and

$$\widetilde{I}_{n-1}(g_n) :=$$

$$n! \int_0^T (\int_0^{t_{n-1}} (\cdots \int_0^{t_3} (\int_0^{t_2} f_n(t_1, ...t_{n-1}, t) dW_{t_1}) dW_{t_2}) \cdots) dW_{t_{n-1}}) dW_t.$$

Many years later, Dellacherie ([2]) gave a simple new proof of Ito's theorem using Hilbert space techniques. Many other articles were written afterwards on this problem and its applications but one of the pioneer work on explicit descriptions of the integrand is certainly the one by Clark ([3]). Those of Haussmann ([4]), Ocone ([5]), Ocone and Karatzas ([6]) and Karatzas, Ocone and Li ([7]) were also particularly significant.

In spite of the fact that this problem is closely related to important issues in applications, for example, finding hedging portfolios in finance, much of the work on the subject did not seem to consider explicitness of the representation as the ultimate goal. In many papers using Malliavin calculus or some kind of differential calculus for stochastic processes, the results are quite general but unsatisfactory from the explicitness point of view: the integrands in the stochastic integral representations always involve predictable projections or conditional expectations and some kind of gradients.

Shiryaev and Yor ([8]) proposed a method based on Ito's formula to find explicit martingale representations for Brownian functionals which yields, in particular, the explicit martingale representation of the running maximum of Brownian motion. Even though they consider Clark-Ocone formula ([5]) as a general way to find stochastic integral representations, they raise the question if it is possible to handle it efficiently even in simple cases.

Note now that in all the cases mentioned above, the functionals under study are stochastically (in Malliavin sense) smooth. It has turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation. The second author of the present paper with prof. O. Glonti in [9] considered Brownian functionals which are not stochastically differentiable.

In particular, we generalized the Clark-Ocone formula in case, when the functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable and established the method of finding the integrand. Here we will consider functionals that do not satisfy even these weakened conditions. Such functionals include, for example, the Lebesgue integral (with respect to the time variable) of stochastically non-smooth square-integrable processes.

Let B_t be a Brownian motion on a standard filtered probability space $(\Omega, \Im, \Im_t, P)$ and let $\Im_t = \Im_t^B$ be the augmentation of the filtration generated by B.

Definition 1.1: Let H be the class of functions $f : [0, T] \times \Omega \longrightarrow R$ such that
(i) the mapping $(t, \omega) \longrightarrow f(t, \omega)$ is $\mathcal{B}([0, T]) \otimes \Im$-measurable;
(ii) $f(t, \omega)$ is $\Im_t$-adapted;
(iii) $\int_\Omega [\int_0^T f^2(t, \omega) dt] dP(\omega) < \infty$.

Remark 1: One important property of the Ito stochastic integral: if $f \in H$ then the process $\xi_t = \int_0^t f(s,\omega)dB_s(\omega)$ is a martingale with respect to the filtration $\{\Im_t\}$.

On the other hand, according to the well-known Clark formula ([3]), the inverse statement (so-called martingale representation theorem) is also true. Indeed, if F is a square integrable $\Im_T$-measurable random variable, then (due to the Clark formula) there exists a square integrable $\Im_t$-adapted random process $\varphi(t,\omega)$ such that

$$F = EF + \int_0^T \varphi(t,\omega)dB_t(\omega).$$

Taking the conditional mathematical expectation from the both sides of the last relation we obtain that for the associated to F Levy's martingale $M_t = E[F|\Im_t]$ the following stochastic integral representation is true

$$M_t = M_0 + \int_0^t \varphi(s,\omega)dB_s(\omega).$$

It should be noted that finding the explicit expression for $\varphi(t,\omega)$ is a very difficult problem. In this direction, one general result is known, called Clark-Ocone formula ([5]), according to which $\varphi(t,\omega) = E[D_t^B F|\Im_t]$, where D_t^B is the so called Malliavin stochastic derivative.

Definition 1.2: The class of smooth Brownian functionals S is the class of random variables which has the form

$$F = f(B_{t_1}, ..., B_{t_n}),\ f \in C_p^\infty(R^n),\ t_i \in [0,T],\ n \geq 1,$$

where $C_p^\infty(R^n)$ is the set of all infinitely continuously differentiable functions $f : R^n \to R$ such that f and all of its partial derivatives have polynomial growth.

Definition 1.3: ([10]) The stochastic (Malliavin) derivative of a smooth random variable $F \in S$ is the stochastic process $D_tF := D_t^B F$ given by

$$D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B_{t_1}, ..., B_{t_n})I_{[0,t_i]}(t).$$

In order to interpret $D.F$ as a directional derivative, note that for any element $h \in L_2([0,T])$ we have

$$<D.F, h>_{L_2([0,T])} = \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon}[f(B(h_1) + \epsilon <h_1, h>_{L_2([0,T])}), ...,$$

$$B(h_n) + \epsilon <h_n, h>_{L_2([0,T])})) - f(B(h_1), ..., B(h_n))],$$

(where $B(h) := \int_0^T h(s)dB_s$).

Roughly speaking, the scalar product $<D.F, h>_{L_2([0,T])}$ is the derivative at $\epsilon = 0$ of the random variable F composed with shifted process $\{B(g) + \epsilon <g, h>_{L_2([0,T])},\ \ g \in L_2([0,T])\}$.

Definition 1.4: ([10]) $D.$ is closable as an operator from $L_2(\Omega)$ to $L_2(\Omega; L_2([0,T]))$. We will denote its domain by $D_{2,1} := D_{2,1}^B$. That means, $D_{2,1}$

is equal to the adherence of the class of smooth random variables with respect to the norm

$$||F||_{2,1} = \{E[|F|^2 + (||D.F||^2_{L_2([0,T])})]\}^{1/2}.$$

In general, as it known from Malliavin calculus, we introduce the norm

$$||F||_{p,1} = \{E[|F|^p + (||D.F||^2_{L_2([0,T])})^{p/2}]\}^{1/p},$$

where $D.$ is the Malliavin derivative operator and $D_{p,1} := D^B_{p,1}$ denotes the Banach space which is the closure of the class of smooth Brownian functionals S with respect to the norm $||\cdot||_{p,1}$ $(p \geq 1)$.

In fact, above, we have defined the Malliavin derivative as an "inverse" of the Ito stochastic integral (with deterministic integrand) in the sense that $DB(h) = h$ (where

$$B(h) := \int_0^T h(s)dB_s \text{ and } D^B_t\Big(\int_0^T h(s)dB_s\Big) = h(t),$$

as well as it's clear that $B_\theta = B(I_{[0,\theta]}(\cdot))$ and $D_t B_\theta = I_{[0,\theta]}(t))$.

In the white noise case, we can go a bit deeper in this way by showing that the Malliavin derivative of an iterated integral of order n is an iterated integral of order $n-1$.

Definition 1.5: ([10]) Let F be an element of $D_{2,1}$ with chaotic expansion $F = EF + \sum_{n=1}^{\infty} I_n(f_n)$, where f_n is a symmetric element of $L_2([0,T]^n)$ for every $n \geq 1$. Then the Malliavin derivative of F is of the form

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \quad t \in [0,T].$$

When the random variable F belongs to the Hilbert space $D_{2,1}$, it turns out that the integrand in the Clark representation (1) can be identified as the optional projection of the derivative of F.

Theorem 1.6: *([5]) If F is differentiable in the Malliavin sense, $F \in D_{2,1}$, then the following stochastic integral representation is fulfilled*

$$F = E[F] + \int_0^T E[D_t F|\Im_t]\mathrm{d}B_t. \tag{1}$$

Remark 2: On the basis of the "good λ-inequality" of Burkholder-Gundy this result was extended by Karatzas, Ocone and Li ([7]) on fuctionals from the Banach space $D_{1,1}$.

A different method for finding the process $\varphi(t,\omega)$ was proposed by Shiryaev, Yor ([8]) and Shiryaev, Yor and Graversen ([11]), which was based on the Ito (generalized) formula and the Levy theorem for the Levy martingale $M_t = E[F|\Im_t]$ associated with F.

Theorem 1.7: *([8]) Let $M_T = \sup_{0\leq t\leq T} B_t$. Then the following stochastic inte-*

gral representation holds

$$M_T = EM_T + 2\int_0^T [1 - \Phi(\frac{M_t - B_t}{\sqrt{T-t}})]\mathrm{d}B_t,$$

where Φ *is a standard normal distribution function.*

Theorem 1.8 : *([11]) Let* $g_T = sup\{0 < t \leq T : B_t = 0\}$, $M_u = \max_{t \leq u} B_t$ *and* $M_{g_T} = \max_{t \leq g_T} W_t$. *Then we have*

$$M_{g_T} = \frac{1}{2}EM_T + \int_0^T [\frac{1}{2}\Psi(\frac{2B_u - B_u}{\sqrt{T-u}}) - (M_u - M_{g_u})\varphi_{T-u}(B_u)]\mathrm{d}B_u,$$

where

$$EM_T = \sqrt{2T/\pi},$$

$$\Psi(x) = 2[1 - \Phi(x)] = 2[1 - \int_{-\infty}^x \varphi(u)du],$$

$$\varphi_{T-u}(x) = \frac{1}{\sqrt{T-u}}\varphi(\frac{x}{\sqrt{T-u}})$$

and

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

Later on, using the Clark-Ocone formula, Renaud and Remillard ([12]) have established explicit martingale representations for path-dependent Brownian functionals (a direct consequences of which are explicit martingale representations of geometric Brownian motion).

Let us define $B_t^\theta = B_t + \theta t$; $m_t^\theta = \inf_{0 \leq s \leq t} B_s^\theta$; $M_t^\theta = \sup_{0 \leq s \leq t} B_s^\theta$; $m_t = m_t^0$; $M_t = M_t^0$; $Div(G) = \partial_x G + \partial_y G + \partial_z G$; $Div_{x,y}(G) = \partial_x G + \partial_y G$; $Div_{x,z}(G) = \partial_x G + \partial_z G$; for $b < a < c$, $b < 0$, $c > 0$, and $\tau = T - t$:

$$\begin{aligned} f(a,b,c;t) = e^{-\frac{1}{2}\theta^2\tau}E[&DivG(B_\tau + a, m_\tau + a, M_\tau + a)e^{\theta B_\tau}I_{\{m_\tau \leq b-a, c-a \leq M_\tau\}} + \\ &+ Div_{x,y}G(B_\tau + a, m_\tau + a, c)e^{\theta B_\tau}I_{\{m_\tau \leq b-a, M_\tau \leq c-a\}} + \\ &+ Div_{x,z}G(B_\tau + a, b, M_\tau + a)e^{\theta B_\tau}I_{\{b-a \leq m_\tau, c-a \leq M_\tau\}} + \\ &+ \partial_x G(B_\tau + a, b, c)e^{\theta B_\tau}I_{\{b-a \leq m_\tau, M_\tau \leq c-a\}}]. \end{aligned}$$

Theorem 1.9 : *([12]) If* $G : R^3 \longrightarrow R$ *is a continuously differentiable function with bounded partial derivatives or a Lipschitz function, then the Brownian functional* $X = G(B_T^\theta, m_T^\theta, M_T^\theta)$ *admits the following martingale representation:*

$$X = EX + \int_0^T f(B_t^\theta, m_t^\theta, M_t^\theta; t)\mathrm{d}B_t.$$

Corollary 1.10: *Taking here $G(x,y,z)=z$ and using the fact that the density function of M_t is given by $z \longmapsto \sqrt{\frac{2}{\pi t}} e^{-\frac{z^2}{2t}} I_{\{z\geq 0\}}$, one obtains (see, Theorem 2) the martingale representation of M_T :*

$$M_T = EM_T + 2\int_0^T [1-\Phi(\frac{M_t - B_t}{\sqrt{T-t}})]\mathrm{d}B_t.$$

Our approach (with Dr. V. Jaoshvili) within the classical Ito's calculus allows to construct $\varphi(t,\omega)$ explicitly, by using both the standard L_2 theory and the theories of weighted Sobolev spaces, for some class of functionals F that do not have a stochastic derivative. For example, we have proved

Theorem 1.11: *([13]) Let the function $f \in L_{2,T/\alpha}, 0<\alpha<1$, and it has the generalized derivative of the first order $\partial f/\partial x$, such that $\partial f/\partial x \in L_{2,T/\beta}$, $0<\beta<1/2$, then the following integral representation holds*

$$f(B_T) = Ef(B_T) + \int_0^T E[\frac{\partial f}{\partial x}(B_T)|\Im_t]\mathrm{d}B_t, \tag{2}$$

where $L_{2,T}$ denotes the set of measurable functions $u : R \to R$, such that $u(\cdot)\rho(\cdot,T) \in L_2 := L_2(R, \mathcal{B}(R), \lambda)$, (where $\mathcal{B}(R)$ is the Borel $\sigma-$ algebra on R, λ is the Lebesgue measure and $\rho(x,T) = \exp\{-\frac{x^2}{2T}\}$).

As already noted, the class of martingales to which the Clark-Ocone formula can be applied, however, limited by the condition that the terminal value of the martingale must be Malliavin differentiable. It is obvious that in all above-mentioned works this requirement is fulfilled.

On the other hand, in spite of the fact that Clark-Ocone formula gives construction of integrand, there are problems with practical realizations. In particular, even in case of smoothness of F, calculation of its Malliavin derivative and then conditional mathematical expectation (or predictable projection in general case) of obtained expression are rather difficult.

We studied the questions of the stochastic integral representation of stochastically non-smooth functionals interesting from the point of view of their practical application in the problem of the European Option. In particular, we generalized the Clark-Ocone formula in case, when the functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable and established the method of finding of its integrand. In addition, we also consider non-smooth Brownian functionals that do not satisfy even weakened conditions.

2. Generalized Clark-Ocone formula

It is well-known, that if a random variable is stochastically differentiable in Malliavin sense, then its conditional mathematical expectation is differentiable too ([14]).

Lemma 2.1: *(Lemma 2. 4 (II) [14]) In f $F \in D_{2,1}$, then $E(F|\Im_s) \in D_{2,1}$ and*

$$D_t[E(F|\Im_s)] = E(D_tF|\Im_s)I_{[0,s]}(t).$$

On the other hand, it is possible that conditional expectation can be smooth even if a random variable is not stochastically smooth. For example, it is well known that $I_{\{B_T \le C\}} \notin D_{2,1}$, but for all $t \in [0,T)$:

$$E[I_{\{B_T \le C\}} | \Im_t] = \Phi\left(\frac{C - B_t}{\sqrt{T-t}}\right) \in D_{2,1}.$$

Remark 1 : It should be noted that the indicator of event A is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one (see Proposition 1.2.6 [14]).

Theorem 2.2 : *([9]) Suppose that $G_t = E(F|\Im_t)$ is Malliavin differentiable ($G_t(\cdot) \in D_{2,1}$) for almost all $t \in [0,T)$. Then we have the stochastic integral representation*

$$G_T = F = EF + \int_0^T \nu_s \mathrm{d}B_s \quad (P - a.s.),$$

where

$$\nu_s := \lim_{t \uparrow T} E[D_s G_t | \Im_s] \quad in \ \ the \quad L_2([0,T] \times \Omega).$$

Proposition 2.3: *Let $F \in D_{2,1}$. Then Theorem 6 implies the Clark-Ocone representation (1) and that the following relation is valid:*

$$\lim_{t_n \to T} E(D_s G_{t_n} | \Im_s) = E(D_s \lim_{t_n \to T} G_{t_n} | \Im_s) \ \ in \ \ L_2([0,T] \times \Omega).$$

Proof: According to the Lemma 1 $G_{t_n} = E(F|\Im_{t_n}) \in D_{2,1}$ for any sequence $t_n \uparrow T$ and

$$D_s(G_{t_n}) = E(D_s F | \Im_{t_n}) I_{[0,t_n]}(s) = \begin{cases} 0, & s > t_n; \\ E(D_s F | \Im_{t_n}), & 0 \le s \le t_n. \end{cases}$$

Hence, Due to the Theorem 6, using the telescopic property of conditional mathematical expectation, we easily obtain the Clark-Ocone representation

$$\begin{aligned} F = EF + \int_0^T \lim_{t_n \to T} E(D_s G_{t_n} | \Im_s) \mathrm{d}B_s = \\ = EF + \int_0^T \lim_{t_n \to T} \{E[E(D_s F | \Im_{t_n}) | \Im_s] I_{[0,t_n]}(s)\} \mathrm{d}B_s = \\ = EF + \int_0^T \lim_{t_n \to T} \{E(D_s F | \Im_s) I_{[0,t_n]}(s)\} \mathrm{d}B_s = \\ = EF + \int_0^T E(D_s F | \Im_s) \mathrm{d}B_s. \end{aligned}$$

Moreover, substituting $F = \lim_{t_n \to T} G_{t_n}$ into the last relation, we conclude that

$$\lim_{t_n \to T} E[D_s(G_{t_n}) | \Im_s] = E[D_s(\lim_{t_n \to T} G_{t_n}) | \Im_s].$$

$\square$

□

Remark 2 : It should be noted that, despite the fact that the operator of stochastic derivative is not a continuous operator, in our case, we have "continuity" in a weak sense.

Proposition 2.4: *For any real $x \in R$ the non-smooth Brownian functional $F(x) = I_{\{B_T \le x\}}$ have the representation*

$$I_{\{B_T \le x\}} = \Phi(\frac{x}{\sqrt{T}}) - \int_0^T \frac{1}{\sqrt{T-s}} \varphi(\frac{x-B_s}{\sqrt{T-s}}) \mathrm{d}B_s.$$

Proof: On the one hand, it is clear that

$$EF(x) = P\{B_T \le x\} = \Phi(\frac{x}{\sqrt{T}}).$$

On the other hand, for all $t \in [0, T)$, we have

$$G_t := G_t(x) = E[F(x)|\Im_t] = \Phi(\frac{x-B_t}{\sqrt{T-t}}).$$

Hence, according to the rule of stochastic differentiation of a composite function (Proposition 1.2.3 [14]) the Malliavin derivative of the functional G_t $(t \in [0, T))$ has the form

$$D_s G_t = -I_{[0,t]}(s) \frac{1}{\sqrt{T-t}} \varphi(\frac{x-B_t}{\sqrt{T-t}}).$$

Therefore, due to the well-known properties of Brownian motion, using the standard integration technique, it is not difficult to see that

$$\begin{aligned} E(D_s G_t|\Im_s) &= -I_{[0,t]}(s) \frac{1}{\sqrt{T-t}} E[\varphi(\frac{x-B_t}{\sqrt{T-t}})|\Im_s] = \\ &= -\frac{I_{[0,t]}(s)}{\sqrt{T-t}} \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} \varphi(\frac{x-y}{\sqrt{T-t}}) \exp\{-\frac{(y-B_s)^2}{2(t-s)}\} \mathrm{d}y = \\ &= -I_{[0,t]}(s) \frac{1}{\sqrt{2\pi(t-s)}} \frac{1}{\sqrt{2\pi(T-t)}} \times \\ &\times \int_{-\infty}^{\infty} \exp\{-\frac{(x-y)^2}{2(T-t)}\} \exp\{-\frac{(y-B_s)^2}{2(t-s)}\} \mathrm{d}y = \\ &= -I_{[0,t]}(s) \exp\{-\frac{(x-B_s)^2}{2(T-s)}\} \frac{1}{\sqrt{2\pi(t-s)}} \frac{1}{\sqrt{2\pi(T-t)}} \times \\ &= -I_{[0,t]}(s) \frac{1}{\sqrt{T-s}} \varphi(\frac{x-B_s}{\sqrt{T-s}}). \end{aligned}$$

Further, it is evident that in this case there exists a sequence $t_n \in [0, T)$, $t_n \uparrow T$, such that

$$\nu_s := \lim_{t \uparrow T} E[D_s G_t|\Im_s] =$$

$$= -I_{[0,T]}(s)\frac{1}{\sqrt{T-s}}\varphi(\frac{x-B_s}{\sqrt{T-s}}) \quad in \quad L_2([0,T]\times\Omega),$$

which, based on Theorem 6, completes the proof of the proposition. □

□

Remark 3: It should be noted that Proposition 2 can also be obtained from Theorem 5. On the other hand, Theorem 6 can also be used for smooth functionals (see Proposition 3 below).

Proposition 2.5: *The smooth Brownian functional* $F = B_T^+ := max\{0, B_T\}$ *have the following stochastic integral representation*

$$B_T^+ = \sqrt{\frac{T}{2\pi}} + \int_0^T \Phi(\frac{B_s}{\sqrt{T-s}})\mathrm{d}B_t.$$

Proof: It is easy to see that

$$EB_T^+ = \sqrt{\frac{T}{2\pi}}.$$

Further, using the Glonti-Purtukhia approach, we have

$$G_t = E(B_T^+|\Im_t) = E[I_{\{B_T>0\}}B_T|\Im_t] =$$

$$= \frac{1}{\sqrt{2\pi(T-t)}}\int_0^\infty x\exp\{-\frac{(x-B_t)^2}{2(T-t)}\}\mathrm{d}x.$$

Hence, due to the rule of stochastic differentiation and the standard integration technique, we obtain

$$\begin{aligned}
D_sG_t &= I_{[0,t]}(s)\frac{1}{\sqrt{2\pi(T-t)}}\int_0^\infty \frac{x(x-B_t)}{T-t}\exp\{-\frac{(x-B_t)^2}{2(T-t)}\}\mathrm{d}x = \\
&= I_{[0,t]}(s)\frac{1}{\sqrt{2\pi(T-t)}}\int_{-\frac{B_t}{\sqrt{T-t}}}^\infty x(\sqrt{T-t}x+B_t)\exp\{-\frac{x^2}{2}\}\mathrm{d}x = \\
&= -I_{[0,t]}(s)\frac{1}{\sqrt{2\pi(T-t)}}\{\sqrt{T-t}\int_{-\frac{B_t}{\sqrt{T-t}}}^\infty x\mathrm{d}(\exp\{-\frac{x^2}{2}\}) + \\
&+ B_t\int_{-\frac{B_t}{\sqrt{T-t}}}^\infty \mathrm{d}(\exp\{-\frac{x^2}{2}\})\} = I_{[0,t]}(s)\Phi(\frac{B_t}{\sqrt{T-t}}).
\end{aligned}$$

Therefore

$$E(D_sG_t|\Im_s) = I_{[0,t]}(s)\frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^\infty \Phi(\frac{x}{\sqrt{T-t}})\exp\{-\frac{(x-B_s)^2}{2(t-s)}\}\mathrm{d}x.$$

Now, using the relation

$$\lim_{t\to T}\Phi(\frac{x}{\sqrt{T-t}}) = \begin{cases} 0, & x<0; \\ 1, & x>0 \end{cases}$$

it is easy to check that

$$\lim_{t\to T} E(D_sG_t|\Im_s) = \lim_{t\to T}\{I_{[0,t]}(s)\frac{1}{\sqrt{2\pi(t-s)}} \times$$
$$\times \int_{-\infty}^{\infty} \Phi(\frac{x}{\sqrt{T-t}}) \exp\{-\frac{(x-B_s)^2}{2(t-s)}\}\mathrm{d}x\} =$$
$$= I_{[0,T]}(s)\frac{1}{\sqrt{2\pi(T-s)}}\int_{-\infty}^{\infty} \lim_{t\to T}[\Phi(\frac{x+B_s}{\sqrt{T-t}}) \exp\{-\frac{x^2}{2(T-s)}\}]\mathrm{d}x =$$
$$= I_{[0,T]}(s)\frac{1}{\sqrt{2\pi(T-s)}}\int_{-B_s}^{\infty} \exp\{-\frac{x^2}{2(T-s)}\}\mathrm{d}x =$$
$$= I_{[0,T]}(s)[1-\Phi(\frac{-B_s}{\sqrt{T-s}})] = I_{[0,T]}(s)\Phi(\frac{B_s}{\sqrt{T-s}}),$$

which, on the basis of Theorem 6, together with the above relations, completes the proof of the proposition. □

□

3. Stochastic integral representation of past-dependent non-smooth Brownian functionals

It should be noted that there are also such functionals which don't satisfy even the above-mentioned weakened conditions, i.e. the non-smooth functionals whose conditional mathematical expectation is not stochastically differentiable either. In particular, to such functional belongs the integral type functional $\int_0^T u_s(\omega)ds$ with the non-smooth integrand $u_s(\omega)$.

It is well known that if $u_s(\omega) \in D_{2,1}$ for all s, then $\int_0^T u_s(\omega)\mathrm{d}s \in D_{2,1}$ and

$$D_t\{\int_0^T u_s(\omega)\mathrm{d}s\} = \int_0^T D_t u_s(\omega)\mathrm{d}s.$$

But if $u_s(\omega)$ is not differentiable in the Malliavin sense, then the Lebesgue average (with respect to ds) is not either differentiable in the Malliavin sense (see, Theorem 2 [15]).

Indeed, in this case the conditional mathematical expectation is not stochastically smooth, because we have:

$$E[\int_0^T u_s(\omega)\mathrm{d}s|\Im_t] = \int_0^t u_s(\omega)\mathrm{d}s + \int_t^T E[u_s(\omega)|\Im_t]\mathrm{d}s,$$

where the first summand (integral) is analogous that the initial integral and therefore it is not Malliavin differentiable, but the second summand is differentiable in the Malliavin sense when u_s satisfied our weakened condition (if $E[u_s(\omega)|\Im_t] \in D_{2,1}$ for almost all s and $E[u_s(\omega)|\Im_t]$ is Lebesgue integrable for a.a. ω, then

$$\int_t^T E[u_s(\omega)|\Im_t]\mathrm{d}s \in D_{2,1}).$$

It should be noted that integral functionals of this type were considered in the works of Glonti and Purtukhia ([16]) and Glonti, Jaoshvili and Purtukhia ([17]).

In particular, they developed the method of obtaining the integral representation using the Trotter-Meyer Theorem which establishes the relation between the predictable square variation of semimartingale and its local time.

Theorem 3.1 : *(Trotter-Meyer Theorem) For any measurable and bounded real function ψ the following relation*

$$\int_0^T \psi(S_t)\mathrm{d}\langle S\rangle_t = \int_{-\infty}^{\infty} l_T^x(S)\psi(x)\mathrm{d}x,$$

is true, where $\langle S\rangle_t$ is the predictable square variation of the S.

For this, at the first stage, the Clark stochastic integral representation for local time was obtained, and then, using the Trotter-Meyer theorem, based on the Fubini theorem of stochastic type, Clark integral representations of European options for the payoff functions of integral type

$$\int_0^T I_{\{a\le S_t\le b\}}\mathrm{d}t \quad \textit{and} \quad \int_0^T I_{\{a\le S_t\le b\}}S_t^2\mathrm{d}t$$

have been obtained and the corresponding hedging problems have been solved in the cases of the Bachelier and Black-Scholes models, respectively, with a zero interest rate.

Theorem 3.2 : *([16]) In the case of Bachelier market model for any real numbers $C_1 < C_2$ we have the following stochastic integral representation*

$$\int_0^T I_{\{C_1\le S_t\le C_2\}}\mathrm{d}t = \int_0^T \big[\Phi(\frac{C-1-rt}{\sigma\sqrt{t}})\big]|_{C=C_1}^{C_2}\mathrm{d}t-$$

$$-\int_0^T\int_s^T \frac{1}{\sqrt{t-s}}\big[\varphi(\frac{C-1-rt-\sigma\widetilde{B}_s}{\sigma\sqrt{t-s}})\big]|_{C=C_1}^{C_2}\mathrm{d}t\mathrm{d}\widetilde{B}_s.$$

Theorem 3.3 : *([16]) In the case of Black-Scholes model for the functional*

$$F = \int_0^T I_{\{C_1\le S_t\le C_2\}}S_t^2\mathrm{d}t$$

the following integral representation formula is fulfilled

$$F = \frac{1}{\sigma^2}\int_{C_1}^{C_2}[\widetilde{E}(|S_T-x|)-|1-x|]dx + \int_0^T v_t\mathrm{d}\widetilde{B}_t,$$

where

$$v_t = \frac{1}{\sigma}S_t\int_{C_1}^{C_2}\{1-2\Phi\big[\frac{\ln x-\sigma\widetilde{B}_t-\sigma^2(T/2-t)}{\sigma\sqrt{T-t}}\big]-sign(S_t-x)\}\mathrm{d}x.$$

In [18], the functional was studied, which can be considered as the payoff function of an exotic option (that is, a certain combination of binary and Asian options), and the problem of hedging was investigated. Unfortunately, the above approach based on the Trotter-Meier theorem is not applicable here. In particular, in [18] the European Option with payoff function

$$\int_0^T I_{\{C_1 \le S_t \le C_2\}} ln(S_t)\mathrm{d}t,$$

was considered where S_t is a geometrical Brownian motion and $C_1 < C_2$ is some real number.

Theorem 3.4: *([18]) In the scheme of Black-Scholes model, for any real positive numbers $C_1 < C_2$, the following stochastic integral representation is true:*

$$\int_0^T I_{\{C_1 \le S_t \le C_2\}} ln(S_t)\mathrm{d}t = \int_0^T \Big[bt\Phi(h_4(t)) - \sigma\sqrt{t}\varphi(h_4(t))\Big] |_{C=C_1}^{C_2} \mathrm{d}t +$$

$$+\int_0^T \left\{ \int_u^T \left[\sigma\Phi(h_5(t,u)) - \frac{lnC}{\sqrt{t-u}}\varphi(h_5(t,u))\right] |_{C=C_1}^{C_2} dt \right\} \mathrm{d}\widetilde{B}_u,$$

where S_t denotes the risky asset price,

$$h_4(t) = \frac{lnC - bt}{\sigma\sqrt{t}},$$

$$h_5(t,u) = \frac{lnC - bt - \sigma\widetilde{B}_u}{\sigma\sqrt{t-u}}.$$

Besides, Glonti and Purtukhia ([19]) and Livinska and Purtukhia ([20]) also considered a path-dependent, stochastically non-smooth Brownian functionals of the type

$$(B_T - K)^+ I_{\{M_T \le L\}} \quad and \quad (B_T - C_1)^- I_{\{m_T \le C_2\}}$$

(wher $M_T = \sup_{0\le t\le T} B_t$ and $m_T = \inf_{0\le t\le T} B_t$) respectively and obtained formulas for the stochastic integral representation with an explicit form of the integrand. For this, the conditional distribution density function of the joint distribution of the Brownian motion and its maximum (respectively, minimum) process was investigated for a given value of the Brownian motion, the conditional mathematical expectation of the corresponding functional was calculated, its stochastic smoothness was checked, and the above-mentioned Glonti-Purtukhia generalization of the Clark-Ocone formula was applied.

Theorem 3.5: *([19]) For the functional $(B_T - K)^+ I_{\{M_T \le L\}}$ the following integral representation is fulfilled*

$$F = EF - \int_0^T \frac{2(L-K)}{\sqrt{T-t}}\varphi(\frac{L-B_t}{\sqrt{T-t}})\mathrm{d}B_t +$$

$$+\int_0^T \{\Phi(\frac{B_t - K}{\sqrt{T-t}}) - \Phi[\frac{B_t - (2L-K)}{\sqrt{T-t}}]\}\mathrm{d}B_t.$$

Theorem 3.6: *([20]) For the Brownian functional $F = (B_T - C_1)^- I_{\{m_T \le C_2\}}$ $(C_2 \le 0,\ C_2 \le C_1)$ the following stochastic integral representation holds*

$$F = EF - \int_0^T \Phi\Big(\frac{2C_2 - C_1 - B_t}{\sqrt{T-t}}\Big)\mathrm{d}B_t.$$

Remark 1: Note that this functional is a typical example of payoff function of so called European barrier[1] and lookback[2] Options. Hence, obtained here stochastic integral representation formula could be used to compute the explicit hedging portfolio of such barrier and lookback option.

Consider now the Brownian functional of integral type $F = \int_0^T f(B_t)\mathrm{d}t$. We introduce the notation

$$V(t,x) := E[\int_t^T f(B_s)\mathrm{d}s | B_t = x].$$

Theorem 3.7: *If $f(x)$ is a bounded measurable function on R^1, then the function $V(t,x)$ satisfies the requirements of the Ito formula and the following stochastic integral representation is valid*

$$\int_0^T f(B_t)\mathrm{d}t = \int_0^T E[f(B_t)]\mathrm{d}t + \int_0^T V_x'(t,B_t)\mathrm{d}B_t.$$

Proof: It is well known that for all measurable bounded functions h and $t > s$ we have

$$E[h(B_t)|\Im_s] = \int_{-\infty}^{\infty} h(y)p(s,t,B_s,\mathrm{d}y), \tag{3}$$

where for any Borel subset A of $(-\infty,\infty)$: $p(s,t,B_s,A) = P(B_t \in A|\Im_s)$ the transition probability of Brownian motion and

$$p(s,t,x,A) = \frac{1}{\sqrt{2\pi(t-s)}}\int_A \exp\{-\frac{(x-y)^2}{2(t-s)}\}\mathrm{d}y.$$

Due to the relation (3), using the well-known properties of conditional mathematical expectation and Brownian motion, we can write

$$V(t,x) = \{E[\int_t^T f(B_s)\mathrm{d}s|B_t]\}|_{B_t=x} =$$

$$= \{\int_t^T E[f(B_s)|B_t]\mathrm{d}s\}|_{B_t=x} = \{\int_t^T E[f(B_s)|\Im_t]\mathrm{d}s\}|_{B_t=x} =$$

[1]The barrier option is either nullified, activated or exercised when the underlying asset price breaches a barrier during the life of the option.
[2]The payoff of a lookback option depends on the minimum or maximum price of the underlying asset attained during certain period of the life of the option.

$$= \{\int_t^T [\int_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{2\pi(s-t)}}\exp\{-\frac{(B_t-y)^2}{2(s-t)}\mathrm{d}y]\mathrm{d}s\}|_{B_t=x} =$$

$$= \int_t^T \{\frac{1}{\sqrt{2\pi(s-t)}}[\int_{-\infty}^{\infty} f(y)\exp\{-\frac{(x-y)^2}{2(s-t)}\mathrm{d}y]\}\mathrm{d}s.$$

The last relation shows that, on the one hand, $V(t,x)$ is an integral with variable boundary with respect to t, and on the other hand, with respect to x, it is an integral that depends on a parameter. Therefore, it is easy to verify that in our case $V(t,x)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to x, that is, $V(t,x)$ satisfies the conditions of the It's formula.

According to Ito's formula, we have

$$V(t,B_t) = V(0,B_0) + \int_0^t [V_s^{'}(s,B_s) + \frac{1}{2}V_{xx}^{''}(s,B_s)]\mathrm{d}s +$$

$$+\int_0^t V_x^{'}(s,B_s)\mathrm{d}B_s \quad (P-a.s.). \tag{4}$$

On the other hand, due to the Markov property of the Brownian motion

$$V(t,B_t) = \{E[\int_t^T f(B_s)\mathrm{d}s|B_t = x]\}|_{x=B_t} =$$

$$= E[\int_t^T f(B_s)\mathrm{d}s|B_t] = E[\int_t^T f(B_s)\mathrm{d}s|\Im_t] \quad (P-a.s.)$$

Therefore, under the conditions of the theorem, the process

$$\int_0^t f(B_s)\mathrm{d}s + V(t,B_t) = E[\int_0^t f(B_s)\mathrm{d}s|\Im_t] +$$

$$+E[\int_t^T f(B_s)\mathrm{d}s|\Im_t] = E\big[\int_0^T f(B_s)\mathrm{d}s|\Im_t\big] := M_t$$

is a martingale.

Further, according to Levy's theorem, it is obvious that M_t is a continuous martingale. On the other hand, a continuous martingale of bounded variation starting from 0 is identically equal to 0. Therefore, in equality (4), the term of bounded variation in total with an additional term ($\int_0^t f(s,B_s)\mathrm{d}s$) of bounded variation of martingale M is equal to zero.

Hence, taking into account the equality

$$M_0 = V(0,B_0) = E[\int_0^T f(B_s)\mathrm{d}s|B_0] =$$

$$= E[\int_0^T f(B_s)\mathrm{d}s|\Im_0] = E[\int_0^T f(B_s)\mathrm{d}s] \quad (P-a.s.)$$

we easily complete the proof of the theorem.

□

□

Remark 2 : It should be noted that the result of Theorem 13 is especially interesting for stochastically non-smooth $f(B_t)$, although it is also useful for smooth $f(B_t)$. On the other hand, if $f(B_t) \in D_{2,1}$ for almost all t, then the Clark-Ocone representation for the functional $F = \int_0^T f(B_t)\mathrm{d}t$ follows from Theorem 13. As for the non-smooth functionals, if we consider, for example, the function $f(B_t) = I_{\{B_t \leq C\}}$ (for some constunt C) which is not a Malliavin differentiable, then the path-dependent functional $F = \int_0^T I_{\{B_t \leq C\}}\mathrm{d}t$ is also stochastically non-smooth, for which even the weakened Glonti-Purtukhia requirement fails.

Proposition 3.8: *For any real numbers $C_1 < C_2$, the following stochastic integral representation is fulfilled*

$$\int_0^T I_{\{C_1 \leq B_t \leq C_2\}}\mathrm{d}t = \int_0^T [\Phi(x/\sqrt{t})]|_{x=C_1}^{x=C_2}\mathrm{d}t-$$

$$-\int_0^T (\int_t^T \frac{1}{\sqrt{s-t}}\varphi(\frac{x-B_t}{\sqrt{s-t}})|_{x=C_1}^{x=C_2}\mathrm{d}s)\mathrm{d}B_t.$$

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