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Iterative algorithms for finding minimum-norm fixed point of a finite family of nonexpansive mappings and applications

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Abstract

This paper deals with iterative methods for approximating the minimum-norm common fixed point of nonexpansive mappings. The proposed cyclic iterative algorithms and simultaneous iterative algorithms combined with a relaxation factor, which make them more flexible to solve the considered problem. Under certain conditions on the parameters, we prove that the sequences generated by the proposed iteration scheme converge strongly to the minimum-norm common fixed point of a finite family of nonexpansive mappings. Furthermore, as applications, we obtain several new strong convergence theorems for solving the multiple-set split feasibility problem which has been found application in intensity modulated radiation therapy. Our results extend and improve some known results in the literature.

Keywords: Common fixed point, nonexpansive mappings, minimum-norm, cyclic iteration method, simultaneous iteration method.
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1. Introduction

Many problems in engineering, signal and image processing can be modeled by finding fixed points of nonexpansive mappings. To find these fixed points, it usually needs to develop an efficient iterative algorithm. Since the nonexpansive mappings may have more than one fixed point, it is a very interesting

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problem to construct iterative algorithms to find the minimum-norm fixed point of it. A unique $x^* \in \text{Fix}(T)$ is said to be the minimum-norm fixed point of T, which satisfies:

$$\|x^*\| = \min\{\|x\| : x \in \text{Fix}(T)\}.$$

In other words, x^* is the metric projection of the origin onto Fix(T), i.e., $x^* = P_{\text{Fix}(T)}0$. In this paper, Fix(T) denotes the fixed points set of mapping T. A number of iterative algorithms for finding the fixed point of nonexpansive mappings have been proposed. See for example [6, 12–15, 18, 24, 26, 27, 29]. In particular, Cui and Liu [8] and Yao and Xu [28] independently introduced two iterative algorithms. One is implicit iterative algorithm, which is defined by the following

$$x_t = P_C((1-t)Tx_t), \quad t \in (0,1). \tag{1.1}$$

The other is explicit iterative algorithm, which is defined as

$$x_{n+1} = P_C((1-t_n)Tx_n), \quad t_n \in (0,1),\ n \geq 0. \tag{1.2}$$

They proved that both the net $\{x_t\}$ and the sequence $\{x_n\}$ converge strongly to the minimum-norm fixed point of nonexpansive mapping T in real Hilbert spaces. These iterative algorithms (1.1) and (1.2) can be viewed as a modification of the well-known iterative algorithms of Browder [1] and Halpern [9], respectively. Yang et al. [25] proposed two relaxed iterative algorithms below,

$$x_t = (1-\beta)P_C[(1-t)x_t] + \beta Tx_t, \quad t, \beta \in (0,1),$$

and

$$x_{n+1} = (1-\beta)P_C[(1-t_n)x_n] + \beta Tx_n, \quad n \geq 0,$$

where $\beta \in (0,1)$, $\{t_n\} \subset (0,1)$. They proved that the two iterative sequences converge strongly to the minimum-norm fixed point of the nonexpansive mapping T provided certain conditions on the parameters. Cai et al. [4] proposed two new implicit and explicit iteration methods, they proved strong convergence of the iterative sequence to the minimum-norm fixed point of nonexpansive mappings T under weaker assumptions about the iterative parameters than in [8] and [28]. If the iterative sequence $\{Tx_n\}$ involved in [4] is replaced by $\{TP_Cx_n\}$, then the self-mapping T defined on nonempty closed convex cone in [4] could be relaxed to nonempty closed convex set. It is different from those iterative algorithms proposed by Yang et al. [25], Tang and Liu [21] introduced a new relaxed implicit and explicit iterative scheme to approximate the minimum-norm fixed point of nonexpansive mappings in a real Hilbert spaces. Sunthrayuth et al. [16] proved several iteration schemes converges strongly to a fixed point of nonexpansive mappings T, which is a unique solution of some variational inequalities. As a direct result, they obtained the unique minimum-norm fixed point of T. The corresponding results recovered the main results of Tang and Liu [21]. In addition to the nonexpansive mappings, Zegeye et al. [31] proposed an implicit and explicit iteration process to approximate a minimum fixed point of pseudocontractive mappings.

There are also some real application problems, which require to find a common fixed point of nonexpansive mappings. In [11], Liu and Cui extended the iterative algorithms for finding the minimum-norm fixed point of a single nonexpansive mapping to the case of a finite family of nonexpansive mappings. They proposed the following iteration methods:

A cyclic iteration method

$$x_{n+1} = P_C((1-t_n)T_{[n+1]}x_n), \quad n \geq 0,$$

where $\{t_n\} \subset (0,1)$, $T_{[n]} := T_{n \bmod N}$ with the mod N function taking values in the set $\{1, 2, \cdots, N\}$.

A simultaneous iteration method

$$x_{n+1} = P_C((1-t_n)\sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n), \quad n \geq 0,$$

where $\lambda_i^{(n)} > 0$ for all $n \geq 0$, $i = 1, 2, \cdots, N$, and $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ for all $n \geq 0$. They proved that the sequence $\{x_n\}$ generated by the cyclic, and the simultaneous iteration method converges strongly to the minimum-norm common fixed point of nonexpansive mappings $\{T_i\}_{i=1}^{N}$ provided appropriate conditions on the parameters. The cyclic algorithmic structures cater for the row action approach while the simultaneous algorithmic structures favor parallel computing platforms. A relaxed simultaneous iterative algorithm for finding minimum-norm common fixed point of (asymptotically) nonexpansive mapping was proposed by Zegeye and Shahzad [30]. For the nonexpansive mappings, the sequence $\{x_n\}$ was given by

$$x_{n+1} = \beta_{n,0} x_n + \sum_{i=1}^{N} \beta_{n,i} T_i P_C((1 - \alpha_n) x_n), \quad n \geq 0. \tag{1.3}$$

They obtained that the sequence $\{x_n\}$ converges strongly to the minimum-norm common fixed point of mappings $\{T_i\}_{i=1}^{N}$ provided certain conditions for the iterative parameters. To find minimum-norm common fixed point of infinite noncountable nonexpansive mappings, Tang et al. [20] proposed a relaxed iteration method for finding the minimum-norm common fixed point of a nonexpansive semigroup.

We are interested to combine the relaxation iteration method with the cyclic and simultaneous iteration methods. The proposed simultaneous iterative sequences are different from the simultaneous iteration scheme of (1.3). Under appropriate conditions on the iterative parameters, we prove that the iterative sequences generated by the proposed iteration methods converge strongly to the minimum-norm common fixed point of a finite family of nonexpansive mappings. Furthermore, we apply these results to solve the multiple-set split feasibility problem.

The paper is organized as follows. In the next section, we introduce notations and provide preliminary results. In Section 3, we propose a relaxed cyclic iteration method and prove strong convergence of the sequences generated by the proposed methods to the minimum-norm common fixed point of nonexpansive mappings. Further, we introduce a relaxed simultaneous iteration method. Under appropriate conditions on the iterative parameters, we prove that the iterative sequences converge strongly to the minimum-norm common fixed point of nonexpansive mappings. In Section 4, we present the relaxed cyclic and simultaneous methods for solving the linear inverse problem of the multiple-set split feasibility problem. Finally, we give some conclusions.

2. Preliminaries

Throughout this paper, H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C is a nonempty closed convex subset of H. We use the following notions in the sequel: (i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence; (ii) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak $\omega-$limit set of $\{x_n\}$. For any $x, y \in H$ and $\alpha \in \mathbb{R}$, it is easy to check that

$$\|\alpha x + (1 - \alpha) y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha (1 - \alpha) \|x - y\|^2. \tag{2.1}$$

Recall that the orthogonal projection $P_C x$ of x onto C is defined by

$$P_C x = \arg\min_{y \in C} \|x - y\|.$$

The orthogonal projection has the following properties. For any $x \in H$,

(i) $\langle x - P_C x, z - P_C x \rangle \leq 0$ for all $z \in C$;

(ii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$ for all $x, y \in H$.

Definition 2.1. A mapping $T : C \to C$ is called

(1) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$;

(2) averaged if $T := (1-\lambda)I + \lambda S$, where $\lambda \in (0,1)$ and S is nonexpansive;

(3) firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle x-y, Tx-Ty\rangle$ for all $x, y \in C$.

It is obviously observed the following relationship

$$\text{projection operator} \Longrightarrow \text{firmly nonexpansive} \Longrightarrow \text{averaged} \Longrightarrow \text{nonexpansive}.$$

The following lemma was proved in [19]. See also [11].

Lemma 2.2. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T_1, T_2, \cdots, T_N : C \to C$ be mappings such that*

$$\emptyset \neq \bigcap_{i=1}^{N} Fix(T_i) = Fix(T_N T_{N-1} \cdots T_1).$$

Then we must have that the relation

$$\begin{aligned}\emptyset \neq \bigcap_{i=1}^{N} Fix(T_i) &= Fix(T_N T_{N-1} \cdots T_1)\\ &= Fix(T_1 T_N \cdots T_2)\\ &\vdots\\ &= Fix(T_{N-1} \cdots T_1 T_N)\end{aligned}$$

holds.

We denote by $\mathbb{N}$ the set of all positive integers. Bruck [2] proved the following lemma.

Lemma 2.3 ([2])**.** *Let C be a closed convex subset of a strictly convex Banach space X. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap Fix(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum \lambda_n = 1$. Then a mapping S on C defined by*

$$Sx = \sum \lambda_n T_n x$$

for all $x \in C$ is well-defined, nonexpansive and $Fix(S) = \bigcap Fix(T_n)$ holds.

We shall make use of the following lemmas.

Lemma 2.4 (Demiclosedness principle of nonexpansive mapping)**.** *Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $x_n \rightharpoonup x$ and $(I-T)x_n \to 0$, then $x = Tx$.*

The following lemma was proved in [17].

Lemma 2.5 ([17])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

The next lemma was established in [22].

Lemma 2.6 ([22])**.** *Let $\{a_n\}$ be a sequence of nonnegative real sequences satisfying the following inequality:*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=0}^{\infty} \gamma_n = +\infty$;
(2) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n \delta_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we propose several relaxed cyclic and simultaneous iterative algorithms to approximate the minimum-norm common fixed point of nonexpansive mappings. Then, we prove strong convergence of these iterative sequences under appropriate conditions on the iterative parameters.

3.1. Relaxed cyclic iterative algorithms

In this subsection, we introduce two relaxed cyclic iterative algorithms for finding the minimum-norm common fixed point of a finite family of nonexpansive mappings. Then, we prove the iterative sequences converge strongly to the minimum-norm common fixed point of $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$. For any initial value x_0, we define the following two iterative algorithms,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C((1-t_n)T_N \cdots T_1 x_n), \quad n \geq 0, \tag{3.1}$$

and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C((1-t_n)T_{[n+1]} x_n), \quad n \geq 0, \tag{3.2}$$

where $\{\alpha_n\}, \{t_n\} \subset (0,1)$.

First, we investigate the convergence analysis of the iterative sequence $\{x_n\}$ generated by (3.1).

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i : i = 1, 2, \cdots, N\} : C \to C$ be a family of nonexpansive mappings and satisfying the condition:*

$$\emptyset \neq F := \bigcap_{i=1}^{N} Fix(T_i) = Fix(T_N T_{N-1} \cdots T_1).$$

Let the iterative sequence $\{x_n\}$ generated by (3.1), where $\{\alpha_n\}$ and $\{t_n\} \subset (0,1)$, satisfy the conditions:

(i) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=0}^{\infty} t_n = +\infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to the minimum-norm common fixed point of $\{T_i\}_{i=1}^{N}$.

Proof. Let $U := T_N \cdots T_1$, since $T_i : C \to C$ is nonexpansive mapping for $i = 1, 2, \cdots, N$, then the composition $T_N \cdots T_1$ is nonexpansive mapping from C to C, i.e., U is nonexpansive mapping. Therefore, the iterative scheme (3.1) can be equivalently rewritten as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C((1-t_n)Ux_n) \triangleq (1-\alpha_n)x_n + \alpha_n z_n, \tag{3.3}$$

where $z_n = P_C((1-t_n)Ux_n)$. Given any $p \in F$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)(x_n - p) + \alpha_n(P_C((1-t_n)Ux_n) - p)\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|(1-t_n)Ux_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n(1-t_n)\|Ux_n - p\| + \alpha_n t_n\|p\| \\
&\leq (1-\alpha_n t_n)\|x_n - p\| + \alpha_n t_n\|p\| \\
&\leq \max\{\|x_n - p\|, \|p\|\}.
\end{aligned}$$

By the induction, we obtain

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \|p\|\}, \quad n \geq 0.$$

Hence $\{x_n\}$ is bounded. So are $\{T_i x_n\}$ for each $i = 1, 2, \cdots, N$ and $\{Ux_n\}$. Let $M > 0$ satisfy

$$\sup\{\|x_n\|, \|Ux_n\|\} \leq M.$$

Notice that $z_n = P_C((1-t_n)Ux_n)$, we immediately have

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|P_C((1-t_{n+1})Ux_{n+1}) - P_C((1-t_n)Ux_n)\| \\ &\le \|(1-t_{n+1})Ux_{n+1} - (1-t_n)Ux_n\| \\ &\le \|(1-t_{n+1})Ux_{n+1} - (1-t_{n+1})Ux_n\| + \|(1-t_{n+1})Ux_n - (1-t_n)Ux_n\| \\ &\le (1-t_{n+1})\|x_{n+1}-x_n\| + |t_n - t_{n+1}|M,\end{aligned}$$

which leads to

$$\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le |t_n - t_{n+1}|M.$$

By the condition (i), we deduce

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0.$$

With the help of Lemma 2.5, we obtain

$$\lim_{n\to\infty}\|x_n - z_n\| = 0.$$

Furthermore, by (3.3), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\alpha_n\|x_n - z_n\| = 0.$$

Then we have

$$\begin{aligned}\|x_n - Ux_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Ux_n\| \\ &\le \|x_n - x_{n+1}\| + (1-\alpha_n)\|x_n - Ux_n\| + \alpha_n\|P_C((1-t_n)Ux_n) - Ux_n\| \\ &\le \|x_n - x_{n+1}\| + (1-\alpha_n)\|x_n - Ux_n\| + \alpha_n t_n M,\end{aligned}$$

which implies that

$$\|x_n - Ux_n\| \le \frac{\|x_n - x_{n+1}\|}{\alpha_n} + t_n M \to 0 \quad \text{as } n\to\infty. \tag{3.4}$$

Next, we prove $\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle \le 0$, where $x^* = P_F 0$. Indeed, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \lim_{j\to\infty}\langle x^* - x_{n_j}, x^*\rangle.$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence of $\{x_{n_j}\}$ which converges weakly to a point $\tilde{x}$. Without loss of generality, we may assume that $\{x_{n_j}\}$ converges weakly to $\tilde{x}$. Therefore, from (3.4) and Lemma 2.4, we have $x_{n_j} \rightharpoonup \tilde{x} \in \mathrm{Fix}(U)$. Thus $\tilde{x} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Since $x^* = P_F 0$, it follows from the properties of projection operator that

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \langle x^* - \tilde{x}, x^*\rangle \le 0. \tag{3.5}$$

Finally, we prove that $x_n \to x^*$. It is easy to observe that

$$\langle x^* - Ux_n, x^*\rangle = \langle x^* - x_n, x^*\rangle + \langle x_n - Ux_n, x^*\rangle \le \langle x^* - x_n, x^*\rangle + \|x_n - Ux_n\|\|x^*\|.$$

Taking the limsup on the both sides of the above inequality and together with (3.4) and (3.5) yield

$$\limsup_{n\to\infty}\langle x^* - Ux_n, x^*\rangle \le 0.$$

Consequently, from (2.1) and (3.3), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(1-\alpha_n)(x_n-x^*)+\alpha_n(P_C((1-t_n)Ux_n)-x^*)\|^2\\ &\le (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|P_C((1-t_n)Ux_n)-x^*\|\\ &\le (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|(1-t_n)(Ux_n-x^*)-t_nx^*\|^2\\ &= (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n(1-t_n)^2\|Ux_n-x^*\|^2+2\alpha_n(1-t_n)t_n\langle x^*-Ux_n,x^*\rangle+\alpha_nt_n^2\|x^*\|^2\\ &\le (1-\alpha_nt_n)\|x_n-x^*\|^2+2\alpha_n(1-t_n)t_n\langle x^*-Ux_n,x^*\rangle+\alpha_nt_n^2\|x^*\|^2.\end{aligned}$$

It is clear that all conditions of Lemma 2.6 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Next, we prove strong convergence of the iterative algorithm (3.2).

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N : C \to C$ be a family of nonexpansive mappings and satisfying the condition:*

$$\emptyset \neq F := \bigcap_{i=1}^N Fix(T_i) = Fix(T_NT_{N-1}\cdots T_1).$$

Let the iterative sequence $\{x_n\}$ given by (3.2), where the parameters $\{\alpha_n\}$ and $\{t_n\} \subset (0,1)$ and satisfy the following conditions:

(i) $0 < a < \alpha_n \le 1$ *and* $\lim_{n\to\infty}\alpha_n = 1$;
(ii) *either* $\sum_{n=0}^\infty |\alpha_n-\alpha_{n+N}| < +\infty$ *or* $\lim_{n\to\infty}\alpha_n/\alpha_{n+N} = 1$;
(iii) $\sum_{n=0}^\infty t_n = +\infty$ *and* $\lim_{n\to\infty} t_n = 0$;
(iv) *either* $\sum_{n=0}^\infty |t_n-t_{n+N}| < +\infty$ *or* $\lim_{n\to\infty} t_n/t_{n+N} = 1$.

Then the sequence $\{x_n\}$ converges strongly to the minimum-norm common fixed point of $\{T_i\}_{i=1}^N$.

Proof. For convenience, we divide the proof into six steps.

Step 1. We prove that the sequence $\{x_n\}$ is bounded. In fact, let $p \in F$. Then, by (3.2), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\alpha_n)(x_n-p)+\alpha_n(P_C((1-t_n)T_{[n+1]}x_n)-p)\|\\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|(1-t_n)T_{[n+1]}x_n-p\|\\ &= (1-\alpha_n)\|x_n-p\|+\alpha_n\|(1-t_n)(T_{[n+1]}x_n-p)-t_np\|\\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n(1-t_n)\|x_n-p\|+\alpha_nt_n\|p\|\\ &= (1-\alpha_nt_n)\|x_n-p\|+\alpha_nt_n\|p\|\\ &\le \max\{\|x_n-p\|,\|p\|\}.\end{aligned}$$

By the induction, we get

$$\|x_n-p\| \le \max\{\|x_0-p\|,\|p\|\}, \quad n \ge 0.$$

Thus $\{x_n\}$ is bounded. Since the mappings $\{T_i\}_{i=1}^N$ are nonexpansive, they are continuous and then $\{T_{[n+1]}x_n\}$ are also bounded. Let $M > 0$ such that $M \ge \sup\{\|x_n\|,\|T_{[n+1]}x_n\|\}$.

Step 2. We show that $\|x_{n+1}-T_{[n+1]}x_n\| \to 0$ as $n \to \infty$. From (3.2), we have

$$\begin{aligned}\|x_{n+1}-T_{[n+1]}x_n\| &= \|(1-\alpha_n)x_n+\alpha_nP_C((1-t_n)T_{[n+1]}x_n)-T_{[n+1]}x_n\|\\ &\le (1-\alpha_n)\|x_n-T_{[n+1]}x_n\|+\alpha_n\|(1-t_n)T_{[n+1]}x_n-T_{[n+1]}x_n\|\\ &\le (1-\alpha_n)\|x_n-T_{[n+1]}x_n\|+\alpha_nt_n\|T_{[n+1]}x_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Notice the conditions (i) and (ii) that $\lim_{n\to\infty}\alpha_n = 1$ and $\lim_{n\to\infty} t_n = 0$, and the sequences $\{x_n\}$ and $\{T_{[n+1]}x_n\}$ are bounded. Then we have the above conclusion.

Step 3. We claim that $\|x_{n+N} - x_n\| \to 0$ as $n \to \infty$. Indeed, by the facts that $T_{[n+N]} = T_{[n]}$ and P_C and T_i are nonexpansive, we obtain

$$\begin{aligned}
\|x_{n+N} - x_n\| &= \|(1-\alpha_{n+N-1})x_{n+N-1} + \alpha_{n+N-1}P_C((1-t_{n+N-1})T_{[n+N]}x_{n+N-1}) \\
&\quad - (1-\alpha_{n-1})x_{n-1} - \alpha_{n-1}P_C((1-t_{n-1})T_{[n]}x_{n-1})\| \\
&\le (1-\alpha_{n+N-1})\|x_{n+N-1} - x_{n-1}\| + |\alpha_{n+N-1} - \alpha_{n-1}|\|x_{n-1}\| \\
&\quad + \alpha_{n+N-1}\|(1-t_{n+N-1})T_{[n+N]}x_{n+N-1} - (1-t_{n-1})T_{[n]}x_{n-1}\| \\
&\quad + |\alpha_{n+N-1} - \alpha_{n-1}|\|P_C((1-t_{n-1})T_{[n]}x_{[n-1]})\| \\
&\le (1-\alpha_{n+N-1})\|x_{n+N-1} - x_{n-1}\| + |\alpha_{n+N-1} - \alpha_{n-1}|\|x_{n-1}\| \\
&\quad + \alpha_{n+N-1}(1-t_{n+N-1})\|x_{n+N-1} - x_{n-1}\| + |t_{n-1} - t_{n+N-1}|\|T_{[n]}x_{n-1}\| \\
&\quad + |\alpha_{n+N-1} - \alpha_{n-1}|\|P_C((1-t_{n-1})T_{[n]}x_{[n-1]})\| \\
&= (1-\alpha_{n+N-1}t_{n+N-1})\|x_{n+N-1} - x_{n-1}\| + |t_{n-1} - t_{n+N-1}|\|T_{[n]}x_{n-1}\| \\
&\quad + |\alpha_{n+N-1} - \alpha_{n-1}|\|x_{n-1}\| + |\alpha_{n+N-1} - \alpha_{n-1}|\|P_C((1-t_{n-1})T_{[n]}x_{[n-1]})\| \\
&\le (1-\alpha_{n+N-1}t_{n+N-1})\|x_{n+N-1} - x_{n-1}\| + |t_{n-1} - t_{n+N-1}|\|T_{[n]}x_{n-1}\| \\
&\quad + |\alpha_{n+N-1} - \alpha_{n-1}|\|x_{n-1}\| + |\alpha_{n+N-1} - \alpha_{n-1}|\|P_C((1-t_{n-1})T_{[n]}x_{[n-1]})\|.
\end{aligned}$$

This together with Lemma 2.6 implies that

$$\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0.$$

Step 4. We prove that $\|x_n - T_{[n+N]} \cdots T_{[n+1]}x_n\| \to 0$ as $n \to \infty$. By Step 2, we have

$$\|x_{n+N} - T_{[n+N]}x_{n+N-1}\| \to 0 \quad \text{as } n \to \infty.$$

Repeatedly, using Step 2 and the nonexpansivity of T_i, we have

$$\begin{aligned}
\|T_{n+N}x_{n+N-1} - T_{[n+N]}T_{[n+N-1]}x_{n+N-2}\| &\to 0, \\
\|T_{[n+N]}T_{[n+N-1]}x_{n+N-2} - T_{[n+N]}T_{[n+N-1]}T_{[n+N-2]}x_{n+N-3}\| &\to 0, \\
&\vdots \\
\|T_{[n+N]}T_{[n+N-1]} \cdots T_{[n+2]}x_{n+1} - T_{[n+N]}T_{[n+N-1]} \cdots T_{[n+1]}x_n\| &\to 0.
\end{aligned}$$

By the triangle inequality, we obtain

$$\|x_{n+N} - T_{[n+N]}T_{[n+N-1]} \cdots T_{[n+1]}x_n\| \to 0 \text{ as } n \to \infty.$$

Notice that Step 3 tells that $\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0$ and

$$\begin{aligned}
\|x_n - T_{[n+N]}T_{[n+N-1]} \cdots T_{[n+1]}x_n\| &\le \|x_n - x_{n+N}\| + \|x_{n+N} - T_{[n+N]}T_{[n+N-1]} \cdots T_{[n+1]}x_n\| \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Step 5. We show that $\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle \le 0$, where $x^* = P_F 0$. To attain this, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \lim_{j\to\infty}\langle x^* - x_{n_j}, x^*\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may further assume that $x_{n_j} \rightharpoonup \overline{x}$. Consequently, we have

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \langle x^* - \overline{x}, x^*\rangle. \tag{3.6}$$

Since the pool of mappings is finite, we may assume (via a subsequence if necessary) that $T_{[n'+1]} = T_l$ for some $l \in \{1, 2, \cdots, N\}$ and for all $n' \geq 1$. It then follows that

$$T_{[n'+N]} \cdots T_{n'+1} = T_{[l+N-1]} \cdots T_{l-1} T_l =: \overline{T}.$$

Observe that $\overline{T}$ is nonexpansive and, by Lemma 2.2, $\mathrm{Fix}(\overline{T}) = F$.

By Step 4, we get

$$x_{n_j} - \overline{T} x_{n_j} \to 0 \text{ as } j \to \infty.$$

This together with Lemma 2.4 implies that $\overline{x} \in \mathrm{Fix}(\overline{T}) = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = F$.

Since $x^* = P_F(0)$, we have $\langle x^*, x - x^* \rangle \geq 0$ for any $x \in F$. In particular, letting $x = \overline{x}$ and using (3.6), we obtain

$$\limsup_{n\to\infty} \langle x^* - x_n, x^* \rangle \leq 0.$$

Step 6. Finally, we prove $x_n \to x^*$ as $n \to \infty$. In fact, by (3.2), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n (P_C((1-t_n)T_{[n+1]}x_n) - x^*)\| \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|P_C((1-t_n)T_{[n+1]}x_n) - x^*\|^2 \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|(1-t_n)(T_{[n+1]}x_n - x^*) - t_n x^*\|^2 \\
&= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n (1-t_n)^2 \|T_{[n+1]}x_n - x^*\|^2 \\
&\quad + 2\alpha_n (1-t_n) t_n \langle T_{[n+1]}x_n - x^*, -x^* \rangle + \alpha_n t_n^2 \|x^*\|^2 \\
&\leq (1-\alpha_n t_n)\|x_n - x^*\|^2 + 2\alpha_n (1-t_n) t_n \langle x^* - T_{[n+1]}x_n, x^* \rangle + \alpha_n t_n^2 \|x^*\|^2 \\
&= (1-\delta_n)\|x_n - x^*\|^2 + \delta_n \theta_n,
\end{aligned}$$

where $\delta_n = \alpha_n t_n$ and $\theta_n = 2(1-t_n)\langle x^* - T_{[n+1]}x_n, x^* \rangle + t_n \|x^*\|^2$. In addition, we have

$$\begin{aligned}
\langle x^* - T_{[n+1]}x_n, x^* \rangle &= \langle x^* - x_{n+1}, x^* \rangle + \langle x_{n+1} - T_{[n+1]}x_n, x^* \rangle \\
&\leq \langle x^* - x_{n+1}, x^* \rangle + \|x_{n+1} - T_{[n+1]}x_n\| \|x^*\|.
\end{aligned}$$

By (i), Step 2 and Step 5, it can be easily found that $\sum_{n=0}^{\infty} \delta_n = +\infty$ and $\limsup_{n\to\infty} \theta_n \leq 0$. We can therefore apply Lemma 2.6 and conclude that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Remark 3.3.

(1) Theorem 3.1 generalizes the corresponding results of Combettes [7] from firmly nonexpansive mappings to nonexpansive mappings.

(2) A simple example of sequence $\{\alpha_n\}_{n\geq 0}$ which satisfies the conditions (i)-(ii) of Theorem 3.2 is $\alpha_n = \frac{n}{n+1}$ for all $n \geq 0$.

3.2. Relaxed simultaneous iterative algorithms

In the following, we establish two relaxed simultaneous iterative algorithms and prove strong convergence of iterative sequences to the minimum-norm common fixed point of nonexpansive mappings $\{T_i\}_{i=1}^{N}$. For any initial x_0, define the following iterative sequences:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C((1-t_n)\sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n), \quad n \geq 0, \tag{3.7}$$

and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \sum_{i=1}^{N} \lambda_i^{(n)} P_C((1-t_n)T_i x_n), \quad n \geq 0, \tag{3.8}$$

where $\{\alpha_n\}, \{t_n\} \subset (0,1)$ and $\lambda_i^{(n)} > 0$ for all $n \geq 0$ and $1 \leq i \leq N$ such that $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ for all $n \geq 0$.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N : C \to C$ be a family of nonexpansive mappings, and $\emptyset \neq F := \bigcap_{i=1}^N Fix(T_i)$. Assume that the sequence $\{x_n\}$ is defined by (3.7). Let the parameters $\{\alpha_n\}$ and $\{t_n\} \subset (0,1)$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=0}^\infty t_n = +\infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\liminf_{n\to\infty} \lambda_i^{(n)} > 0$ *for all* $1 \le i \le N$ *and* $\lim_{n\to\infty}(\sum_{i=1}^N |\lambda_i^{(n+1)} - \lambda_i^{(n)}|) = 0$.

Then the sequence $\{x_n\}$ converges strongly to the minimum-norm common fixed point of $\{T_i\}_{i=1}^N$.

Proof. Define

$$A_n := \sum_{i=1}^N \lambda_i^{(n)} T_i.$$

It is easy to see that A_n is nonexpansive. Then the iterative scheme (3.7) can be rewritten as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C((1-t_n)A_n x_n) \triangleq (1-\alpha_n)x_n + \alpha_n z_n, \tag{3.9}$$

where $z_n = P_C((1-t_n)A_n x_n)$.

First, we show that the sequence $\{x_n\}$ is bounded. In fact, let $p \in F$. Then, by (3.9), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + P_C((1-t_n)A_n x_n) - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|(1-t_n)A_n x_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|(1-t_n)(A_n x_n - p) - t_n p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n(1-t_n)\|x_n - p\| + \alpha_n t_n\|p\| \\ &= (1-\alpha_n t_n)\|x_n - p\| + \alpha_n t_n\|p\| \\ &\le \max\{\|x_n - p\|, \|p\|\}. \end{aligned}$$

By the induction, we get

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \|p\|\}, \quad n \ge 0.$$

This implies that the sequence $\{x_n\}$ is bounded.

Second, we prove that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since $\{x_n\}$ is bounded, there exists a constant $M > 0$ such that $\|T_i x_n\| \le M$ for all $i = 1, 2, \cdots, N$. With $z_n = P_C((1-t_n)A_n x_n)$, we have

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|P_C((1-t_{n+1})A_{n+1}x_{n+1}) - P_C((1-t_n)A_n x_n)\| \\ &= \Big\|P_C((1-t_{n+1})\sum_{i=1}^N \lambda_i^{(n+1)} T_i x_{n+1}) - P_C((1-t_n)\sum_{i=1}^N \lambda_i^{(n)} T_i x_n)\Big\| \\ &\le \|(1-t_{n+1})A_{n+1}x_{n+1} - (1-t_n)A_n x_n\| \\ &\le \|(1-t_{n+1})A_{n+1}x_{n+1} - (1-t_{n+1})A_{n+1}x_n\| + \|(1-t_{n+1})A_{n+1}x_n - (1-t_n)A_n x_n\| \\ &\le (1-t_{n+1})\|x_{n+1} - x_n\| + \|(1-t_{n+1})A_{n+1}x_n - (1-t_{n+1})A_n x_n\| \\ &\quad + \|(1-t_{n+1})A_n x_n - (1-t_n)A_n x_n\| \\ &\le (1-t_{n+1})\|x_{n+1} - x_n\| + \Big[(1-t_{n+1})\sum_{i=1}^N |\lambda_i^{(n+1)} - \lambda_i^{(n)}| + |t_n - t_{n+1}|\Big]M, \end{aligned}$$

that is,

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le \Big[(1-t_{n+1})\sum_{i=1}^N |\lambda_i^{(n+1)} - \lambda_i^{(n)}| + |t_n - t_{n+1}|\Big]M.$$

Notice from the conditions (i) and (iii),

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

With the help of Lemma 2.5, we obtain $\lim_{n\to\infty}\|x_n-z_n\|=0$. Consequently, by (3.9), we deduce that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\alpha_n\|x_n-z_n\|=0.$$

Third, we show that $\|x_n-A_nx_n\|\to 0$ as $n\to\infty$. In fact, we have

$$\begin{aligned}\|x_n-A_nx_n\|&\leq\|x_n-x_{n+1}\|+\|x_{n+1}-A_nx_n\|\\&=\|x_n-x_{n+1}\|+\|(1-\alpha_n)x_n+\alpha_nP_C((1-t_n)A_nx_n)-A_nx_n\|\\&=\|x_n-x_{n+1}\|+\|(1-\alpha_n)(x_n-A_nx_n)+\alpha_n(P_C((1-t_n)A_nx_n)-A_nx_n)\|\\&\leq\|x_n-x_{n+1}\|+(1-\alpha_n)\|x_n-A_nx_n\|+\alpha_n\|(1-t_n)A_nx_n-A_nx_n\|\\&\leq\|x_n-x_{n+1}\|+(1-\alpha_n)\|x_n-A_nx_n\|+\alpha_nt_nM.\end{aligned}$$

From the above inequality, we obtain

$$\|x_n-A_nx_n\|\leq\frac{\|x_n-x_{n+1}\|}{\alpha_n}+t_nM.$$

Since $\lim_{n\to\infty}t_n=0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\lim_{n\to\infty}\|x_n-A_nx_n\|=0$.

Fourth, we prove that $\limsup_{n\to\infty}\langle x^*-x_n,x^*\rangle\leq 0$, where $x^*=P_F0$. To achieve this, we take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^*-x_n,x^*\rangle=\lim_{j\to\infty}\langle x^*-x_{n_j},x^*\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_j}\rightharpoonup\tilde{x}$. Consequently, we have

$$\limsup_{n\to\infty}\langle x^*-x_n,x^*\rangle=\langle x^*-\tilde{x},x^*\rangle.$$

Next, we show that $\omega_w(x_n)\subset F$. To see this, we may assume that $\lambda_i^{n_j}\to\lambda_i$ as $j\to\infty$ for all $i=1,2,\cdots,N$. Due to the assumption (iii), we have $\lambda_i>0$ for all $i=1,2,\cdots,N$. Notice that $\sum_{i=1}^N\lambda_i=1$. Set $A:=\sum_{i=1}^N\lambda_iT_i$. It is easily checked that A is nonexpansive, $A_{n_j}x\to Ax$ for all $x\in C$ and $\mathrm{Fix}(A)=F$ (due to Lemma 2.3). Moreover, we have

$$\|x_{n_j}-Ax_{n_j}\|\leq\|x_{n_j}-A_{n_j}x_{n_j}\|+\|A_{n_j}x_{n_j}-Ax_{n_j}\|\to 0\quad\text{as } j\to\infty.$$

By the demiclosedness of nonexpansive mapping, we obtain $\tilde{x}\in F$. Since $x^*=P_F0$, it follows from the properties of projection operator that

$$\limsup_{n\to\infty}\langle x^*-x_n,x^*\rangle=\langle x^*-\tilde{x},x^*\rangle\leq 0.$$

Finally, by the iterative scheme (3.7), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2&=\|(1-\alpha_n)(x_n-x^*)+\alpha_n(P_C((1-t_n)A_nx_n)-x^*)\|^2\\&\leq(1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|P_C((1-t_n)A_nx_n)-x^*\|^2\\&\leq(1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|(1-t_n)(A_nx_n-x^*)-t_nx^*\|^2\\&=(1-\alpha_n)\|x_n-x^*\|^2+\alpha_n(1-t_n)^2\|A_nx_n-x^*\|^2\\&\quad+2\alpha_n(1-t_n)t_n\langle A_nx_n-x^*,-x^*\rangle+\alpha_nt_n^2\|x^*\|^2\\&\leq(1-\alpha_nt_n)\|x_n-x^*\|^2+2\alpha_n(1-t_n)t_n\langle x^*-A_nx_n,x^*\rangle+\alpha_nt_n^2\|x^*\|^2.\end{aligned}\tag{3.10}$$

Observe that

$$\langle x^* - A_n x_n, x^*\rangle = \langle x^* - x_n, x^*\rangle + \langle x_n - A_n x_n, x^*\rangle \le \langle x^* - x_n, x^*\rangle + \|x_n - A_n x_n\|\|x^*\|.$$

Taking the limsup on the both sides of the above inequality and noticing Step 3 and Step 4, we have

$$\limsup_{n\to\infty}\langle x^* - A_n x_n, x^*\rangle \le 0.$$

Therefore, using (3.10) together with the conditions (i), (ii), and Lemma 2.6, it follows that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

The difference between the iterative algorithms (3.7) and (3.8) is that the sum of weighted parameters is inside the projection operator in (3.7), while the sum of weighted parameters is outside the iterative algorithm (3.8). We can prove the strong convergence of iterative sequence generated by (3.8) based on the same proof method before. Then we have the following convergence theorem.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N : C \to C$ be a family of nonexpansive mappings, and $\emptyset \ne F := \bigcap_{i=1}^N Fix(T_i)$. Assume that the iterative sequence $\{x_n\}$ is defined by (3.8), where $\{\alpha_n\}$ and $\{t_n\} \subset (0, 1)$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=0}^{\infty} t_n = +\infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\liminf_{n\to\infty} \lambda_i^{(n)} > 0$ *for all* $1 \le i \le N$ *and* $\lim_{n\to\infty}(\sum_{i=1}^N |\lambda_i^{(n+1)} - \lambda_i^{(n)}|) = 0$.

Then the sequence $\{x_n\}$ converges strongly to the minimum-norm common fixed point of $\{T_i\}_{i=1}^N$.

Proof. For notation simplicity, we set $A_n := \sum_{i=1}^N \lambda_i^{(n)} T_i$ and $z_n := \sum_{i=1}^N \lambda_i^{(n)} P_C((1 - t_n)T_i x_n)$. With this notation, the iterative algorithm (3.8) can be written as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z_n.$$

The remainder of the proof is the same as Theorem 3.4. So it is omitted here. □

Remark 3.6. Theorems 3.4 and 3.5 improve the results of [11] in two aspects: (i) we relax the assumption "$\sum_{n=0}^{\infty}\sum_{i=1}^N |\lambda_i^{(n+1)} - \lambda_i^{(n)}| < +\infty$" required in [11]; (ii) the condition "either $\sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty$ or $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$" was removed in our theorems.

4. Applications

In this section, we present an application of the obtained results to solve the multiple-set split feasibility problem. Recall that the multiple-set split feasibility problem (MSSFP, for short), which was first introduced by Censor et al. [5]:

$$\text{Find a point } x^* \in \bigcap_{i=1}^N C_i \quad \text{such that } Ax^* \in \bigcap_{j=1}^M Q_j, \tag{4.1}$$

where $N, M \ge 1$ are integers, $\{C_i\}_{i=1}^N, \{Q_j\}_{j=1}^M$ are closed convex subset of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. The multiple-set split feasibility problem is a generalization of the split feasibility problem as follows (say SFP, for short):

$$\text{Find a point } x^* \in C \quad \text{such that } Ax^* \in Q, \tag{4.2}$$

where C and Q are closed convex subset of Hilbert spaces H_1 and H_2, respectively. To solve the MSSFP (4.1), Censor et al. [5] first proposed a gradient projection algorithm. However, this iterative algorithm could not reduce to the CQ iterative algorithm [3] for solving the SFP (4.2). Xu [23] proved that the MSSFP

(4.1) is equivalent to find a common fixed point of mappings $\{T_i\}_{i=1}^N$, where $T_i = P_{C_i}(I - \gamma\nabla q)$, $\nabla q(x) = \sum_{j=1}^M \beta_j A^*(I - P_{Q_j})Ax$, $\{\beta_j\}_{j=1}^M$ are nonnegative real numbers and $\gamma > 0$. Under mild assumption on the parameter γ, Xu [23] proposed several iterative algorithms included cyclic iteration scheme and simultaneous iteration scheme to solve the MSSFP (4.1). The main characteristic of these iterative algorithms is that they will reduce to the CQ iterative algorithm when the MSSFP (4.1) is reduced to the SFP (4.2). Further, He et al. [10] generalized the iterative algorithms introduced by Xu [23] to the relaxed iteration method. It is worth mentioning that these iterative algorithms have only weak convergence except for finite dimensional space. Thus, we establish several strong convergence theorems, and its limit is the minimum-norm solution of the MSSFP (4.1).

Theorem 4.1. *Assume that the MSSFP* (4.1) *is consistent. Let* $\{x_n\}$ *be the sequence generated by the iteration scheme*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \left((1-t_n)P_{C_N}(I-\gamma\nabla q)\cdots P_{C_1}(I-\gamma\nabla q)x_n\right),$$

where $0 < \gamma < 2/L$ *with* $L = \|A\|^2 \sum_{j=1}^M \beta_j$ *and the parameters* $\{\alpha_n\}$ *and* $\{t_n\}$ *satisfy the conditions in Theorem* 3.1. *Then the sequence* $\{x_n\}$ *converges strongly to the minimum-norm solution of the MSSFP* (4.1).

Proof. It is known that the mapping $T_i = P_{C_i}(I - \gamma\nabla q) : H_1 \to C_i$ for any $i = 1, 2, \cdots, N$ is averaged. So $\{T_i\}_{i=1}^N$ are nonexpansive mappings. Consequently, it is followed from the results of Xu [23] that the MSSFP (4.1) is equivalent to the common fixed point of nonexpansive mappings $\{T_i\}_{i=1}^N$. Substituting the projection operator P_C in (3.1) with an identity operator I, by Theorem 3.1, we can conclude that the sequence $\{x_n\}$ converges strongly to the minimum-norm solution of the MSSFP (4.1). □

Similarly, with the help of Theorem 3.2, we have the following strong convergence theorem.

Theorem 4.2. *Assume that the MSSFP* (4.1) *is consistent. Let* $\{x_n\}$ *be defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Big((1-t_n)P_{C_{[n+1]}}(I-\gamma\nabla q)x_n\Big),$$

where $0 < \gamma < 2/L$ *with* $L = \|A\|^2 \sum_{j=1}^M \beta_j$ *and the parameters* $\{\alpha_n\}$ *and* $\{t_n\}$ *satisfy the conditions in Theorem* 3.2. *Then the sequence* $\{x_n\}$ *converges strongly to the minimum-norm solution of the MSSFP* (4.1).

From Theorem 4.1, we know that $T_i = P_{C_i}(I - \gamma\nabla q)$ is averaged. It is easy to check that the convex combination of $S := \sum_{j=1}^M \lambda_i^{(n)} T_i$ is also averaged. Thus S is nonexpansive. The simultaneous iterative algorithms (3.7) and (3.8) are indeed equivalent by taking the projection operator P_C with the identity operator. By Theorem 3.4 or Theorem 3.5, we have the following strong convergence theorem for finding the minimum-norm solution of the MSSFP (4.1).

Theorem 4.3. *Assume that the MSSFP* (4.1) *is consistent. Define the iterative sequence* $\{x_n\}$ *as follows:*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \left((1-t_n)\sum_{i=1}^N \lambda_i^{(n)} P_{C_i}(I-\gamma\nabla q)x_n\right),$$

where $\{\alpha_n\}$, $\{t_n\}$, *and* $\lambda_i^{(n)}$ *for all* $i = 1, \cdots N$ *satisfy the assumptions* (i)-(iii) *in Theorem* 3.4, *and* $0 < \gamma < 2/L$ *with* $L = \|A\|^2 \sum_{j=1}^M \beta_j$. *Then the sequence* $\{x_n\}$ *converges strongly to the minimum-norm solution of the MSSFP* (4.1).

5. Conclusions

Iterative methods of finding fixed points or common fixed points of nonexpansive mappings is a very challenging problem. There are several methods have been studied to approximate the minimum-norm fixed point of nonexpansive mappings. In this paper, we have proposed cyclic iteration methods and simultaneous iteration methods with relaxation factors. The strong convergence of the proposed iterative sequences has been proved under weaker assumptions of the parameters than existing results. Based on these results, we have obtained several new strong convergence theorems for solving the multiple-set split feasibility problem.

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