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Presentations for the punctured mapping class groups in terms of Artin groups

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Abstract Consider an oriented compact surface F of positive genus, possibly with boundary, and a finite set $\mathcal{P}$ of punctures in the interior of F, and define the punctured mapping class group of F relatively to $\mathcal{P}$ to be the group of isotopy classes of orientation-preserving homeomorphisms $h : F \to F$ which pointwise fix the boundary of F and such that $h(\mathcal{P}) = \mathcal{P}$. In this paper, we calculate presentations for all punctured mapping class groups. More precisely, we show that these groups are isomorphic with quotients of Artin groups by some relations involving fundamental elements of parabolic subgroups.

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1 Introduction

Throughout the paper $F = F_{g,r}$ will denote a compact oriented surface of genus g with r boundary components, and $\mathcal{P} = \mathcal{P}_n = \{P_1, \dots, P_n\}$ a finite set of points in the interior of F, called *punctures*. We denote by $\mathcal{H}(F, \mathcal{P})$ the group of orientation-preserving homeomorphisms $h : F \to F$ that pointwise fix the boundary of F and such that $h(\mathcal{P}) = \mathcal{P}$. The *punctured mapping class group* $\mathcal{M}(F, \mathcal{P})$ of F relatively to $\mathcal{P}$ is defined to be the group of isotopy classes of elements of $\mathcal{H}(F, \mathcal{P})$. Note that the group $\mathcal{M}(F, \mathcal{P})$ only depends up to isomorphism on the genus g, on the number r of boundary components, and on the cardinality n of $\mathcal{P}$. If $\mathcal{P}$ is empty, then we write $\mathcal{M}(F) = \mathcal{M}(F, \emptyset)$, and call $\mathcal{M}(F)$ the *mapping class group* of F.

The *pure mapping class group* of F relatively to $\mathcal{P}$ is defined to be the subgroup $\mathcal{PM}(F, \mathcal{P})$ of isotopy classes of elements of $\mathcal{H}(F, \mathcal{P})$ that pointwise fix $\mathcal{P}$. Let Σ_n denote the symmetric group of $\{1, \dots, n\}$. Then the punctured mapping class group and the pure mapping class group are related by the following exact sequence.

$$1 \to \mathcal{PM}(F, \mathcal{P}_n) \to \mathcal{M}(F, \mathcal{P}_n) \to \Sigma_n \to 1 \ .$$

A *Coxeter matrix* is a matrix $M = (m_{i,j})_{i,j=1,\dots,l}$ satisfying:

- $m_{i,i} = 1$ for all $i = 1, \dots, l$;
- $m_{i,j} = m_{j,i} \in \{2, 3, 4, \dots, \infty\}$, for $i \neq j$.

A Coxeter matrix $M = (m_{i,j})$ is usually represented by its *Coxeter graph* Γ. This is defined by the following data:

- Γ has l vertices: $x_1, \dots, x_l$;
- two vertices x_i and x_j are joined by an edge if $m_{i,j} \geq 3$;
- the edge joining two vertices x_i and x_j is labelled by $m_{i,j}$ if $m_{i,j} \geq 4$.

For $i, j \in \{1, \dots, l\}$, we write:

$$\mathrm{prod}(x_i, x_j, m_{i,j}) = \begin{cases} (x_i x_j)^{m_{i,j}/2} & \text{if } m_{i,j} \text{ is even,} \\ (x_i x_j)^{(m_{i,j}-1)/2} x_i & \text{if } m_{i,j} \text{ is odd.} \end{cases}$$

The *Artin group* $A(\Gamma)$ associated with Γ (or with M) is the group given by the presentation:

$$A(\Gamma) = \langle x_1, \dots, x_l \mid \mathrm{prod}(x_i, x_j, m_{i,j}) = \mathrm{prod}(x_j, x_i, m_{i,j}) \text{ if } i \neq j \text{ and } m_{i,j} < \infty \rangle.$$

The *Coxeter group* $W(\Gamma)$ associated with Γ is the quotient of $A(\Gamma)$ by the relations $x_i^2 = 1$, $i = 1, \dots, l$. We say that Γ or $A(\Gamma)$ is of *finite type* if $W(\Gamma)$ is finite.

For a subset X of the set $\{x_1, \dots, x_l\}$ of vertices of Γ, we denote by Γ_X the Coxeter subgraph of Γ generated by X, by W_X the subgroup of $W(\Gamma)$ generated by X, and by A_X the subgroup of $A(\Gamma)$ generated by X. It is a non-trivial but well known fact that W_X is the Coxeter group associated with Γ_X (see [3]), and A_X is the Artin group associated with Γ_X (see [16], [19]). Both W_X and A_X are called *parabolic subgroups* of $W(\Gamma)$ and of $A(\Gamma)$, respectively.

Define the *quasi-center* of an Artin group $A(\Gamma)$ to be the subgroup of elements α in $A(\Gamma)$ satisfying $\alpha X \alpha^{-1} = X$, where X is the natural generating set of $A(\Gamma)$. If Γ is of finite type and connected, then the quasi-center is an infinite cyclic group generated by a special element of $A(\Gamma)$, called *fundamental element*, and denoted by $\Delta(\Gamma)$ (see [8], [4]).

The most significant work on presentations for mapping class groups is certainly the paper [10] of Hatcher and Thurston. In this paper, the authors introduced a simply connected complex on which the mapping class group $\mathcal{M}(F_{g,0})$ acts, and, using this action and following a method due to Brown [5], they obtained a presentation for $\mathcal{M}(F_{g,0})$. However, as pointed out by Wajnryb [25], this presentation is rather complicated and requires many generators and relations. Wajnryb [25] used this presentation of Hatcher and Thurston to calculate new presentations for $\mathcal{M}(F_{g,1})$ and for $\mathcal{M}(F_{g,0})$. He actually presented $\mathcal{M}(F_{g,1})$ as the quotient of an Artin group by two relations, and presented $\mathcal{M}(F_{g,0})$ as the quotient of the same Artin group by the same two relations plus another one. In [18], Matsumoto showed that these three relations are nothing else than equalities among powers of fundamental elements of parabolic subgroups. Moreover, he showed how to interpret these powers of fundamental elements inside the mapping class group. Once this interpretation is known, the relations in Matsumoto's presentations become trivial. At this point, one has "good" presentations for $\mathcal{M}(F_{g,1})$ and for $\mathcal{M}(F_{g,0})$, in the sence that one can remember them. Of course, the definition of a "good" presentation depends on the memory of the reader and on the time he spends working on the presentation.

One can find in [17] another presentation for $\mathcal{M}(F_{g,1})$ as the quotient of an Artin group by relations involving fundamental elements of parabolic subgroups. Recently, Gervais [9] found another "good" presentation for $\mathcal{M}(F_{g,r})$ with many generators but simple relations.

In the present paper, starting from Matsumoto's presentations, we calculate presentations for all punctured mapping class groups $\mathcal{M}(F_{g,r},\mathcal{P}_n)$ as quotients of Artin groups by some relations which involve fundamental elements of parabolic subgroups. In particular, $\mathcal{M}(F_{g,0},\mathcal{P}_n)$ is presented as the quotient of an Artin group by five relations, all of them being equalities among powers of fundamental elements of parabolic subgroups.

The generators in our presentations are Dehn twists and braid twists. We define them in Subsection 2.1, and we show that they verify some "braid" relations that allow us to define homomorphisms from Artin groups to punctured mapping class groups. The main algebraic tool we use is Lemma 2.5, stated in Subsection 2.2, which says how to find a presentation for a group G from an exact sequence $1 \to K \to G \to H \to 1$ and from presentations of K and H. We also state in Subsection 2.2 some exact sequences involving punctured mapping class groups on which Lemma 2.5 will be applied. In order to find our presentations, we first need to investigate some homomorphisms from finite type Artin groups to punctured mapping class groups, and to calculate the images under these homomorphisms of some powers of fundamental elements. This is the object of Subsection 2.3. Once these images are known, one can easily verify that the relations in our presentations hold. Of course, it remains to prove that no other relation is needed. We state our presentation for $\mathcal{M}(F_{g,r+1}, \mathcal{P}_n)$ (where $g \ge 1$, and $r, n \ge 0$) in Theorem 3.1, and we state our presentation for $\mathcal{M}(F_{g,0}, \mathcal{P}_n)$ (where $g, n \ge 1$) in Theorem 3.2. Then, Subsection 3.1 is dedicated to the proof of Theorem 3.1, and Subsection 3.2 is dedicated to the proof of Theorem 3.2.

2 Preliminaries

2.1 Dehn twists and braid twists

We introduce in this subsection some elements of the punctured mapping class group, the Dehn twists and the braid twists, which will play a prominent rôle throughout the paper. In particular, the generators for the punctured mapping class group will be chosen among them.

By an *essential circle* in $F \setminus \mathcal{P}$ we mean an embedding $s : S^1 \to F \setminus \mathcal{P}$ of the circle whose image is in the interior of $F \setminus \mathcal{P}$ and does not bound a disk in $F \setminus \mathcal{P}$. Two essential circles s, s' are called *isotopic* if there exists $h \in \mathcal{H}(F, \mathcal{P})$ which represents the identity in $\mathcal{M}(F, \mathcal{P})$ and such that $h \circ s = s'$. Isotopy of circles is an equivalence relation which we denote by $s \simeq s'$. Let $s : S^1 \to F \setminus \mathcal{P}$ be an essential circle. We choose an embedding $A : [0,1] \times S^1 \to F \setminus \mathcal{P}$ of the annulus such that $A(\frac{1}{2}, z) = s(z)$ for all $z \in S^1$, and we consider the homeomorphism $T \in \mathcal{H}(F, \mathcal{P})$ defined by

$$(T \circ A)(t, z) = A(t, e^{2i\pi t} z), \quad t \in [0,1],\ z \in S^1,$$

and T is the identity on the exterior of the image of A (see Figure 1). The *Dehn twist* along s is defined to be the element $\sigma \in \mathcal{M}(F, \mathcal{P})$ represented by T. Note that:

- the definition of σ does not depend on the choice of A;
- the element σ does not depend on the orientation of s;
- if s and s' are isotopic, then their corresponding Dehn twists are equal;
- if s bounds a disk in F which contains exactly one puncture,then $\sigma = 1$; otherwise, σ is of infinite order;
- if $\xi \in \mathcal{M}(F, \mathcal{P})$ is represented by $f \in \mathcal{H}(F, \mathcal{P})$, then $\xi \sigma \xi^{-1}$ is the Dehn twist along $f(s)$.

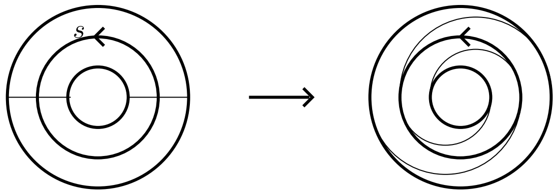

Figure 1: Dehn twist along s

By an *arc* we mean an embedding $a : [0,1] \to F$ of the segment whose image is in the interior of F, such that $a((0,1)) \cap \mathcal{P} = \emptyset$, and such that both $a(0)$ and $a(1)$ are punctures. Two arcs a, a' are called *isotopic* if there exists $h \in \mathcal{H}(F, \mathcal{P})$ which represents the identity in $\mathcal{M}(F, \mathcal{P})$ and such that $h \circ a = a'$. Note that $a(0) = a'(0)$ and $a(1) = a'(1)$ if a and a' are isotopic. Isotopy of arcs is an equivalence relation which we denote by $a \simeq a'$. Let a be an arc. We choose an embedding $A : D^2 \to F$ of the unit disk satisfying:

- $a(t) = A(t - \frac{1}{2})$ for all $t \in [0,1]$,
- $A(D^2) \cap \mathcal{P} = \{a(0), a(1)\}$,

and we consider the homeomorphism $T \in \mathcal{H}(F, \mathcal{P})$ defined by

$$(T \circ A)(z) = A(e^{2i\pi|z|} z), \quad z \in D^2,$$

and T is the identity on the exterior of the image of A (see Figure 2). The *braid twist* along a is defined to be the element $\tau \in \mathcal{M}(F, \mathcal{P})$ represented by T. Note that:

- the definition of τ does not depend on the choice of A;
- if a and a' are isotopic, then their corresponding braid twists are equal;
- if $\xi \in \mathcal{M}(F, \mathcal{P})$ is represented by $f \in \mathcal{H}(F, \mathcal{P})$, then $\xi \tau \xi^{-1}$ is the braid twist along $f(a)$;
- if $s : S^1 \to F \setminus \mathcal{P}$ is the essential circle defined by $s(z) = A(z)$ (see Figure 2), then τ^2 is the Dehn twist along s.

We turn now to describe some relations among Dehn twists and braid twists which will be essential to define homomorphisms from Artin groups to punctured mapping class groups.

The first family of relations are known as "braid relations" for Dehn twists (see [2]).

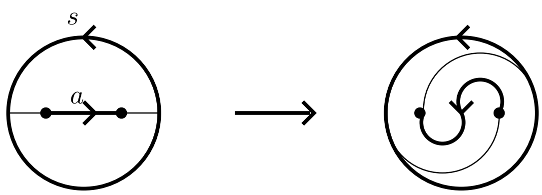

Figure 2: Braid twist along a

Lemma 2.1 *Let s and s' be two essential circles which intersect transversely, and let σ and σ' be the Dehn twists along s and s', respectively. Then:*

$$\begin{array}{ll} \sigma\sigma' = \sigma'\sigma & \text{if } s \cap s' = \emptyset, \\ \sigma\sigma'\sigma = \sigma'\sigma\sigma' & \text{if } |s \cap s'| = 1. \end{array}$$ □

The next family of relations are simply the usual braid relations viewed inside the punctured mapping class group.

Lemma 2.2 *Let a and a' be two arcs, and let τ and τ' be be the braid twists along a and a', respectively. Then:*

$$\begin{array}{ll} \tau\tau' = \tau'\tau & \text{if } a \cap a' = \emptyset, \\ \tau\tau'\tau = \tau'\tau\tau' & \text{if } a(0) = a'(1) \text{ and } a \cap a' = \{a(0)\}. \end{array}$$ □

To our knowledge, the last family of relations does not appear in the literature. However, their proofs are easy and are left to the reader.

Lemma 2.3 *Let s be an essential circle, and let a be an arc which intersects s transversely. Let σ be the Dehn twist along s, and let τ be the braid twist along a. Then:*

$$\begin{array}{ll} \sigma\tau = \tau\sigma & \text{if } s \cap a = \emptyset, \\ \sigma\tau\sigma\tau = \tau\sigma\tau\sigma & \text{if } |s \cap a| = 1. \end{array}$$ □

We finish this subsection by recalling another relation called *lantern relation* (see [13]) which is not used to define homomorphisms between Artin groups and punctured mapping class groups, but which will be useful in the remainder.

We point out first that we use the convention in figures that a letter which appears over a circle or an arc denotes the corresponding Dehn twist or braid twist, and not the circle or the arc itself.

Lemma 2.4 *Consider an embedding of $F_{0,4}$ in $F \setminus \mathcal{P}$ and the Dehn twists $e_1, e_2, e_3, e_4, a, b, c$ represented in Figure 3. Then*

$$e_1e_2e_3e_4 = abc. \qquad \square$$

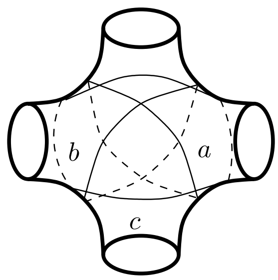

Figure 3: Lantern relation

2.2 Exact sequences

Now, we introduce in Lemma 2.5 our main tool to obtain presentations for the punctured mapping class groups. Briefly, this lemma says how to find a presentation for a group G from an exact sequence $1 \to K \to G \to H \to 1$ and from presentations of H and K. This lemma will be applied to the exact sequences (2.1), (2.2), and (2.3) given after Lemma 2.5.

Consider an exact sequence

$$1 \to K \to G \xrightarrow{\rho} H \to 1$$

and presentations $H = \langle S_H | R_H \rangle$, $K = \langle S_K | R_K \rangle$ for H and K, respectively. For all $x \in S_H$, we fix some $\tilde{x} \in G$ such that $\rho(\tilde{x}) = x$, and we write

$$\tilde{S}_H = \{\tilde{x}\ ;\ x \in S_H\}.$$

Let $r = x_1^{\varepsilon_1} \dots x_l^{\varepsilon_l}$ in R_H. Write $\tilde{r} = \tilde{x}_1^{\varepsilon_1} \dots \tilde{x}_l^{\varepsilon_l} \in G$. Since r is a relator of H, we have $\rho(\tilde{r}) = 1$. Thus, S_K being a generating set of the kernel of ρ, one may choose a word w_r over S_K such that both $\tilde{r}$ and w_r represent the same element of G. Set

$$R_1 = \{\tilde{r}w_r^{-1}\ ;\ r \in R_H\}.$$

Let $\tilde{x} \in \tilde{S}_H$ and $y \in S_K$. Since K is a normal subgroup of G, $\tilde{x}y\tilde{x}^{-1}$ is also an element of K, thus one may choose a word $v(x,y)$ over S_K such that both $\tilde{x}y\tilde{x}^{-1}$ and $v(x,y)$ represent the same element of G. Set

$$R_2 = \{\tilde{x}y\tilde{x}^{-1}v(x,y)^{-1}\ ;\ \tilde{x} \in \tilde{S}_H \text{ and } y \in S_K\}.$$

The proof of the following lemma is left to the reader.

Lemma 2.5 *G admits the presentation*

$$G = \langle \tilde{S}_H \cup S_K \mid R_1 \cup R_2 \cup R_K \rangle. \qquad \square$$

The first exact sequence on which we will apply Lemma 2.5 is the one given in the introduction:

$$1 \to \mathcal{PM}(F, \mathcal{P}_n) \to \mathcal{M}(F, \mathcal{P}_n) \to \Sigma_n \to 1, \tag{2.1}$$

where Σ_n denotes the symmetric group of $\{1, \dots, n\}$.

The inclusion $\mathcal{P}_{n-1} \subset \mathcal{P}_n$ gives rise to a homomorphism $\varphi_n : \mathcal{PM}(F, \mathcal{P}_n) \to \mathcal{PM}(F, \mathcal{P}_{n-1})$. By [1], if $(g,r,n) \neq (1,0,1)$, then we have the following exact sequence:

$$1 \to \pi_1(F \setminus \mathcal{P}_{n-1}, P_n) \xrightarrow{\iota_n} \mathcal{PM}(F, \mathcal{P}_n) \xrightarrow{\varphi_n} \mathcal{PM}(F, \mathcal{P}_{n-1}) \to 1. \tag{2.2}$$

We will need later a more precise description of the images by ι_n of certain elements of $\pi_1(F \setminus \mathcal{P}_{n-1}, P_n)$. Consider an essential circle $\alpha : S^1 \to F \setminus \mathcal{P}_{n-1}$ such that $\alpha(1) = P_n$. Here, we assume that α is oriented. Let ξ be the element of $\pi_1(F \setminus \mathcal{P}_{n-1}, P_n)$ represented by α. We choose an embedding $A : [0,1] \times S^1 \to F \setminus \mathcal{P}_{n-1}$ of the annulus such that $A(\frac{1}{2}, z) = \alpha(z)$ for all $z \in S^1$ (see Figure 4). Let $s_0, s_1 : S^1 \to F \setminus \mathcal{P}_n$ be the essential circles defined by

$$s_0(z) = A(0,z), \quad s_1(z) = A(1,z), \quad z \in S^1,$$

and let σ_0, σ_1 be the Dehn twists along s_0 and s_1, respectively. Then the following holds.

Lemma 2.6 *We have $\iota_n(\xi) = \sigma_0^{-1}\sigma_1$.* $\square$

Now, consider a surface $F_{g,r+m}$ of genus g with $r+m$ boundary components, and a set $\mathcal{P}_n = \{P_1, \dots, P_n\}$ of n punctures in the interior of $F_{g,r+m}$. Choose m boundary curves $c_1, \dots, c_m : S^1 \to \partial F_{g,r+m}$. Let $F_{g,r}$ be the surface of genus g with r boundary components obtained from $F_{g,r+m}$ by gluing a disk D_i^2 along c_i, for all $i = 1, \dots, m$, and let $\mathcal{P}_{n+m} = \{P_1, \dots, P_n, Q_1, \dots, Q_m\}$

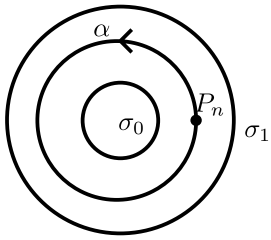

Figure 4: Image of a simple circle by ι_n

be a set of punctures in the interior of $F_{g,r}$, where Q_i is chosen in the interior of D_i^2, for all $i = 1, \dots, m$. The proof of the following exact sequence can be found in [21].

Lemma 2.7 *Assume that $(g, r, m) \notin \{(0, 0, 1), (0, 0, 2)\}$. Then we have the exact sequence:*

$$1 \to \mathbf{Z}^m \to \mathcal{PM}(F_{g,r+m}, \mathcal{P}_n) \to \mathcal{PM}(F_{g,r}, \mathcal{P}_{n+m}) \to 1 \ , \tag{2.3}$$

where $\mathbf{Z}^m$ stands for the free abelian group of rank m generated by the Dehn twists along the c_i's. □

2.3 Geometric representations of Artin groups

Define a *geometric representation* of an Artin group $A(\Gamma)$ to be a homomorphism from $A(\Gamma)$ to some punctured mapping class group. In this subparagraph, we describe some geometric representations of Artin groups whose properties will be used later in the paper.

The first family of geometric representations has been introduced by Perron and Vannier for studying geometric monodromies of simple singularities [22]. A *chord diagram* in the disk D^2 is a family $S_1, \dots, S_l : [0,1] \to D^2$ of segments satisfying:

- $S_i : [0,1] \to D^2$ is an embedding for all $i = 1, \dots, l$;
- $S_i(0), S_i(1) \in \partial D^2$, and $S_i((0,1)) \cap \partial D^2 = \emptyset$, for all $i = 1, \dots, l$;
- either S_i and S_j are disjoint, or they intersect transversely in a unique point in the interior of D^2, for $i \neq j$.

From this data, one can first define a Coxeter matrix $M = (m_{i,j})_{i,j=1,\dots,l}$ by seting $m_{i,j} = 2$ if S_i and S_j are disjoint, and $m_{i,j} = 3$ if S_i and S_j intersect transversely in a point. The Coxeter graph Γ associated with M is called *intersection diagram* of the chord diagram. It is an "ordinary" graph in the sence that none of the edges has a label. From the chord diagram we can also define a surface F by attaching to D^2 a handle H_i which joins both extremities of S_i, for all $i = 1, \dots, l$ (see Figure 5). Let σ_i be the Dehn twist along the circle made up with the segment S_i together with the central curve of H_i. By Lemma 2.1, one has a geometric representation $A(\Gamma) \to \mathcal{M}(F)$ which sends x_i on σ_i for all $i = 1, \dots, l$. This geometric representation will be called *Perron-Vannier representation*.

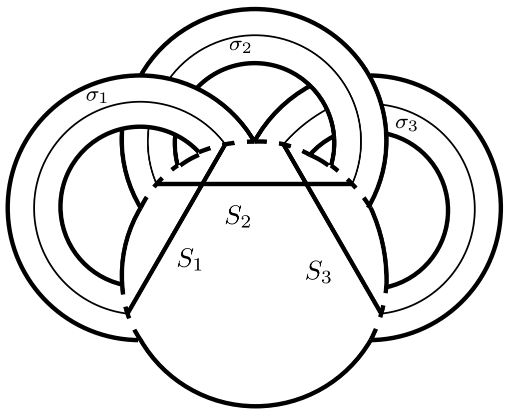

Figure 5: Chord diagram and associated surface and Dehn twists

If Γ is connected, then the Perron-Vannier representation is injective if and only if Γ is of type A_l or D_l [15], [26]. In the case where Γ is of type A_l, D_l, E_6, or E_7, the vertices of Γ will be numbered according to Figure 6, and the Dehn twists $\sigma_1, \dots, \sigma_l$ are those represented in Figures 7, 8, 9.

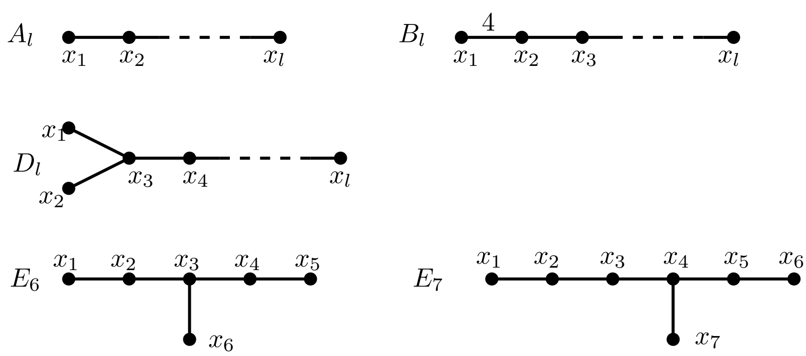

Figure 6: Some finite type Coxeter graphs

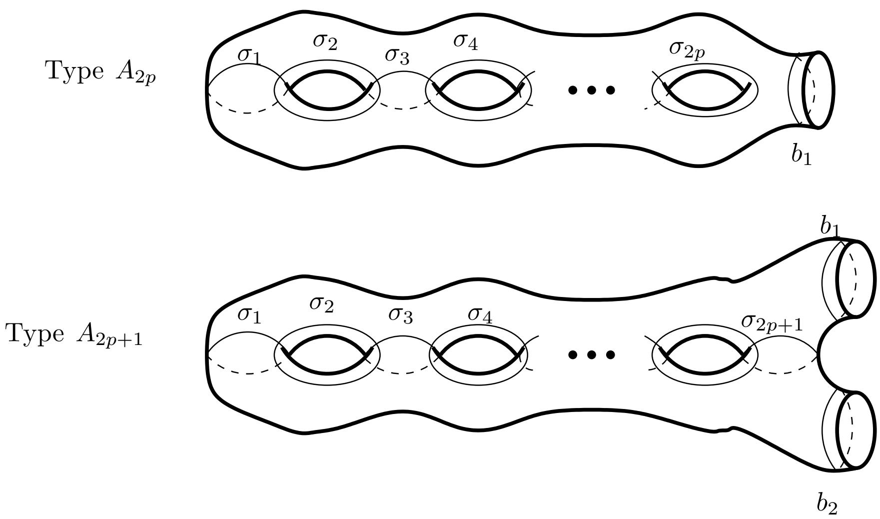

Figure 7: Perron-Vannier representations of type A_l

The Perron-Vannier representation of the Artin group of type A_{l-1} can be extended to a geometric representation of the Artin group of type B_l as follows. First, we number the vertices of B_l according to Figure 6. Then A_{l-1} is the subgraph of B_l generated by the vertices $x_2, \dots, x_l$. We start from a chord diagram $S_2, \dots, S_l$ whose intersection diagram is A_{l-1}, and we denote by F the associated surface. For $i = 2, \dots, l$, we denote by s_i the essential circle of F made up with S_i and the central curve of the handle H_i. We can choose two points P_1, P_2 in the interior of F and an arc a_1 from P_1 to P_2 satisfying:

- $\{P_1, P_2\} \cap s_i = \emptyset$ for all $i = 2, \dots, l$;

- $a_1 \cap s_i = \emptyset$ for all $i = 3, \dots, l$, and a_1 and s_2 intersect transversely in a unique point (see Figure 10).

Let τ_1 be the braid twist along a_1, and let σ_i be the Dehn twist along s_i, for $i = 2, \dots, l$. By Lemma 2.3, there is a well defined homomorphism $A(B_l) \to \mathcal{M}(F, \{P_1, P_2\})$ which sends x_1 on τ_1, and x_i on σ_i for $i = 2, \dots, l$. It is shown in [14] that this geometric representation is injective.

Now, consider a graph G embedded in a surface F. Here, we assume that G has no loop and no multiple-edge. Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be the set of vertices of G, and let $a_1, \dots, a_l$ be the edges. Define the Coxeter matrix $M = (m_{i,j})_{i,j=1,\dots,l}$ by $m_{i,j} = 3$ if a_i and a_j have a common vertex, and $m_{i,j} = 2$ otherwise. Denote by Γ the Coxeter graph associated with M. By Lemma 2.2, one has

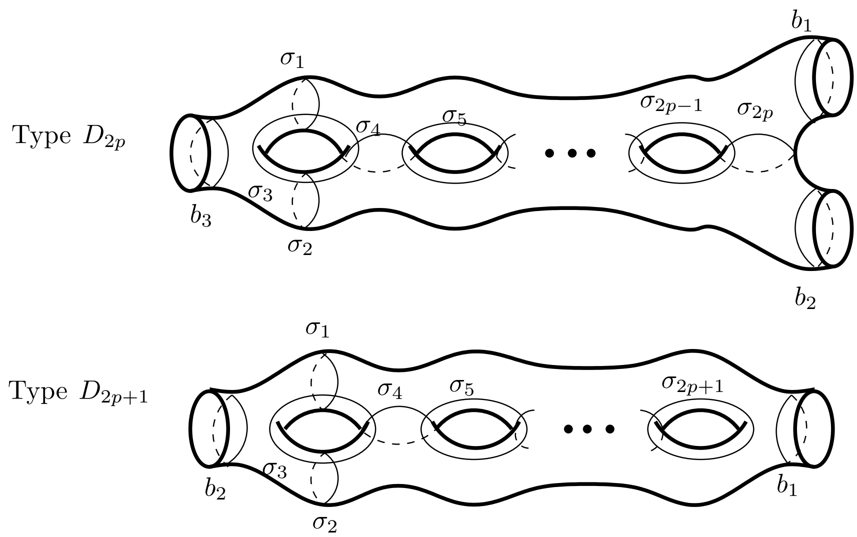

Figure 8: Perron-Vannier representations of type D_l

a homomorphism $A(\Gamma) \to \mathcal{M}(F, \mathcal{P})$ which associates with x_i the braid twist τ_i along a_i, for all $i = 1, \dots, l$. This homomorphism will be called *graph representation* of $A(\Gamma)$. Its image clearly belongs to the surface braid group of F based at $\mathcal{P}$. The particular case where F is a disk has been studied by Sergiescu [23] to find new presentations for the Artin braid groups. Graph representations have been also used by Humphries [12] to solve some Tits' conjecture.

Assume now that G is a line in a cylinder $F = S^1 \times I$. Let $a_2, \dots, a_l$ be the edges of G, and let $\mathcal{P}_l = \{P_1, \dots, P_l\}$ be the set of vertices. Choose an essential circle $s_1 : S^1 \to F \setminus \mathcal{P}$ such that:

- s_1 does not bound a disk in F;
- $s_1 \cap a_i = \emptyset$ for all $i = 3, \dots, l$, and s_1 and a_2 intersect transversely in a unique point (see Figure 11).

Let σ_1 be the Dehn twist along s_1, and let τ_i be the braid twist along a_i for $i = 2, \dots, l$. By Lemma 2.3, there is a well defined homomorphism $A(B_l) \to \mathcal{M}(S^1 \times I, \mathcal{P}_l)$ which sends x_1 on σ_1, and x_i on τ_i for $i = 2, \dots, l$. This homomorphism is clearly an extension of the graph representation of $A(A_{l-1})$ in $\mathcal{M}(S^1 \times I, \mathcal{P}_l)$.

Let Γ be a finite type connected graph. Recall that the *quasi-center* of $A(\Gamma)$ is the subgroup of elements α in $A(\Gamma)$ satisfying $\alpha X \alpha^{-1} = X$, where X is

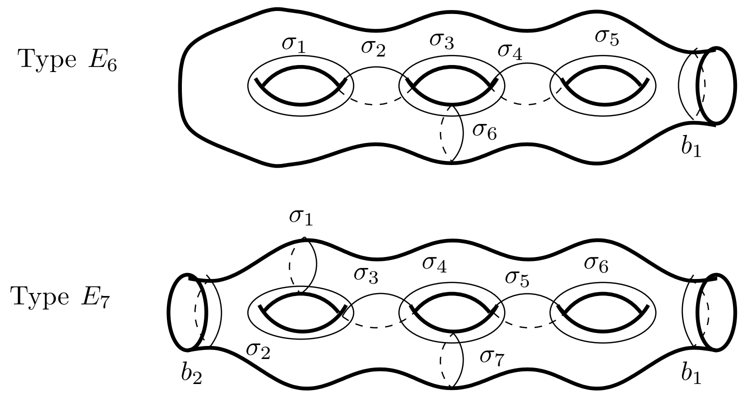

Figure 9: Perron-Vannier representations of type E_6 and E_7

the natural generating set of $A(\Gamma)$, and that this subgroup is an infinite cyclic group generated by some special element of $A(\Gamma)$, called *fundamental element*, and denoted by $\Delta(\Gamma)$. (see [4] and [8]). The center of $A(\Gamma)$ is an infinite cyclic group generated by $\Delta(\Gamma)$ if Γ is B_l, D_l (l even), E_7, E_8, F_4, H_3, H_4, and $I_2(p)$ (p even), and by $\Delta^2(\Gamma)$ if Γ is A_l, D_l (l odd), E_6, and $I_2(p)$ (p odd). Explicit expressions of $\Delta(\Gamma)$ and of $\Delta^2(\Gamma)$ can be found in [4]. In the remainder, we will need the following ones.

Proposition 2.8 (Brieskorn, Saito [4]) *We number the vertices of A_l, B_l, D_l, E_6, and E_7 according to Figure 6.*

$$\begin{aligned}
\Delta^2(A_l) &= (x_1x_2\dots x_l)^{l+1} , \\
\Delta(B_l) &= (x_1x_2\dots x_l)^{l} , \\
\Delta(D_{2p}) &= (x_1x_2\dots x_{2p})^{2p-1} , \\
\Delta^2(D_{2p+1}) &= (x_1x_2\dots x_{2p+1})^{4p} , \\
\Delta^2(E_6) &= (x_1x_2\dots x_6)^{12} , \\
\Delta(E_7) &= (x_1x_2\dots x_7)^{15} .
\end{aligned}$$

□

We will also need the following well known equalities (see [20]).

Proposition 2.9 *We number the vertices of A_l, B_l, and D_l according to Figure 6. Then:*

$$\begin{aligned}
\Delta(A_l) &= x_1\dots x_l\cdot\Delta(A_{l-1}), \\
\Delta(B_l) &= x_l\dots x_2x_1x_2\dots x_l\cdot\Delta(B_{l-1}), \\
\Delta(D_l) &= x_l\dots x_3x_1x_2x_3\dots x_l\cdot\Delta(D_{l-1}).
\end{aligned}$$

□

Our goal now is to determine the images under Perron-Vannier representations and under graph representations of some powers of fundamental elements

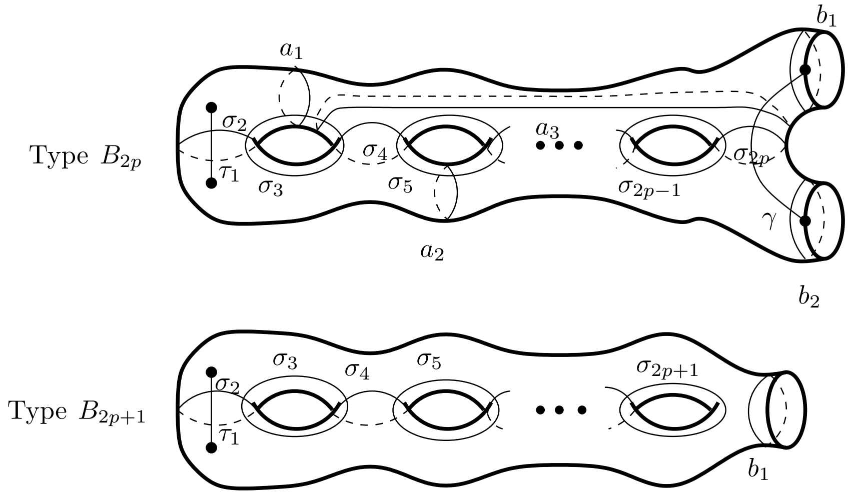

Figure 10: Perron-Vannier representation of type B_l

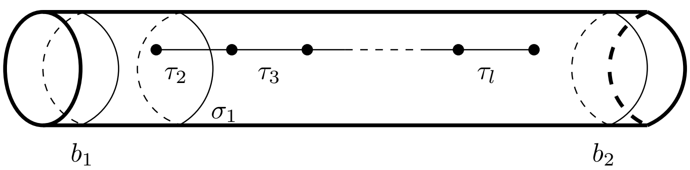

Figure 11: Graph representation of type B_l

(Proposition 2.12). To do so, we first need to know generating sets for the punctured mapping class groups. So, we prove the following.

Proposition 2.10 *Let $g \geq 1$ and $r, n \geq 0$.*

(i) *$\mathcal{PM}(F_{g,r+1}, \mathcal{P}_n)$ is generated by the Dehn twists $a_0, \dots, a_{n+r}, b_1, \dots, b_{2g-1}$, c, $d_1, \dots, d_r$ represented in Figure 12.*

(ii) *$\mathcal{M}(F_{g,r+1}, \mathcal{P}_n)$ is generated by the Dehn twists $a_0, \dots, a_r, a_{r+1}$, $b_1, .., b_{2g-1}$, c, $d_1, \dots, d_r$, and the braid twists $\tau_1, \dots, \tau_{n-1}$ represented in Figure 12.*

Corollary 2.11 *Let $g \geq 1$ and $n \geq 0$.*

(i) *$\mathcal{PM}(F_{g,0}, \mathcal{P}_n)$ is generated by the Dehn twists $a_0, \dots, a_n$, $b_1, \dots, b_{2g-1}$, c represented in Figure 13.*

(ii) *$\mathcal{M}(F_{g,0}, \mathcal{P}_n)$ is generated by the Dehn twists a_0, a_1, $b_1, \dots, b_{2g-1}$, c, and the braid twists $\tau_1, \dots, \tau_{n-1}$ represented in Figure 13.*

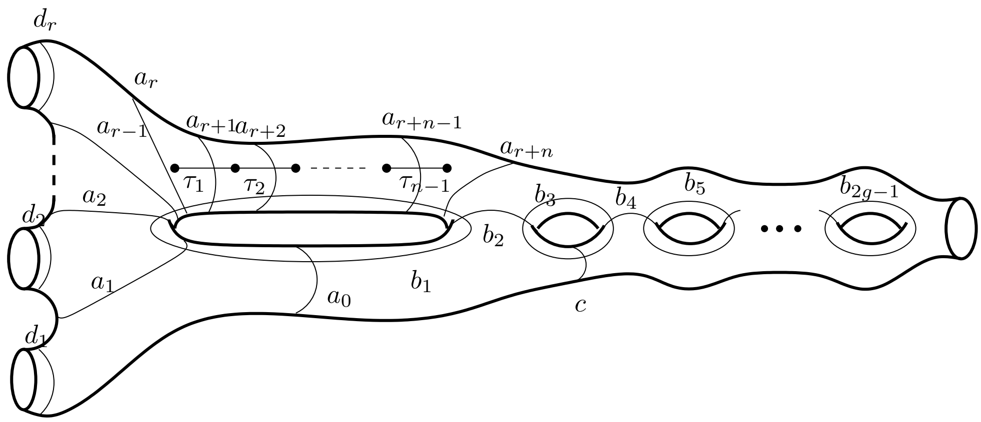

Figure 12: Generators for $\mathcal{PM}(F_{g,r+1},\mathcal{P}_n)$ and $\mathcal{M}(F_{g,r+1},\mathcal{P}_n)$

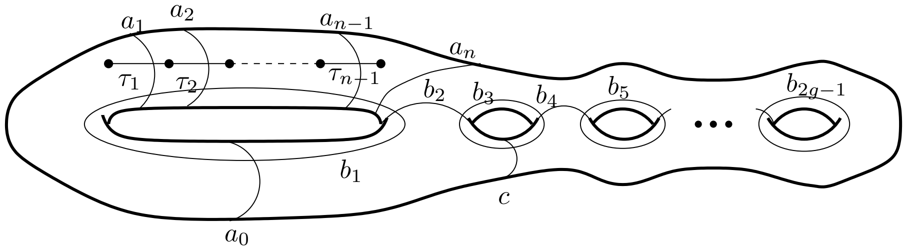

Figure 13: Generators for $\mathcal{PM}(F_{g,0},\mathcal{P}_n)$ and $\mathcal{M}(F_{g,0},\mathcal{P}_n)$

Proof The key argument of the proof of Proposition 2.10 is the following remark stated as Assertion 1, and which we apply to the exact sequences (2.1), (2.2), and (2.3) of Subsection 2.2.

Assertion 1 *Let*

$$1 \to K \to G \xrightarrow{\rho} H \to 1$$

be an exact sequence, and let S_H, S_K be generating sets of H and K, respectively. For each $x \in S_H$ we choose $\tilde{x} \in G$ such that $\rho(\tilde{x}) = x$, and we write $\tilde{S}_H = \{\tilde{x}; x \in S_H\}$. Then $S_K \cup \tilde{S}_H$ generates G.

First, we prove by induction on n that $\mathcal{PM}(F_{g,1},\mathcal{P}_n)$ is generated by $a_0,\dots,a_n$, $b_1,\dots,b_{2g-1}$, c. The case $n=0$ is proved in [11]. So, we assume that $n>0$. By the inductive hypothesis, $\mathcal{PM}(F_{g,1},\mathcal{P}_{n-1})$ is generated by $a_0,\dots,a_{n-1}$, $b_1,\dots,b_{2g-1}$, c. On the other hand, $\pi_1(F_{g,1}\setminus\mathcal{P}_{n-1},P_n)$ is the free group generated by the loops $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_{2g-1}$ represented in Figure 14. Applying

Assertion 1 to the exact sequence (2.2), one has that $\mathcal{PM}(F_{g,1}, \mathcal{P}_n)$ is generated by $a_0, \dots, a_{n-1}$, $b_1, \dots, b_{2g-1}$, c, $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_{2g-1}$. One can directly verify the following equalities:

$$\begin{aligned} \alpha_i &= (b_1 a_n a_{i-1} b_1 a_{n-1})^{-1} \alpha_n^{-1} (b_1 a_n a_{i-1} b_1 a_{n-1}), && i = 1, \dots, n-1, \\ \beta_1 &= (b_1 a_{n-1})^{-1} \alpha_n (b_1 a_{n-1}), && \\ \beta_j &= (b_j b_{j-1})^{-1} \beta_{j-1} (b_j b_{j-1}), && j = 2, \dots, 2g-1. \end{aligned}$$

and, from Proposition 2.6, one has:

$$\alpha_n = a_{n-1}^{-1} a_n,$$

thus $\mathcal{PM}(F_{g,1}, \mathcal{P}_n)$ is generated by $a_0, \dots, a_n$, $b_1, \dots, b_{2g-1}$, c.

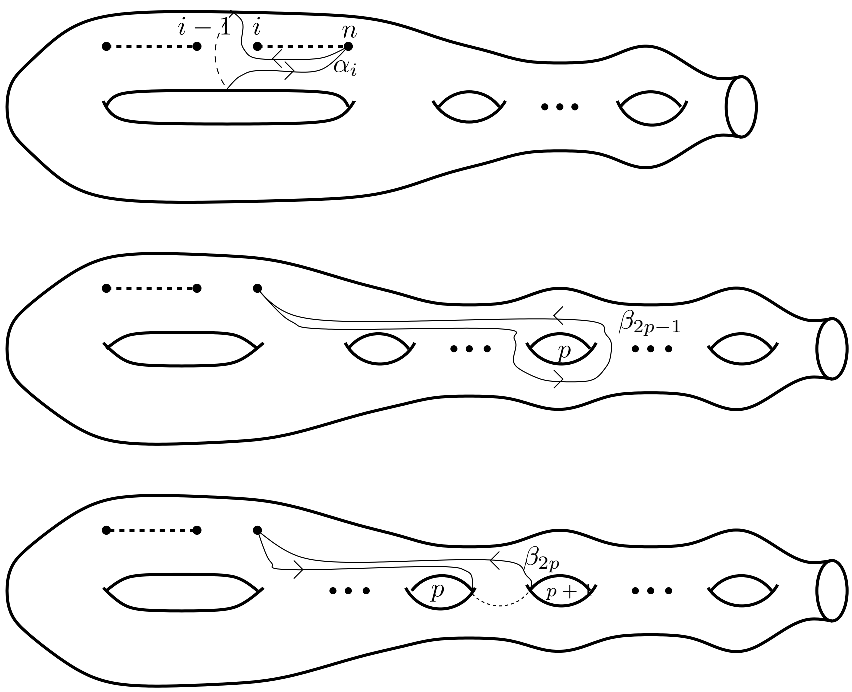

Figure 14: Generators for $\pi_1(F_{g,1} \setminus \mathcal{P}_{n-1}, P_n)$

Now, applying Assertion 1 to (2.3), one has that $\mathcal{PM}(F_{g,r+1}, \mathcal{P}_n)$ is generated by $a_0, \dots, a_{n+r}$, $b_1, \dots, b_{2g-1}$, c, $d_1, \dots, d_r$.

Assertion 2 *Let a_0, a_1, a_2 be the Dehn twists and τ the braid twist in $\mathcal{M}(S^1 \times I, \{P_1, P_2\})$ represented in Figure 15. Then*

$$\tau a_1 \tau a_1 = a_0 a_2 \ .$$

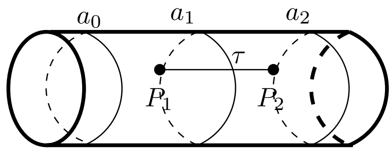

Figure 15: A relation in $\mathcal{M}(S^1 \times I, \{P_1, P_2\})$

Proof of Assertion 2 We consider the Dehn twist a_3 along a circle which bounds a small disk in $S^1 \times I$ which contains P_1, and the Dehn twist a_4 along a circle which bounds a small disk in $S^1 \times I$ which contains P_2. As pointed out in Subsection 2.1, we have $a_3 = a_4 = 1$. The lantern relation of Lemma 2.4 says:

$$\tau^2 \cdot a_1 \cdot \tau a_1 \tau^{-1} = a_0 a_2 a_3 a_4 \ .$$

Thus, since τ commutes with a_0 and a_2, we have:

$$\tau a_1 \tau a_1 = a_0 a_2 \ .$$

Now, we prove (ii). Applying Assertion 1 to (2.1), one has that $\mathcal{M}(F_{g,r+1}, \mathcal{P}_n)$ is generated by $a_0, \dots, a_{n+r}, b_1, \dots, b_{2g-1}, c, d_1, \dots, d_r, \tau_1, \dots, \tau_{n-1}$. But, Assertion 2 implies

$$a_{r+i} = \tau_{i-1} a_{r+i-1} \tau_{i-1} a_{r+i-1} a_{r+i-2}^{-1}$$

for $i = 2, \dots, r$, thus $\mathcal{M}(F_{g,r+1}, \mathcal{P}_n)$ is generated by $a_0, \dots, a_{r+1}$, $b_1, \dots, b_{2g-1}$, c, $d_1, \dots, d_r$, $\tau_1, \dots, \tau_{n-1}$. □

Proposition 2.12 (i) *For Γ equal to A_l, D_l, E_6, or E_7, we denote by $\rho_{PV} : A(\Gamma) \to \mathcal{M}(F)$ the Perron-Vannier representation of $A(\Gamma)$. In each case, b_i denotes the Dehn twist represented in the corresponding figure (Figure 7, 8, or 9), for $i = 1, 2, 3$. Then:*

$$\begin{aligned} \rho_{PV}(\Delta^2(A_{2p+1})) &= b_1 b_2, \\ \rho_{PV}(\Delta^4(A_{2p})) &= b_1, \\ \rho_{PV}(\Delta^2(D_{2p+1})) &= b_1 b_2^{2p-1}, \\ \rho_{PV}(\Delta(D_{2p})) &= b_1 b_2 b_3^{p-1}, \\ \rho_{PV}(\Delta^2(E_6)) &= b_1, \\ \rho_{PV}(\Delta(E_7)) &= b_1 b_2^2. \end{aligned}$$

(ii) *We denote by* $\rho_{PV} : A(B_l) \to \mathcal{M}(F, \{P_1, P_2\})$ *the Perron-Vannier representation of* $A(B_l)$. *In each case,* b_i *denotes the Dehn twist represented in Figure 10, for* $i = 1, 2$. *Then:*

$$\begin{aligned} \rho_{PV}(\Delta(B_{2p})) &= b_1 b_2, \\ \rho_{PV}(\Delta^2(B_{2p+1})) &= b_1. \end{aligned}$$

(iii) *We denote by* $\rho_G : A(B_l) \to \mathcal{M}(S^1 \times I, \mathcal{P}_l)$ *the graph representation of* $A(B_l)$ *in the punctured mapping class group of the cylinder. Let* b_1, b_2 *denote the Dehn twists represented in Figure 11. Then:*

$$\rho_G(\Delta(B_l)) = b_1^{l-1} b_2 \ .$$

Part (i) of Proposition 2.12 is proved in [18] with different techniques from the ones used in this paper. Matsumoto's proof is based on the study of geometric monodromies of simple singularities. Our proof consists first on showing that the image of the considered element lies in the center of the punctured mapping class group, and, afterwards, on identifying this image using the action of the center on some curves.

Proof We only prove the equality

$$\rho(\Delta(B_{2p})) = b_1 b_2$$

of Part (ii): the other equalities can be proved in the same way.

By Proposition 2.10, $\mathcal{M}(F, \{P_1, P_2\})$ is generated by the Dehn twists a_1, a_2, a_3, b_1, $\sigma_2, \dots, \sigma_{2p-1}$ and the braid twist τ_1 represented in Figure 10. Since $\Delta(B_{2p})$ is in the center of $A(B_{2p})$, $\rho_{PV}(\Delta(B_{2p}))$ commutes with $\tau_1, \sigma_2, \dots, \sigma_{2p-1}$. The Dehn twist b_1 belongs to the center of $\mathcal{M}(F, \{P_1, P_2\})$, thus $\rho_{PV}(\Delta(B_{2p}))$ also commutes with b_1. Let s_i be the defining circle of a_i, for $i = 1, 2, 3$. Using the expression of $\Delta(B_{2p})$ given in Proposition 2.8, we verify that $\rho_{PV}(\Delta(B_{2p}))(s_i)$ is isotopic to s_i, thus $\rho_{PV}(\Delta(B_{2p}))$ commutes with a_i.

So, $\rho_{PV}(\Delta(B_{2p}))$ is an element of the center of $\mathcal{M}(F, \{P_1, P_2\})$. By [21], this center is a free abelian group of rank 2 generated by b_1 and b_2. Thus $\rho_{PV}(\Delta(B_{2p})) = b_1^{q_1} b_2^{q_2}$ for some $q_1, q_2 \in \mathbf{Z}$.

Now, consider the curve γ of Figure 10. Clearly, the only element of the center of $\mathcal{M}(F, \{P_1, P_2\})$ which fixes γ up to isotopy is the identity. Using the expression of $\Delta(B_{2p})$ given in Proposition 2.8, we verify that $\rho_{PV}(\Delta(B_{2p}))b_1^{-1}b_2^{-1}$ fixes γ up to isotopy, thus $q_1 = q_2 = 1$ and $\rho_{PV}(\Delta(B_{2p})) = b_1 b_2$. □

2.4 Matsumoto's presentation for $\mathcal{M}(F_{g,1})$ and $\mathcal{M}(F_{g,0})$

This subparagraph is dedicated to the statement of Matsumoto's presentations for $\mathcal{M}(F_{g,1})$ and $\mathcal{M}(F_{g,0})$.

We first introduce some notation. Let Γ be a Coxeter graph, and let X be a subset of the set $\{x_1, \dots, x_l\}$ of vertices of Γ. Recall that Γ_X denotes the Coxeter subgraph generated by X, and A_X denotes the parabolic subgroup of $A(\Gamma)$ generated by X. If Γ_X is a finite type connected Coxeter graph, then we denote by $\Delta(X)$ the fundamental element of A_X, viewed as an element of $A(\Gamma)$.

Theorem 2.13 (Matsumoto [18]). *Let $g \geq 1$, and let Γ_g be the Coxeter graph drawn in Figure 16.*

(i) *$\mathcal{M}(F_{g,1})$ is isomorphic with the quotient of $A(\Gamma_g)$ by the following relations:*

$$\begin{array}{llcll} (1) & \Delta^4(y_1, y_2, y_3, z) & = & \Delta^2(x_0, y_1, y_2, y_3, z) & \text{if } g \geq 2, \\ (2) & \Delta^2(y_1, y_2, y_3, y_4, y_5, z) & = & \Delta(x_0, y_1, y_2, y_3, y_4, y_5, z) & \text{if } g \geq 3. \end{array}$$

(ii) *$\mathcal{M}(F_{g,0})$ is isomorphic with the quotient of $A(\Gamma_g)$ by the relations (1) and (2) above plus the following relation:*

$$(3) \quad \begin{array}{lcll} (x_0 y_1)^6 & = & 1 & \text{if } g = 1, \\ x_0^{2g-2} & = & \Delta^2(y_2, y_3, z, y_4, \dots, y_{2g-1}) & \text{if } g \geq 2. \end{array}$$

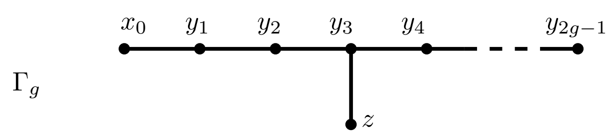

Figure 16: Coxeter graph associated with $\mathcal{M}(F_{g,1})$ and with $\mathcal{M}(F_{g,0})$

Set $r = n = 0$, and consider the Dehn twists a_0, $b_1, \dots, b_{2g-1}$, c of Figure 12. By Lemma 2.1, there is a well defined homomorphism $\rho : A(\Gamma_g) \to \mathcal{M}(F_{g,1})$ which sends x_0 on a_0, y_i on b_i for $i = 1, \dots, 2g-1$, and z on c. By [11] (see Proposition 2.10), this homomorphism is surjective. By Proposition 2.12, both $\rho(\Delta^4(y_1, y_2, y_3, z))$ and $\rho(\Delta^2(x_0, y_1, y_2, y_3, z))$ are equal to the Dehn twist σ_1 of Figure 17. Similarly, both $\rho(\Delta^2(y_1, \dots, y_5, z))$ and $\rho(\Delta(x_0, y_1, \dots, y_5, z))$ are equal to the Dehn twist σ_2 of Figure 17. Let G_g denote the quotient of $A(\Gamma_g)$ by the relations (1) and (2). So, the homomorphism $\rho : A(\Gamma_g) \to \mathcal{M}(F_{g,1})$ induces a surjective homomorphism $\bar{\rho} : G_g \to \mathcal{M}(F_{g,1})$. In order to prove that this homomorphism is in fact an isomorphism, Matsumoto [18] showed that the presentation of G_g as a quotient of $A(\Gamma_g)$ is equivalent to Wajnryb's presentation of $\mathcal{M}(F_{g,1})$ [25].

Similar remarks can be made for the presentation of $\mathcal{M}(F_{g,0})$.

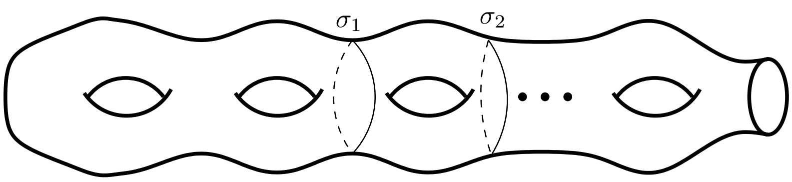

Figure 17: Relations in $\mathcal{M}(F_{g,1})$

3 The presentation

Recall that, if Γ is a finite type connected Coxeter graph, then $\Delta(\Gamma)$ denotes the fundamental element of $A(\Gamma)$. If Γ is any Coxeter graph and X is a subset of the set $\{x_1, \dots, x_l\}$ of vertices of Γ such that Γ_X is finite type and connected, then we denote by $\Delta(X)$ the fundamental element of $A_X = A(\Gamma_X)$ viewed as an element of $A(\Gamma)$.

Theorem 3.1 *Let $g \ge 1$, let $r, n \ge 0$, and let $\Gamma_{g,r,n}$ be the Coxeter graph drawn in Figure 18. Then $\mathcal{M}(F_{g,r+1}, \mathcal{P}_n)$ is isomorphic with the quotient of $A(\Gamma_{g,r,n})$ by the following relations.*

- *Relations from $\mathcal{M}(F_{g,1})$:*

$$\text{(R1)} \quad \Delta^4(y_1, y_2, y_3, z) = \Delta^2(x_0, y_1, y_2, y_3, z) \quad \text{if } g \ge 2,$$
$$\text{(R2)} \quad \Delta^2(y_1, y_2, y_3, y_4, y_5, z) = \Delta(x_0, y_1, y_2, y_3, y_4, y_5, z) \quad \text{if } g \ge 3.$$

- *Relations of commutation:*

$$\begin{aligned} \text{(R3)} \quad & x_k \Delta^{-1}(x_{i+1}, x_j, y_1) x_i \Delta(x_{i+1}, x_j, y_1) \\ &= \Delta^{-1}(x_{i+1}, x_j, y_1) x_i \Delta(x_{i+1}, x_j, y_1) x_k \quad \text{if } 0 \le k < j < i \le r, \\ \text{(R4)} \quad & y_2 \Delta^{-1}(x_{i+1}, x_j, y_1) x_i \Delta(x_{i+1}, x_j, y_1) \\ &= \Delta^{-1}(x_{i+1}, x_j, y_1) x_i \Delta(x_{i+1}, x_j, y_1) y_2 \quad \text{if } 0 \le j < i \le r \text{ and } g \ge 2, \end{aligned}$$

- *Expressions of the u_i's:*

$$\text{(R5)}\quad u_1 = \Delta(x_0,x_1,y_1,y_2,y_3,z)\Delta^{-2}(x_1,y_1,y_2,y_3,z) \qquad \text{if } g\geq 2,$$

$$\text{(R6)}\quad u_{i+1} = \Delta(x_i,x_{i+1},y_1,y_2,y_3,z)\Delta^{-2}(x_{i+1},y_1,y_2,y_3,z)\,\Delta^{2}(x_0,x_{i+1},y_1)\Delta^{-1}(x_0,x_i,x_{i+1},y_1) \quad \text{if } 1\leq i\leq r-1\,,\ g\geq 2.$$

- *Other relations:*

$$\text{(R7)}\quad \Delta(x_r,x_{r+1},y_1,v_1) = \Delta^2(x_{r+1},y_1,v_1) \qquad \text{if } n\geq 2,$$

$$\text{(R8a)}\quad \Delta(x_0,x_1,y_1,y_2,y_3,z) = \Delta^2(x_1,y_1,y_2,y_3,z) \qquad \text{if } n\geq 1, g\geq 2,\ r=0,$$

$$\text{(R8b)}\quad \Delta(x_r,x_{r+1},y_1,y_2,y_3,z)\Delta^{-2}(x_{r+1},y_1,y_2,y_3,z) = \Delta(x_0,x_r,x_{r+1},y_1)\Delta^{-2}(x_0,x_{r+1},y_1) \qquad \text{if } n\geq 1, g\geq 2,\ r\geq 1.$$

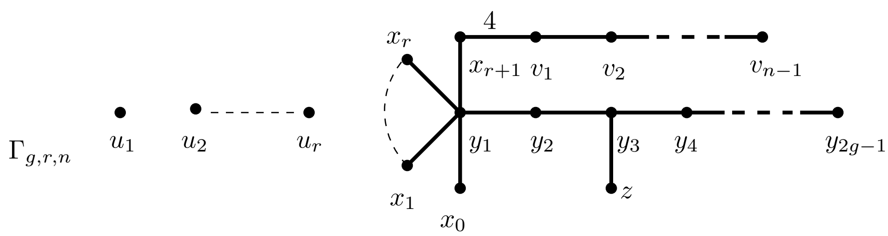

Figure 18: Coxeter graph associated with $\mathcal{M}(F_{g,r+1},\mathcal{P}_n)$

Notice that only the relations (R1), (R2), (R7), and (R8a) remain in the presentation of $\mathcal{M}(F_{g,1},\mathcal{P}_n)$, and (R8a) has to be replaced by (R8b) if $r\geq 1$.

Assume that $g\geq 2$. From the relations (R5) and (R6) we see that we can remove $u_1,\dots,u_r$ from the generating set. However, to do so, one has to add relations comming from the ones in the Artin group $A(\Gamma_{g,r,n})$. For example, one has that $\Delta(x_0,x_1,y_1,y_2,y_3,z)\Delta^{-2}(x_1,y_1,y_2,y_3,z)$ commutes with y_4 in the quotient, since u_1 commutes with y_4 in $A(\Gamma_{g,r,n})$.

Consider the Dehn twists $a_0,\dots,a_{r+1}$, $b_1,\dots,b_{2g-1}$, c, $d_1,\dots,d_r$ and the braid twists $\tau_1,\dots,\tau_{n-1}$ represented in Figure 12. From Subsection 2.1 follows that there is a well defined homomorphism $\rho: A(\Gamma_{g,r,n})\to\mathcal{M}(F_{g,r+1},\mathcal{P}_n)$ which sends x_i on a_i for $i=0,\dots,r+1$, y_i on b_i for $i=1,\dots,2g-1$, z on c, u_i on d_i for $i=1,\dots,r$, and v_i on τ_i for $i=1,\dots,n-1$. This homomorphism is surjective by Proposition 2.10. If $w_1=w_2$ is one of the relations (R1),…,(R7), (R8a), (R8b), then we have $\rho(w_1)=\rho(w_2)$. This fact can be easily proved using Proposition 2.12 in the case of the relations (R1), (R2), (R5), (R6), (R7), (R8a), and (R8b), and comes from the following reason in the case of the relations (R3) and (R4). We have the equality

$$\Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1) = y_1^{-1}x_{i+1}^{-1}x_j^{-1}y_1^{-1}x_iy_1x_jx_{i+1}y_1,$$

and the image by $b_1^{-1}a_{i+1}^{-1}a_j^{-1}b_1^{-1}$ of the defining circle of a_i is disjoint from the defining circle of a_k, up to isotopy, if $k<j$, and is disjoint from the defining circle of b_2, up to isotopy.

Let $G(g,r,n)$ denote the quotient of $A(\Gamma_{g,r,n})$ by the relations (R1),..,(R7), (R8a), (R8b). By the above considerations, the homomorphism :

$$\rho : A(\Gamma_{g,r,n}) \to \mathcal{M}(F_{g,r+1},\mathcal{P}_n)$$

induces a surjective homomorphism $\bar\rho : G(g,r,n) \to \mathcal{M}(F_{g,r+1},\mathcal{P}_n)$. In order to prove Theorem 3.1, it remains to show that this homomorphism is in fact an isomorphism. This will be the object of Subsection 3.1.

Theorem 3.2 *Let $g\ge 1$, let $n\ge 1$, and let $\Gamma_{g,0,n}$ be the Coxeter graph drawn in Figure 18. Then $\mathcal{M}(F_{g,0},\mathcal{P}_n)$ is isomorphic with the quotient of $A(\Gamma_{g,0,n})$ by the following relations.*

- *Relations from $\mathcal{M}(F_{g,1},\mathcal{P}_n)$:*

$$\begin{array}{llcll}
\text{(R1)} & \Delta^4(y_1,y_2,y_3,z) & = & \Delta^2(x_0,y_1,y_2,y_3,z) & \textit{if } g\ge 2,\\
\text{(R2)} & \Delta^2(y_1,y_2,y_3,y_4,y_5,z) & = & \Delta(x_0,y_1,y_2,y_3,y_4,y_5,z) & \textit{if } g\ge 3,\\
\text{(R7)} & \Delta(x_0,x_1,y_1,v_1) & = & \Delta^2(x_1,y_1,v_1) & \textit{if } n\ge 2,\\
\text{(R8a)} & \Delta(x_0,x_1,y_1,y_2,y_3,z) & = & \Delta^2(x_1,y_1,y_2,y_3,z) & \textit{if } n\ge 1 \textit{ and } g\ge 2.
\end{array}$$

- *Other relations:*

$$\begin{array}{lrcll}
\text{(R9a)} & x_0^{2g-n-2}\Delta(x_1,v_1,\dots,v_{n-1}) & = & \Delta^2(z,y_2,\dots,y_{2g-1}) & \textit{if } g\ge 2,\\
\text{(R9b)} & x_0^n & = & \Delta(x_1,v_1,\dots,v_{n-1}) & \textit{if } g=1,\\
\text{(R9c)} & \Delta^4(x_0,y_1) & = & \Delta^2(v_1,\dots,v_{n-1}) & \textit{if } g=1.
\end{array}$$

Note that, in the above presentation, the relation (R9a), which holds if $g\ge 2$, has to be replaced by the relations (R9b) and (R9c) when $g=1$.

Consider the Dehn twists a_0,a_1, $b_1,\dots,b_{2g-1}$, c and the braid twists $\tau_1,..,\tau_{n-1}$ represented in Figure 13. From Subsection 2.1 follows that there is a well defined homomorphism $\rho_0 : A(\Gamma_{g,0,n}) \to \mathcal{M}(F_{g,0},\mathcal{P}_n)$ which sends x_i on a_i for $i=0,1$, y_i on b_i for $i=1,\dots,2g-1$, z on c, and v_i on τ_i for $i=1,\dots,n-1$. This homomorphism is surjective by Corollary 2.11. Let $G_0(g,n)$ denote the quotient of $A(\Gamma_{g,0,n})$ by the relations (R1), (R2), (R7), (R8), (R9a), (R9b), and (R9c). As before, using Proposition 2.12, one can easily prove that the homomorphism $\rho_0 : A(\Gamma_{g,0,n}) \to \mathcal{M}(F_{g,0},\mathcal{P}_n)$ induces a surjective homomorphism $\bar\rho_0 : G_0(g,n) \to \mathcal{M}(F_{g,0},\mathcal{P}_n)$. In order to prove Theorem 3.2, it remains to show that this homomorphism is in fact an isomorphism. This will be the object of Subsection 3.2.

3.1 Proof of Theorem 3.1

The proof of Theorem 3.1 is organized as follows. In the first step, starting from Matsumoto's presentation of $\mathcal{M}(F_{g,1})$ [18] (see Theorem 2.13), we determine by induction on n a presentation of $\mathcal{PM}(F_{g,1},\mathcal{P}_n)$ (Proposition 3.3), applying Lemma 2.5 to the exact sequence (2.2) of Subsection 2.2. In the second step, we determine a presentation of $\mathcal{PM}(F_{g,r+1},\mathcal{P}_n)$ (Proposition 3.7), applying Lemma 2.5 to the exact sequence (2.3). Finally, we prove Theorem 3.1 applying Lemma 2.5 to the exact sequence (2.1).

Proposition 3.3 *Let $g \ge 1$, let $n \ge 0$, and let $P\Gamma_{g,0,n}$ be the Coxeter graph drawn in Figure 19. Then $\mathcal{PM}(F_{g,1},\mathcal{P}_n)$ is isomorphic with the quotient of $A(P\Gamma_{g,0,n})$ by the following relations.*

- *Relations from $\mathcal{M}(F_{g,1})$:*

(PR1) $\Delta^4(y_1,y_2,y_3,z) = \Delta^2(x_0,y_1,y_2,y_3,z)$ *if $g \ge 2$,*

(PR2) $\Delta^2(y_1,y_2,y_3,y_4,y_5,z) = \Delta(x_0,y_1,y_2,y_3,y_4,y_5,z)$ *if $g \ge 3$.*

- *Relations of commutation:*

(PR3)
$$x_k\Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1) = \Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)x_k \quad \text{if } 0 \le k < j < i \le n-1,$$

(PR4)
$$y_2\Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1) = \Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)y_2 \quad \text{if } 0 \le j < i \le n-1,\ g \ge 2.$$

- *Relations between fundamental elements:*

(PR5) $\Delta(x_0,x_1,y_1,y_2,y_3,z) = \Delta^2(x_1,y_1,y_2,y_3,z)$ *if $g \ge 2$,*

(PR6)
$$\Delta(x_i,x_{i+1},y_1,y_2,y_3,z)\Delta^{-2}(x_{i+1},y_1,y_2,y_3,z) = \Delta(x_0,x_i,x_{i+1},y_1)\Delta^{-2}(x_0,x_{i+1},y_1) \quad \text{if } 1 \le i \le n-1,\ g \ge 2.$$

$P\Gamma_{g,0,n}$

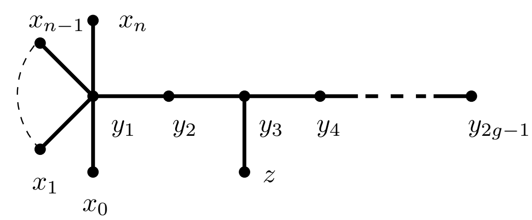

Figure 19: Coxeter graph associated with $\mathcal{PM}(F_{g,1},\mathcal{P}_n)$

The following lemmas 3.4, 3.5, and 3.6 are preliminary results to the proof of Proposition 3.3.

Lemma 3.4 *Let Γ be the Coxeter graph drawn in Figure 20, and let G be the quotient of $A(\Gamma)$ by the following relation:*

$$x_4\Delta^{-1}(x_1,x_3,y)x_2\Delta(x_1,x_3,y) = \Delta^{-1}(x_1,x_3,y)x_2\Delta(x_1,x_3,y)x_4\ .$$

Then the following equalities hold in G.

$$\begin{aligned}
x_3\Delta^{-1}(x_2,x_4,y)x_1\Delta(x_2,x_4,y) &= \Delta^{-1}(x_2,x_4,y)x_1\Delta(x_2,x_4,y)x_3,\\
x_2\Delta^{-1}(x_1,x_3,y)x_4\Delta(x_1,x_3,y) &= \Delta^{-1}(x_1,x_3,y)x_4\Delta(x_1,x_3,y)x_2,\\
x_1\Delta^{-1}(x_2,x_4,y)x_3\Delta(x_2,x_4,y) &= \Delta^{-1}(x_2,x_4,y)x_3\Delta(x_2,x_4,y)x_1.
\end{aligned}$$

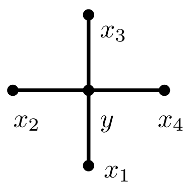

Figure 20

Proof It clearly suffices to prove the first equality.

$$\begin{aligned}
&x_3\Delta^{-1}(x_2,x_4,y)x_1\Delta(x_2,x_4,y)x_3^{-1}\Delta^{-1}(x_2,x_4,y)x_1^{-1}\Delta(x_2,x_4,y)\\
=\ &x_3y^{-1}x_2^{-1}x_4^{-1}y^{-1}x_1yx_2x_4yx_3^{-1}y^{-1}x_2^{-1}x_4^{-1}y^{-1}x_1^{-1}yx_2x_4y\\
=\ &y^{-1}\cdot x_3^{-1}yx_3x_2^{-1}x_4^{-1}x_1yx_1^{-1}x_2x_4x_3^{-1}y^{-1}x_3x_2^{-1}x_4^{-1}x_1y^{-1}x_1^{-1}x_2x_4\cdot y\\
=\ &y^{-1}x_2^{-1}x_3^{-1}\cdot x_2yx_2^{-1}x_1x_3x_4^{-1}yx_4x_1^{-1}x_3^{-1}x_2y^{-1}x_2^{-1}x_1x_3x_4^{-1}y^{-1}x_4x_1^{-1}x_3^{-1}\\
&\cdot x_3x_2y\\
=\ &y^{-1}x_2^{-1}x_3^{-1}\cdot y^{-1}x_2yx_1x_3yx_4y^{-1}x_1^{-1}x_3^{-1}y^{-1}x_2^{-1}yx_1x_3yx_4^{-1}y^{-1}x_1^{-1}x_3^{-1}\cdot x_3x_2y\\
=\ &y^{-1}x_2^{-1}x_3^{-1}y^{-1}\cdot x_2\Delta(x_1,x_3,y)x_4\Delta^{-1}(x_1,x_3,y)x_2^{-1}\Delta(x_1,x_3,y)x_4^{-1}\\
&\Delta^{-1}(x_1,x_3,y)\cdot yx_3x_2y\\
=\ &1.
\end{aligned}$$

□

Lemma 3.5 *We number the vertices of the Coxeter graph D_l according to Figure 6. Then the following equalities hold in $A(D_l)$.*

$$\begin{aligned}
&\Delta^{-1}(x_2,\dots,x_{l-1})x_1^{-1}x_2\Delta(x_2,\dots,x_{l-1})\Delta^{-1}(x_2,\dots,x_l)x_2^{-1}x_1\Delta(x_2,\dots,x_l)\\
=\ &x_l\Delta^{-1}(x_2,\dots,x_{l-1})x_1^{-1}x_2\Delta(x_2,\dots,x_{l-1})x_l^{-1},
\end{aligned}$$

$$\begin{aligned}
&\Delta^{-1}(x_2,\dots,x_l)x_2^{-1}x_1\Delta(x_2,\dots,x_l)\Delta^{-1}(x_2,\dots,x_{l-1})x_2^{-1}x_1\Delta(x_2,\dots,x_{l-1})\\
=\ &x_{l-1}\Delta^{-1}(x_2,\dots,x_l)x_2^{-1}x_1\Delta(x_2,\dots,x_l)x_{l-1}^{-1}.
\end{aligned}$$

Proof

$$\begin{aligned}
&x_l^{-1}\Delta^{-1}(x_2,\dots,x_{l-1})x_1^{-1}x_2\Delta(x_2,\dots,x_{l-1})\Delta^{-1}(x_2,\dots,x_l)x_2^{-1}x_1\\
&\Delta(x_2,\dots,x_l)x_l\quad \Delta^{-1}(x_2,\dots,x_{l-1})x_2^{-1}x_1\Delta(x_2,\dots,x_{l-1})
\end{aligned}$$

$$
\begin{aligned}
&= x_l^{-1}\Delta^{-1}(x_2,\dots,x_{l-2})(x_{l-1}^{-1}\dots x_2^{-1})x_2x_1^{-1}(x_l^{-1}\dots x_2^{-1})x_2^{-1}x_1x_2\Delta(x_2,\dots,x_l)\\
&\qquad \Delta^{-1}(x_2,\dots,x_{l-1})x_1x_2^{-1}(x_2\dots x_{l-1})\Delta(x_2,\dots,x_{l-2})\\
&= \Delta^{-1}(x_2,\dots,x_{l-2})x_l^{-1}(x_{l-1}^{-1}\dots x_3^{-1})x_1^{-1}(x_l^{-1}\dots x_2^{-1})x_1(x_2\dots x_l)x_1\\
&\quad (x_3\dots x_{l-1})\Delta(x_2,\dots,x_{l-2})\\
&= \Delta^{-1}(x_2,\dots,x_{l-2})(x_l^{-1}\dots x_3^{-1})x_1^{-1}(x_l^{-1}\dots x_3^{-1})(x_3\dots x_l)x_1(x_3\dots x_l)\\
&\quad \Delta(x_2,\dots,x_{l-2})\\
&= 1.
\end{aligned}
$$

$$
\begin{aligned}
&\quad \Delta^{-1}(x_2,\dots,x_l)x_2^{-1}x_1\Delta(x_2,\dots,x_l)\Delta^{-1}(x_2,\dots,x_{l-1})x_2^{-1}x_1\Delta(x_2,\dots,x_{l-1})\\
&\quad x_{l-1}\Delta^{-1}(x_2,\dots,x_l)x_1^{-1}x_2\Delta(x_2,\dots,x_l)x_{l-1}^{-1}\\
&= \Delta^{-1}(x_2,\dots,x_l)x_2^{-1}x_1(x_2\dots x_l)x_2^{-1}x_1x_2\Delta(x_2,\dots,x_{l-1})\Delta^{-1}(x_2,\dots,x_l)x_1^{-1}\\
&\quad x_2x_3^{-1}\quad \Delta(x_2,\dots,x_l)\\
&= \Delta^{-1}(x_2,\dots,x_l)x_1(x_3\dots x_l)x_1(x_l^{-1}\dots x_2^{-1})x_1^{-1}x_2x_3^{-1}\Delta(x_2,\dots,x_l)\\
&= \Delta^{-1}(x_2,\dots,x_l)x_3x_1(x_3\dots x_l)(x_l^{-1}\dots x_3^{-1})x_1^{-1}x_3^{-1}\Delta(x_2,\dots,x_l)\\
&= 1. \qquad \square
\end{aligned}
$$

Several algorithms to solve the word problem in finite type Artin groups are known (see [4], [8], [6], [7]). We use the one of [7] implemented in a Maple program to prove the following.

Lemma 3.6 (i) *We number the vertices of D_6 according to Figure 6. Let*

$$
\begin{aligned}
w_1 &= \Delta^{-1}(x_1,x_3)x_1^{-1}x_2\Delta(x_1,x_3)\\
w_2 &= \Delta^{-1}(x_1,x_3,x_4)x_1^{-1}x_2\Delta(x_1,x_3,x_4)\\
w_3 &= \Delta^{-1}(x_1,x_3,x_4,x_5)x_1^{-1}x_2\Delta(x_1,x_3,x_4,x_5)
\end{aligned}
$$

Then the following equality holds in $A(D_6)$.

$$x_2^{-1}x_1w_1^{-1}w_2^{-1}w_3^{-1}x_6w_3x_6^{-1}w_1 = \Delta^{-2}(x_2,x_3,\dots,x_6)\Delta(x_1,x_2,x_3,\dots,x_6).$$

(ii) *We number the vertices of D_4 according to Figure 6. Let*

$$w = x_2^{-1}\Delta^{-1}(x_1,x_3,x_4)x_1^{-1}x_2\Delta(x_1,x_3,x_4)x_2.$$

Then the following equality holds in $A(D_4)$.

$$x_1^{-1}x_2w = \Delta^{-2}(x_1,x_3,x_4)\Delta(x_1,x_2,x_3,x_4). \qquad \square$$

Proof of Proposition 3.3 We set $r = 0$ and we consider the Dehn twists $a_0,\dots,a_n$ $b_1,\dots,b_{2g-1}$, c represented in Figure 12. From Subsection 2.1 follows that there is a well defined homomorphism $\rho : A(P\Gamma_{g,0,n}) \to \mathcal{PM}(F_{g,1},\mathcal{P}_n)$ which sends x_i on a_i for $i = 0,\dots,n$, y_i on b_i for $i = 1,\dots,2g-1$, and z

on c. This homomorphism is surjective by Proposition 2.10. Let $PG(g,0,n)$ denote the quotient of $A(P\Gamma_{g,0,n})$ by the relations (PR1),...,(PR6). One can easily prove using Proposition 2.12 that: if $w_1 = w_2$ is one of the relations (PR1),...,(PR6), then $\rho(w_1) = \rho(w_2)$. So, the homomorphism $\rho : A(P\Gamma_{g,0,n}) \to \mathcal{PM}(F_{g,1}, \mathcal{P}_n)$ induces a surjective homomorphism :

$$\bar{\rho} : PG(g,0,n) \to \mathcal{PM}(F_{g,1}, \mathcal{P}_n).$$

Now, we prove by induction on n that $\bar{\rho}$ is an isomorphism. The case $n=0$ is proved in [18] (see Theorem 2.13). So, we assume that $n > 0$. By the inductive hypothesis, $\mathcal{PM}(F_{g,1}, \mathcal{P}_{n-1})$ is isomorphic with $PG(g,0,n-1)$. On the other hand, $\pi_1(F_{g,1} \setminus \mathcal{P}_{n-1}, P_n)$ is the free group $F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{2g-1})$ freely generated by the loops $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_{2g-1}$ represented in Figure 14. Applying Lemma 2.5 to the exact sequence (2.2) of Subsection 2.2, one has that $\mathcal{PM}(F_{g,1}, \mathcal{P}_n)$ is isomorphic with the quotient of the free product $PG(g,0,n-1) * F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{2g-1})$ by the following relations.

- Relations involving the α_i's:

$$\begin{array}{llll}
\text{(PT1)} & x_j \alpha_i x_j^{-1} = \alpha_i & \text{for } 0 \le j < i \le n, \\
\text{(PT2)} & x_j \alpha_i x_j^{-1} = \alpha_{j+1}^{-1} \alpha_i \alpha_{j+1} & \text{for } 1 \le i \le j \le n-1, \\
\text{(PT3)} & y_1 \alpha_i y_1^{-1} = \beta_1^{-1} \alpha_i & \text{for } 1 \le i \le n, \\
\text{(PT4)} & y_j \alpha_i y_j^{-1} = \alpha_i & \text{for } 1 \le i \le n \text{ and } 2 \le j \le 2g-1, \\
\text{(PT5)} & z \alpha_i z^{-1} = \alpha_i & \text{for } 1 \le i \le n.
\end{array}$$

- Relations involving the β_i's:

$$\begin{array}{llll}
\text{(PT6)} & x_j \beta_1 x_j^{-1} = \beta_1 \alpha_{j+1} & \text{for } 0 \le j \le n-1, \\
\text{(PT7)} & x_j \beta_i x_j^{-1} = \beta_i & \text{for } 0 \le j \le n-1 \text{ and } 2 \le i \le 2g-1, \\
\text{(PT8)} & y_j \beta_i y_j^{-1} = \beta_i & \text{for } j \ne i-1 \text{ and } j \ne i+1, \\
\text{(PT9)} & y_{i-1} \beta_i y_{i-1}^{-1} = \beta_i \beta_{i-1} & \text{for } 2 \le i \le 2g-1, \\
\text{(PT10)} & y_{i+1} \beta_i y_{i+1}^{-1} = \beta_{i+1}^{-1} \beta_i & \text{for } 1 \le i \le 2g-2, \\
\text{(PT11)} & z \beta_3 z^{-1} = \beta_3 \beta_2 \beta_1 \alpha_1 \beta_1^{-1}, & \\
\text{(PT12)} & z \beta_i z^{-1} = \beta_i & \text{for } i \ne 3.
\end{array}$$

Consider the homomorphism $f : PG(g,0,n-1) * F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{2g-1}) \to PG(g,0,n)$ defined by:

$$\begin{array}{ll}
f(x_i) = x_i & \text{for } 0 \le i \le n-1, \\
f(y_i) = y_i & \text{for } 1 \le i \le 2g-1, \\
f(z) = z, & \\
f(\alpha_i) = x_{n-1}^{-1} \Delta^{-1}(x_n, x_{i-1}, y_1) x_n^{-1} x_{n-1} \Delta(x_n, x_{i-1}, y_1) x_{n-1} & \text{for } 1 \le i \le n-1, \\
f(\alpha_n) = x_{n-1}^{-1} x_n, & \\
f(\beta_i) = \Delta^{-1}(x_{n-1}, y_1, \dots, y_i) x_{n-1}^{-1} x_n \Delta(x_{n-1}, y_1, \dots, y_i) & \text{for } 1 \le i \le 2g-1.
\end{array}$$

Assertion 1 *f induces a homomorphism $\bar{f} : \mathcal{PM}(F_{g,1}, \mathcal{P}_n) \to PG(g,0,n)$.*

One can easily verify on the generators of $PG(g,0,n)$ that $\bar{f} \circ \bar{\rho}$ is the identity of $PG(g,0,n)$. So, Assertion 1 shows that $\bar{\rho}$ is injective and, therefore, finishes the proof of Proposition 3.3.

Proof of Assertion 1 We have to show that: if $w_1 = w_2$ is one of the relations (PT1),...,(PT12), then $f(w_1) = f(w_2)$.

By an *easy case* we mean a relation $w_1 = w_2$ such that the equality $f(w_1) = f(w_2)$ in $PG(g,0,n)$ is a direct consequence of the braid relations in $A(P\Gamma_{g,0,n})$. For instance, (PT5), (PT6), and (PT8) are easy cases.

• Relation (PT1): (PT1) is an easy case if either $j = i-1$ or $i = n$. So, we assume that $0 \le j < i-1 < n-1$. Then:

$$\begin{aligned}
&f(x_j \alpha_i x_j^{-1}) f(\alpha_i)^{-1} \\
= \ & x_j x_{n-1}^{-1} \Delta^{-1}(x_n, x_{i-1}, y_1) x_n^{-1} x_{n-1} \Delta(x_n, x_{i-1}, y_1) x_{n-1} x_j^{-1} x_{n-1}^{-1} \\
& \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1}^{-1} x_n \Delta(x_n, x_{i-1}, y_1) x_{n-1} \\
= \ & x_{n-1}^{-1} x_{i-1}^{-1} \cdot x_j \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1} \Delta(x_n, x_{i-1}, y_1) x_j^{-1} \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1}^{-1} \\
& \Delta(x_n, x_{i-1}, y_1) \cdot x_{i-1} x_{n-1} \\
= \ & 1 \quad \text{(by (PR3))}.
\end{aligned}$$

• Relation (PT2): (PT2) is an easy case if $j = n-1$. So, we assume that $j < n-1$. Then:

$$\begin{aligned}
&f(x_j \alpha_i x_j^{-1}) f(\alpha_{j+1}^{-1} \alpha_i \alpha_{j+1})^{-1} \\
= \ & x_j x_{n-1}^{-1} \Delta^{-1}(x_n, x_{i-1}, y_1) x_n^{-1} x_{n-1} \Delta(x_n, x_{i-1}, y_1) x_{n-1} x_j^{-1} x_{n-1}^{-1} \Delta^{-1}(x_n, x_j, y_1) \\
& x_{n-1}^{-1} x_n \Delta(x_n, x_j, y_1) x_{n-1} x_{n-1}^{-1} \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1}^{-1} x_n \Delta(x_n, x_{i-1}, y_1) x_{n-1} \\
& x_{n-1}^{-1} \Delta^{-1}(x_n, x_j, y_1) x_n^{-1} x_{n-1} \Delta(x_n, x_j, y_1) x_{n-1} \\
= \ & x_j x_{n-1}^{-1} x_{i-1}^{-1} \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1} \Delta(x_n, x_{i-1}, y_1) \Delta^{-1}(x_n, x_j, y_1) x_{n-1}^{-1} \\
& \Delta(x_n, x_j, y_1) \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1}^{-1} \Delta(x_n, x_{i-1}, y_1) x_{i-1} \Delta^{-1}(x_n, x_j, y_1) x_{n-1} \\
& \Delta(x_n, x_j, y_1) x_{n-1} x_j^{-1} \\
= \ & x_j x_{n-1}^{-1} x_{i-1}^{-1} \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1} \Delta(x_n, x_{i-1}, y_1) \Delta^{-1}(x_n, x_j, y_1) x_{n-1}^{-1} \\
& \Delta(x_n, x_j, y_1) \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1}^{-1} \Delta(x_n, x_{i-1}, y_1) \Delta^{-1}(x_n, x_j, y_1) x_{n-1} \\
& \Delta(x_n, x_j, y_1) x_{i-1} x_{n-1} x_j^{-1} \text{ (by (PR3))} \\
= \ & x_j x_{n-1}^{-1} x_{i-1}^{-1} y_1^{-1} x_n^{-1} x_{i-1}^{-1} y_1^{-1} x_{n-1} y_1 x_n x_{i-1} y_1 y_1^{-1} x_n^{-1} x_j^{-1} y_1^{-1} x_{n-1}^{-1} y_1 x_n x_j \\
& y_1 y_1^{-1} x_n^{-1} x_{i-1}^{-1} y_1^{-1} x_{n-1}^{-1} y_1 x_n x_{i-1} y_1 y_1^{-1} x_n^{-1} x_j^{-1} y_1^{-1} x_{n-1} y_1 x_n x_j y_1 x_{i-1} x_{n-1} x_j^{-1} \\
= \ & x_j x_{n-1}^{-1} x_{i-1}^{-1} y_1^{-1} x_n^{-1} x_{i-1}^{-1} x_{n-1} y_1 x_{n-1}^{-1} x_{i-1} x_j^{-1} x_{n-1} y_1^{-1} x_{n-1}^{-1} x_j x_{i-1}^{-1} x_{n-1} \\
& y_1^{-1} x_{n-1}^{-1} x_{i-1} x_j^{-1} x_{n-1} y_1 x_{n-1}^{-1} x_j x_n y_1 x_{i-1} x_{n-1} x_j^{-1}
\end{aligned}$$

$$\begin{aligned}
&= x_j x_{n-1}^{-1} x_{i-1}^{-1} y_1^{-1} x_n^{-1} x_{n-1} y_1 x_{i-1} y_1^{-1} y_1 x_j^{-1} y_1^{-1} y_1 x_{i-1}^{-1} y_1^{-1} y_1 x_j y_1^{-1} x_{n-1}^{-1} x_n y_1 x_{i-1} \\
&\quad x_{n-1} x_j^{-1} \\
&= 1.
\end{aligned}$$

• Relation (PT3): (PT3) is an easy case if $i = n$. So, we assume that $i < n$. Then:

$$\begin{aligned}
&\quad f(y_1 \alpha_i y_1^{-1}) f(\beta_1^{-1} \alpha_i)^{-1} \\
&= y_1 x_{n-1}^{-1} \Delta^{-1}(x_n, x_{i-1}, y_1) x_n^{-1} x_{n-1} \Delta(x_n, x_{i-1}, y_1) x_{n-1} y_1^{-1} x_{n-1}^{-1} \\
&\quad \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1}^{-1} x_n \Delta(x_n, x_{i-1}, y_1) x_{n-1} \Delta^{-1}(x_{n-1}, y_1) x_{n-1}^{-1} x_n \Delta(x_{n-1}, y_1) \\
&= y_1 x_{n-1}^{-1} y_1^{-1} x_n^{-1} x_{i-1}^{-1} y_1^{-1} x_n^{-1} x_{n-1} y_1 x_n x_{i-1} y_1 x_{n-1} y_1^{-1} x_{n-1}^{-1} y_1^{-1} x_n^{-1} x_{i-1}^{-1} y_1^{-1} x_{n-1}^{-1} \\
&\quad x_n y_1 x_n x_{i-1} y_1 x_{n-1} x_{n-1}^{-1} y_1^{-1} x_{n-1}^{-1} x_n y_1 x_{n-1} \\
&= x_{n-1}^{-1} y_1^{-1} x_{n-1} x_n^{-1} x_{i-1}^{-1} y_1^{-1} x_n^{-1} x_{n-1} y_1 x_n x_{i-1} y_1 x_{n-1} x_{n-1}^{-1} y_1^{-1} x_{n-1}^{-1} x_n^{-1} x_{i-1}^{-1} y_1^{-1} \\
&\quad x_{n-1}^{-1} x_n y_1 x_n x_{i-1} x_{n-1}^{-1} x_n y_1 x_{n-1} \\
&= x_{n-1}^{-1} y_1^{-1} x_{n-1} x_n^{-1} x_{i-1}^{-1} y_1^{-1} x_n^{-1} x_{n-1} y_1 y_1^{-1} x_{n-1}^{-1} y_1^{-1} y_1 x_n y_1 x_{i-1} x_n x_{n-1}^{-1} y_1 x_{n-1} \\
&= 1.
\end{aligned}$$

• Relation (PT4): (PT4) is an easy case if either $i = n$ or $j \ge 3$. So, we assume that $j = 2$ and $i \le n-1$. Then:

$$\begin{aligned}
&\quad y_2 f(\alpha_i) y_2^{-1} \\
&= y_2 x_{n-1}^{-1} \Delta^{-1}(x_n, x_{i-1}, y_1) x_n^{-1} x_{n-1} \Delta(x_n, x_{i-1}, y_1) x_{n-1} y_2^{-1} \\
&= x_{n-1}^{-1} x_{i-1}^{-1} y_2 \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1} \Delta(x_n, x_{i-1}, y_1) y_2^{-1} x_{n-1} \\
&= x_{n-1}^{-1} x_{i-1}^{-1} \Delta^{-1}(x_n, x_{i-1}, y_1) x_{n-1} \Delta(x_n, x_{i-1}, y_1) x_{n-1} \quad \text{(by (PR4))} \\
&= f(\alpha_i).
\end{aligned}$$

• Relation (PT7): (PT7) is an easy case if $j = n-1$. So, we assume that $j \le n-2$. We prove by induction on $i \ge 2$ that x_j and $f(\beta_i)$ commute. Assume first that $i = 2$. (PR4) and Lemma 3.4 imply:

$$x_j \Delta^{-1}(x_{n-1}, y_1, y_2) x_n \Delta(x_{n-1}, y_1, y_2) = \Delta^{-1}(x_{n-1}, y_1, y_2) x_n \Delta(x_{n-1}, y_1, y_2) x_j,$$

and this last equality implies:

$$x_j f(\beta_2) x_j^{-1} = f(\beta_2).$$

Now, we assume that $i > 2$. The first equality of Lemma 3.5 implies:

$$f(\beta_i) = f(\beta_{i-1}) y_i f(\beta_{i-1})^{-1} y_i^{-1}.$$

Thus, since x_j commutes with y_i and with $f(\beta_{i-1})$ (inductive hypothesis), x_j also commutes with $f(\beta_i)$.

• Relation (PT9): The equality

$$y_{i-1} f(\beta_i) y_{i-1}^{-1} = f(\beta_i) f(\beta_{i-1})$$

is a straightforward consequence of the second equality of Lemma 3.5.

- Relation (PT10): The equality

$$y_{i+1} f(\beta_i) y_{i+1}^{-1} = f(\beta_{i+1})^{-1} f(\beta_i)$$

is a straightforward consequence of the first equality of Lemma 3.5.

- Relation (PT11): Assume first that $n = 1$. Then:

$$\begin{aligned}
& f(\alpha_1)^{-1} f(\beta_1)^{-1} f(\beta_2)^{-1} f(\beta_3)^{-1} z f(\beta_3) z^{-1} f(\beta_1) \\
= \ & \Delta^{-2}(x_1, y_1, y_2, y_3, z) \Delta(x_0, x_1, y_1, y_2, y_3, z) \quad \text{(by Lemma 3.6.(i))} \\
= \ & 1 \quad \text{(by (PR5))}.
\end{aligned}$$

Now, assume that $n \geq 2$. Lemma 3.6.(i) implies:

$$\begin{aligned}
& x_n^{-1} x_{n-1} f(\beta_1)^{-1} f(\beta_2)^{-1} f(\beta_3)^{-1} z f(\beta_3) z^{-1} f(\beta_1) \\
= \ & \Delta^{-2}(x_n, y_1, y_2, y_3, z) \Delta(x_{n-1}, x_n, y_1, y_2, y_3, z),
\end{aligned}$$

and Lemma 3.6.(ii) implies:

$$x_n^{-1} x_{n-1} f(\alpha_1) = \Delta^{-2}(x_0, x_n, y_1) \Delta(x_0, x_{n-1}, x_n, y_1).$$

Thus:

$$\begin{aligned}
& f(\alpha_1)^{-1} f(\beta_1)^{-1} f(\beta_2)^{-1} f(\beta_3)^{-1} z f(\beta_3) z^{-1} f(\beta_1) \\
= \ & \Delta^{-1}(x_0, x_{n-1}, x_n, y_1) \Delta^2(x_0, x_n, y_1) \Delta^{-2}(x_n, y_1, y_2, y_3, z) \Delta(x_{n-1}, x_n, y_1, y_2, y_3, z) \\
= \ & 1 \quad \text{(by (PR6))}.
\end{aligned}$$

- Relation (PT12): (PT12) is an easy case if $i = 1, 2$. We prove by induction on $i \geq 4$ that z and $f(\beta_i)$ commute. Recall first that the first equality of Lemma 3.5 implies:

$$f(\beta_i) = f(\beta_{i-1}) y_i f(\beta_{i-1})^{-1} y_i^{-1}.$$

Assume that $i = 4$. Then:

$$\begin{aligned}
& z f(\beta_4) z^{-1} \\
= \ & z f(\beta_3) y_4 f(\beta_3)^{-1} y_4^{-1} z^{-1} \\
= \ & f(\beta_3) f(\beta_2) f(\beta_1) f(\alpha_1) f(\beta_1)^{-1} y_4 f(\beta_1) f(\alpha_1)^{-1} f(\beta_1)^{-1} f(\beta_2)^{-1} f(\beta_3)^{-1} y_4^{-1} \\
& \hfill \text{by (PT11)} \\
= \ & f(\beta_3) y_4 f(\beta_3)^{-1} y_4^{-1} \quad \text{(by (PT4) and (PT8))} \\
= \ & f(\beta_4).
\end{aligned}$$

Now, we assume that $i > 4$. Then z commutes with $f(\beta_i)$, since it commutes with y_i and with $f(\beta_{i-1})$ (inductive hypothesis). □

Now, in view of Proposition 3.3, and applying Lemma 2.5 to the exact sequences (2.3) of Subsection 2.2, one has immediately the following presentation for $\mathcal{PM}(F_{g,r+1},\mathcal{P}_n)$.

Proposition 3.7 *Let $g,r\ge 1$, let $n\ge 0$, and let $P\Gamma_{g,r,n}$ be the Coxeter graph drawn in Figure 21. Then $\mathcal{PM}(F_{g,r+1},\mathcal{P}_n)$ is isomorphic with the quotient of $A(P\Gamma_{g,r,n})$ by the following relations.*

- *Relations from $\mathcal{M}(F_{g,1})$:*

$$\begin{array}{llcll} \text{(PR1)} & \Delta^4(y_1,y_2,y_3,z) & = & \Delta^2(x_0,y_1,y_2,y_3,z) & \text{if } g\ge 2,\\ \text{(PR2)} & \Delta^2(y_1,y_2,y_3,y_4,y_5,z) & = & \Delta(x_0,y_1,y_2,y_3,y_4,y_5,z) & \text{if } g\ge 3. \end{array}$$

- *Relations of commutation:*

$$\begin{array}{ll} \text{(PR3)} & x_k\Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)\\ & = \Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)x_k \ \text{ if } 0\le k<j<i\le r+n-1,\\ \text{(PR4)} & y_2\Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)\\ & = \Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)y_2 \ \text{ if } 0\le j<i\le r+n-1, \end{array}$$

- *Relations between fundamental elements:*

$$\begin{array}{lll} \text{(PR5a)} & u_1 & = \Delta(x_0,x_1,y_1,y_2,y_3,z)\Delta^{-2}(x_1,y_1,y_2,y_3,z),\\ \text{(PR6a)} & u_{i+1} & = \Delta(x_i,x_{i+1},y_1,y_2,y_3,z)\Delta^{-2}(x_{i+1},y_1,y_2,y_3,z)\\ & & \quad \Delta^2(x_0,x_{i+1},y_1)\Delta^{-1}(x_0,x_i,x_{i+1},y_1) \quad \text{if} \quad 1\le i\le r-1,\\ \text{(PR6b)} & \multicolumn{2}{l}{\Delta(x_i,x_{i+1},y_1,y_2,y_3,z)\Delta^{-2}(x_{i+1},y_1,y_2,y_3,z)}\\ & & = \Delta(x_0,x_i,x_{i+1},y_1)\Delta^{-2}(x_0,x_{i+1},y_1) \quad \text{if} \quad r\le i\le n+r-1. \end{array}$$

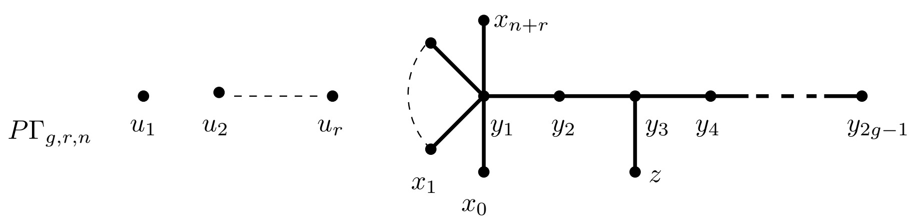

Figure 21: Coxeter graph associated with $\mathcal{PM}(F_{g,r+1},\mathcal{P}_n)$

Let $PG(g,r,n)$ denote the quotient of $A(P\Gamma_{g,r,n})$ by the relations (PR1),(PR2), (PR3),(PR4),(PR5a), (PR6a), (PR6b). Consider the Dehn twists $a_0,\dots,a_{n+r}$, $b_1,\dots,b_{2g-1}$, c, $d_1,\dots,d_r$ represented in Figure 12. Then an isomorphism $\bar\rho : PG(g,r,n)\to\mathcal{PM}(F_{g,r+1},\mathcal{P}_n)$ between $PG(g,r,n)$ and $\mathcal{PM}(F_{g,r+1},\mathcal{P}_n)$

is given by $\bar{\rho}(x_i) = a_i$ for $i = 0, \dots, n+r$, $\bar{\rho}(y_i) = b_i$ for $i = 1, \dots, 2g-1$, $\bar{\rho}(z) = c$, and $\bar{\rho}(u_i) = d_i$ for $i = 1, \dots, r$.

As in Lemma 3.6, we use the algorithm of [7] to prove the following.

Lemma 3.8 (i) *We number the vertices of the Coxeter graph D_6 according to Figure 6. Then the following equality holds in $A(D_6)$.*

$$\Delta^2(x_1, x_3, \dots, x_6)\Delta^{-1}(x_1, x_2, x_3, \dots, x_6) = x_6x_5x_4x_3x_1x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}x_6^{-1}x_5x_4 \\ x_3x_2x_1^{-1}x_3^{-1}x_4^{-1}x_5^{-1}x_4x_3x_1x_2^{-1}x_3^{-1}x_4^{-1}x_2x_3x_2x_1^{-1}x_3^{-1}x_2^{-1}.$$

(ii) *We number the vertices of the Coxeter graph D_4 according to Figure 6. Then the following equality holds in $A(D_4)$.*

$$\Delta(x_1, x_2, x_3, x_4)\Delta^{-2}(x_1, x_3, x_4) = x_2x_3x_2^{-1}x_1x_3^{-1}x_2^{-1}x_4x_3x_2x_1^{-1}x_3^{-1}x_4^{-1}. \qquad \square$$

Proof of Theorem 3.1 Recall that $\Gamma_{g,r,n}$ denotes the Coxeter graph drawn in Figure 18, and that $G(g, r, n)$ denotes the quotient of $A(\Gamma_{g,r,n})$ by the relations (R1),…,(R7), (R8a), (R8b). Recall also that there is a well defined epimorphism $\bar{\rho} : G(g, r, n) \to \mathcal{M}(F_{g,r+1}, \mathcal{P}_n)$ which sends x_i on a_i for $i = 0, \dots, r+1$, y_i on b_i for $i = 1, \dots, 2g-1$, z on c, u_i on d_i for $i = 1, \dots, r$, and v_i on τ_i for $i = 1, \dots, n-1$. Our aim now is to construct a homomorphism $\bar{f} : \mathcal{M}(F_{g,r+1}, \mathcal{P}_n) \to G(g, r, n)$ such that $\bar{f} \circ \bar{\rho}$ is the identity of $G(g, r, n)$. The existence of such a homomorphism clearly proves that $\bar{\rho}$ is an isomorphism.

We set $A_0 = x_r$, $A_1 = x_{r+1}$, and

$$A_i = x_r^{1-i}\Delta(x_{r+1}, v_1, \dots, v_{i-1}) \quad \text{for } i = 2, \dots, n.$$

These expressions are viewed as elements of $G(g, r, n)$. Note that, by Proposition 2.12, we have $\bar{\rho}(A_i) = a_{r+i}$ for all $i = 0, 1, \dots, n$.

Assertion 1 (i) *The following relations hold in $G(g, r, n)$:*

$$\begin{array}{llll}
\text{(T1)} & A_{i-1}A_{i+1} & = v_iA_iv_iA_i & \\
 & & = A_iv_iA_iv_i & \text{for } 1 \le i \le n-1, \\
\text{(T2)} & A_iA_j & = A_jA_i & \text{for } 0 \le i < j \le n, \\
\text{(T3)} & A_iv_j & = v_jA_i & \text{for } i \ne j, \\
\text{(T4)} & y_1A_iy_1 & = A_iy_1A_i & \text{for } 0 \le i \le n.
\end{array}$$

(ii) *The relations (T1),…,(T4) imply that there is a well defined homomorphism $h_i : A(B_4) \to G(g, r, n)$ which sends x_1 on v_i, x_2 on A_i, x_3 on y_1, and x_4 on A_{i-1}. Then the following relation holds in $G(g, r, n)$:*

$$\text{(T5)} \quad h_i(\Delta(x_1, x_2, x_3, x_4)) = h_i(\Delta^2(x_1, x_2, x_3)) \quad \text{for } 1 \le i \le n.$$

Proof of Assertion 1 • Relation (T1):

$$\begin{aligned}
A_{i+1} &= x_r^{-i}\Delta(x_{r+1}, v_1, \dots, v_i) \\
&= x_r^{-i} v_i v_{i-1} \dots v_1 x_{r+1} v_1 \dots v_{i-1} v_i \Delta(x_{r+1}, v_1, \dots, v_{i-1}) \quad \text{(by 2.9)} \\
&= x_r^{-i} v_i \Delta(x_{r+1}, v_1, \dots, v_{i-1}) \Delta^{-1}(x_{r+1}, v_1, \dots, v_{i-2}) v_i \\
&\quad \Delta(x_{r+1}, v_1, \dots, v_{i-1}) \\
&= x_r^{i-2}\Delta^{-1}(x_{r+1}, v_1, \dots, v_{i-2}) v_i x_r^{1-i} \Delta(x_{r+1}, v_1, \dots, v_{i-1}) v_i x_r^{1-i} \\
&\quad \Delta(x_{r+1}, v_1, \dots, v_{i-1}) \\
&= A_{i-1}^{-1} v_i A_i v_i A_i.
\end{aligned}$$

Similarly:

$$A_{i+1} = A_{i-1}^{-1} A_i v_i A_i v_i.$$

• The relations (T2) and (T3) are direct consequences of the "braid" relations in $A(\Gamma_{g,r,n})$.

• Now, we prove (T4) and (T5) by induction on i. First, assume $i=1$. Then (T4) follows from the "braid" relation $y_1 x_{r+1} y_1 = x_{r+1} y_1 x_{r+1}$ in $A(\Gamma_{g,r,n})$, and (T5) follows from the relation (R7) in the definition of $G(g,r,n)$.

Now, assume $i>1$. Then the relation (T4) follows from the following sequence of equalities.

$$\begin{aligned}
&\quad A_i y_1 A_i y_1^{-1} A_i^{-1} y_1^{-1} \\
&= A_{i-2}^{-1} v_{i-1} A_{i-1} v_{i-1} A_{i-1} y_1 A_{i-1} v_{i-1} A_{i-1} v_{i-1} A_{i-2}^{-1} y_1^{-1} A_{i-2} v_{i-1}^{-1} A_{i-1}^{-1} v_{i-1}^{-1} A_{i-1}^{-1} y_1^{-1} \\
&\qquad \text{(by (T1))} \\
&= A_{i-2}^{-1} \cdot v_{i-1} A_{i-1} v_{i-1} A_{i-1} y_1 A_{i-1} v_{i-1} A_{i-1} y_1 A_{i-2}^{-1} y_1^{-1} A_{i-1}^{-1} v_{i-1}^{-1} A_{i-1}^{-1} y_1^{-1} A_{i-2}^{-1} \\
&\quad \cdot A_{i-2} \quad \text{(by (T2), (T3), induction)} \\
&= A_{i-2}^{-1} \cdot h_{i-1}(\Delta^2(x_1, x_2, x_3)\Delta^{-1}(x_1, x_2, x_3, x_4)) \cdot A_{i-2} \quad \text{(by Proposition 2.9)} \\
&= 1 \quad \text{(by induction)}.
\end{aligned}$$

The Relation (T5) follows from the following sequence of equalities.

$$\begin{aligned}
&\quad h_i(\Delta^{-1}(x_1, x_2, x_3, x_4)\Delta^2(x_1, x_2, x_3)) \\
&= A_{i-1}^{-1} y_1^{-1} A_i^{-1} v_i^{-1} A_i^{-1} y_1^{-1} A_{i-1}^{-1} y_1 A_i v_i y_1 A_i v_i y_1 A_i v_i \quad \text{(by Propositions 2.8 , 2.9)} \\
&= A_{i-1}^{-1} y_1^{-1} A_{i-2} v_{i-1}^{-1} A_{i-1}^{-1} v_{i-1}^{-1} A_{i-1}^{-1} v_i^{-1} v_{i-1}^{-1} A_{i-1}^{-1} v_{i-1}^{-1} A_{i-1}^{-1} A_{i-2} y_1^{-1} A_{i-1}^{-1} y_1 A_{i-2}^{-1} A_{i-1} \\
&\quad v_{i-1} A_{i-1} v_{i-1} v_i y_1 A_{i-2}^{-1} A_{i-1} v_{i-1} A_{i-1} v_{i-1} v_i y_1 v_{i-1} A_{i-1} v_{i-1} A_{i-1} A_{i-2}^{-1} v_i \quad (\text{ T1 }) \\
&= A_{i-2} \cdot A_{i-1}^{-1} A_{i-2}^{-1} y_1^{-1} A_{i-2} v_{i-1}^{-1} A_{i-1}^{-1} v_{i-1}^{-1} A_{i-1}^{-1} v_i^{-1} v_{i-1}^{-1} A_{i-1}^{-1} v_{i-1}^{-1} A_{i-1}^{-1} A_{i-2} A_{i-1} y_1^{-1} \\
&\quad A_{i-1}^{-1} A_{i-2}^{-1} A_{i-1} v_{i-1} A_{i-1} v_{i-1} v_i y_1 A_{i-2}^{-1} A_{i-1} v_{i-1} A_{i-1} v_{i-1} v_i y_1 v_{i-1} A_{i-1} v_{i-1} A_{i-1} v_i \\
&\quad \cdot A_{i-2}^{-1} \text{(by (T2), (T3), induction)} \\
&= A_{i-2} A_{i-1}^{-1} v_{i-1}^{-1} \cdot y_1 A_{i-2}^{-1} y_1^{-1} A_{i-1}^{-1} v_{i-1}^{-1} v_i^{-1} A_{i-1}^{-1} v_{i-1}^{-1} A_{i-1}^{-1} y_1^{-1} A_{i-2}^{-1} y_1 A_{i-1} v_{i-1} v_i y_1 \\
&\quad A_{i-2}^{-1} A_{i-1} v_{i-1} A_{i-1} v_{i-1} v_i y_1 v_{i-1} A_{i-1} v_{i-1} v_i v_{i-1}^{-1} \cdot v_{i-1} A_{i-1} A_{i-2}^{-1}
\end{aligned}$$

$$\begin{aligned}
& \text{(by (T2), (T3), induction)} \\
= \;& A_{i-2}A_{i-1}^{-1}v_{i-1}^{-1}y_1 \cdot A_{i-2}^{-1}y_1^{-1}A_{i-1}^{-1}v_{i-1}^{-1}v_i^{-1}y_1A_{i-2} \cdot h_{i-1}(\Delta^{-1}(x_1,x_2,x_3,x_4) \\
& \Delta(x_1,x_2,x_3)) \cdot y_1A_{i-1}v_{i-1}v_iy_1A_{i-2}^{-1}A_{i-1}v_{i-1}A_{i-1}v_{i-1}v_iv_{i-1}y_1A_{i-1}v_i^{-1}v_{i-1}v_iy_1 \\
& \cdot y_1^{-1}v_{i-1}A_{i-1}A_{i-2}^{-1} \text{(by Proposition 2.9)} \\
= \;& A_{i-2}A_{i-1}^{-1}v_{i-1}^{-1}y_1 \cdot A_{i-2}^{-1}y_1^{-1}A_{i-1}^{-1}v_{i-1}^{-1}v_i^{-1}y_1A_{i-2}v_{i-1}^{-1}A_{i-1}^{-1}y_1^{-1}v_{i-1}^{-1}A_{i-1}^{-1}y_1^{-1}v_{i-1}^{-1} \\
& A_{i-1}^{-1}y_1^{-1}y_1A_{i-1}v_{i-1}v_iy_1A_{i-2}^{-1}A_{i-1}v_{i-1}A_{i-1}v_{i-1}v_iv_{i-1}y_1A_{i-1}v_i^{-1}v_{i-1}v_iy_1 \\
& y_1^{-1}v_{i-1}A_{i-1}A_{i-2}^{-1} \quad \text{(by induction)} \\
= \;& A_{i-2}A_{i-1}^{-1}v_{i-1}^{-1}y_1 \cdot A_{i-2}^{-1}y_1^{-1}A_{i-1}^{-1}y_1v_{i-1}^{-1}v_i^{-1}v_{i-1}^{-1}A_{i-1}^{-1}A_{i-2}y_1^{-1}A_{i-2}^{-1}v_{i-1}^{-1}v_iv_{i-1} \\
& A_{i-1}v_iv_{i-1}v_iy_1A_{i-1}v_i^{-1}y_1v_{i-1}v_i \cdot y_1^{-1}v_{i-1}A_{i-1}A_{i-2}^{-1}(\text{ (T2), (T3), induction)} \\
= \;& A_{i-2}A_{i-1}^{-1}v_{i-1}^{-1}y_1 \cdot A_{i-2}^{-1}A_{i-1}y_1^{-1}A_{i-1}^{-1}v_{i-1}^{-1}v_i^{-1}v_{i-1}^{-1}A_{i-1}^{-1}y_1^{-1}A_{i-2}^{-1}y_1v_iv_{i-1}v_i^{-1}A_{i-1} \\
& v_iv_{i-1}y_1A_{i-1}y_1v_{i-1}v_i \cdot y_1^{-1}v_{i-1}A_{i-1}A_{i-2}^{-1} \quad \text{(by (T2), (T3), induction)} \\
= \;& A_{i-2}A_{i-1}^{-1}v_{i-1}^{-1}y_1A_{i-1} \cdot A_{i-2}^{-1}y_1^{-1}A_{i-1}^{-1}v_{i-1}^{-1}v_i^{-1}v_{i-1}^{-1}A_{i-1}^{-1}y_1^{-1}A_{i-2}^{-1}v_i \cdot y_1v_{i-1}A_{i-1}y_1 \\
& v_{i-1}A_{i-1}y_1v_{i-1}A_{i-1} \cdot v_i \cdot A_{i-1}^{-1}y_1^{-1}v_{i-1}A_{i-1}A_{i-2}^{-1} \quad \text{((T2), (T3), induction)} \\
= \;& A_{i-2}A_{i-1}^{-1}v_{i-1}^{-1}y_1A_{i-1} \cdot A_{i-2}^{-1}y_1^{-1}A_{i-1}^{-1}v_{i-1}^{-1}v_i^{-1}v_{i-1}^{-1}A_{i-1}^{-1}y_1^{-1}A_{i-2}^{-1}v_i \\
& \cdot h_{i-1}(\Delta(x_1,x_2,x_3)) \cdot v_i \cdot A_{i-1}^{-1}y_1^{-1}v_{i-1}A_{i-1}A_{i-2}^{-1} \quad \text{(by Proposition 2.8)} \\
= \;& A_{i-2}A_{i-1}^{-1}v_{i-1}^{-1}y_1A_{i-1} \cdot A_{i-2}^{-1}y_1^{-1}A_{i-1}^{-1}v_{i-1}^{-1}v_i^{-1}v_{i-1}^{-1}A_{i-1}^{-1}y_1^{-1}A_{i-2}^{-1}v_iA_{i-2}y_1A_{i-1} \\
& v_{i-1}A_{i-1}y_1A_{i-2}v_i \cdot A_{i-1}^{-1}y_1^{-1}v_{i-1}A_{i-1}A_{i-2}^{-1} \quad \text{(by induction)} \\
= \;& A_{i-2}A_{i-1}^{-1}v_{i-1}^{-1}y_1A_{i-1} \cdot A_{i-2}^{-1}y_1^{-1}A_{i-1}^{-1}v_{i-1}^{-1}v_i^{-1}v_iv_{i-1}v_i^{-1}A_{i-1}y_1A_{i-2}v_i \cdot A_{i-1}^{-1}y_1^{-1} \\
& v_{i-1}A_{i-1}A_{i-2}^{-1} \text{(by (T2), (T3), induction)} \\
= \;& 1 \quad \text{(by (T2), (T3), induction)}
\end{aligned}$$

Assertion 2 *Recall that $P\Gamma_{g,r,n}$ denotes the Coxeter graph drawn in Figure 21. There is a well defined homomorphism $g : A(P\Gamma_{g,r,n}) \to G(g,r,n)$ which sends x_i on x_i for $i = 0, \dots, r+1$, x_{r+i} on A_i for $i = 2, \dots, n$, y_i on y_i for $i = 1, \dots, 2g-1$, z on z, and u_i on u_i for $i = 1, \dots, r$.*

Proof of Assertion 2 We have to verify that the following relations hold in $G(g,r,n)$.

$$\begin{array}{lrll}
\text{(T6)} & A_iA_j = & A_jA_i & \text{for } 1 \le i \le j \le n, \\
\text{(T7)} & x_iA_j = & A_jx_i & \text{for } 0 \le i \le r \text{ and } 1 \le j \le n, \\
\text{(T8)} & y_1A_iy_1 = & A_iy_1A_i & \text{for } 1 \le i \le n, \\
\text{(T9)} & A_iy_j = & y_jA_i & \text{for } 1 \le i \le n \text{ and } 2 \le j \le 2g-1, \\
\text{(T10)} & A_iz = & zA_i & \text{for } 1 \le i \le n, \\
\text{(T11)} & A_iu_j = & u_jA_i & \text{for } 1 \le i \le n \text{ and } 1 \le j \le r.
\end{array}$$

The relations (T6) and (T8) hold by Assertion 1, and the other relations are direct consequences of the "braid" relations in $A(\Gamma_{g,r,n})$.

Recall that $PG(g,r,n)$ denotes the quotient of $A(P\Gamma_{g,r,n})$ by the relations (PR1),...,(PR4), (PR5a), (PR6a), (PR6b), and that this quotient is isomorphic with $\mathcal{PM}(F_{g,r+1},\mathcal{P}_n)$ (see Proposition 3.7).

Assertion 3 *The homomorphism $g : A(P\Gamma_{g,r,n}) \to G(g,r,n)$ induces a homomorphism $\bar{g} : PG(g,r,n) \to G(g,r,n)$.*

Proof of Assertion 3 It suffices to show that the following relations hold in $G(g,r,n)$.

$$\begin{aligned}
\text{(T12)}\quad & g(x_k\Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)) \\
&= g(\Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)x_k) \text{ for } 0\le k<j<i\le r+n-1, \\
\text{(T13)}\quad & g(y_2\Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)) \\
&= g(\Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)y_2) \text{ for } 0\le j<i\le r+n-1, \\
\text{(T14)}\quad & g(\Delta(x_i,x_{i+1},y_1,y_2,y_3,z)\Delta^{-2}(x_{i+1},y_1,y_2,y_3,z)) \\
&= g(\Delta(x_0,x_i,x_{i+1},y_1)\Delta^{-2}(x_0,x_{i+1},y_1)) \text{ for } \ r+1\le i\le r+n-1.
\end{aligned}$$

- Relation (T12): for $i\ge r+1$ and $j<i-1$, we have:

$$\begin{aligned}
\text{(E1)}\quad & g(\Delta^{-1}(x_{i+1},x_j,y_1)x_i\Delta(x_{i+1},x_j,y_1)) \\
&= y_1^{-1}g(x_j)^{-1}A_{i-r+1}^{-1}y_1^{-1}A_{i-r}y_1A_{i-r+1}g(x_j)y_1 \\
&= y_1^{-1}g(x_j)^{-1}A_{i-r-1}v_{i-r}^{-1}A_{i-r}^{-1}v_{i-r}^{-1}A_{i-r}^{-1}y_1^{-1}A_{i-r}y_1A_{i-r}v_{i-r}A_{i-r}v_{i-r} \\
&\quad A_{i-r-1}^{-1}g(x_j)y_1 \quad \text{(by (T1))} \\
&= v_{i-r}^{-1}y_1^{-1}g(x_j)^{-1}A_{i-r}^{-1}A_{i-r-1}v_{i-r}^{-1}A_{i-r}^{-1}A_{i-r}y_1A_{i-r}^{-1}A_{i-r}v_{i-r}A_{i-r-1}^{-1}A_{i-r} \\
&\quad g(x_j)y_1v_{i-r} \quad \text{(by (T2), (T3), (T4))} \\
&= v_{i-r}^{-1}y_1^{-1}g(x_j)^{-1}A_{i-r}^{-1}y_1^{-1}A_{i-r-1}y_1A_{i-r}g(x_j)y_1v_{i-r} \quad \text{(by (T2), (T3), (T4))} \\
&= v_{i-r}^{-1}g(\Delta^{-1}(x_i,x_j,y_1)x_{i-1}\Delta(x_i,x_j,y_1))v_{i-r}.
\end{aligned}$$

For $i\ge r+1$ and $j=i-1$ we have:

$$\begin{aligned}
\text{(E2)}\quad & g(\Delta^{-1}(x_{i+1},x_{i-1},y_1)x_i\Delta(x_{i+1},x_{i-1},y_1)) \\
&= y_1^{-1}A_{i-r-1}^{-1}A_{i-r+1}^{-1}y_1^{-1}A_{i-r}y_1A_{i-r+1}A_{i-r-1}y_1 \\
&= y_1^{-1}A_{i-r-1}^{-1}A_{i-r-1}v_{i-r}^{-1}A_{i-r}^{-1}v_{i-r}^{-1}A_{i-r}^{-1}y_1^{-1}A_{i-r}y_1A_{i-r}v_{i-r}A_{i-r}v_{i-r} \\
&\quad A_{i-r-1}^{-1}A_{i-r-1}y_1 \quad \text{(by (T1))} \\
&= v_{i-r}^{-1}y_1^{-1}A_{i-r}^{-1}v_{i-r}^{-1}A_{i-r}^{-1}A_{i-r}y_1A_{i-r}^{-1}A_{i-r}v_{i-r}A_{i-r}y_1v_{i-r} \\
&\quad \text{(by (T2), (T3), (T4))} \\
&= v_{i-r}^{-1}y_1^{-1}y_1A_{i-r}y_1^{-1}y_1v_{i-r} \quad \text{(by (T2), (T3), (T4))} \\
&= v_{i-r}^{-1}A_{i-r}v_{i-r}.
\end{aligned}$$

First, assume that $i\le r$. Then the relation (T12) follows from the relation (R3) in the definition of $G(g,r,n)$. Now, we assume that $j<r\le i\le r+n-1$,

and we prove by induction on i that the relation (T12) holds. The case $i = r$ follows from the relation (R3) in the definition of $G(g, r, n)$, and the case $i > r$ follows from the inductive hypothesis and from the equality (E1) above. Now, we assume that $r \le j < i \le r + n - 1$, and we prove, again by induction on i, that the relation (T12) holds. The case $i = j + 1$ follows from the equality (E2) above, and the case $i > j + 1$ follows from the inductive hypothesis and from the equality (E1).

- The relation (T13) can be shown in the same manner as the relation (T12).

- Relation (T14): We prove by induction on $i \ge \sup\{r, 1\}$ that the relation (T14) holds in $G(g, r, n)$. If $i = r \ge 1$, then the relation (T14) follows from the relation (R8b) in the definition of $G(g, r, n)$. Assume $r = 0$ and $i = 1$. Then:

$$\begin{aligned}
& g(\Delta^2(x_2, y_1, y_2, y_3, z)\Delta^{-1}(x_1, x_2, y_1, y_2, y_3, z)\Delta(x_0, x_1, x_2, y_1)\Delta^{-2}(x_0, x_2, y_1)) \\
= \ & z y_3 y_2 y_1 A_2 A_1^{-1} y_1^{-1} y_2^{-1} y_3^{-1} z^{-1} y_3 y_2 y_1 A_1 A_2^{-1} y_1^{-1} y_2^{-1} y_3^{-1} y_2 y_1 A_2 A_1^{-1} y_1^{-1} y_2^{-1} A_1 y_1 \\
& A_1 A_2^{-1} y_1^{-1} A_1^{-1} \cdot A_1 y_1 A_1^{-1} A_2 y_1^{-1} A_1^{-1} A_0 y_1 A_1 A_2^{-1} y_1^{-1} A_0^{-1} \quad \text{(by Lemma 3.8)} \\
= \ & z y_3 y_2 y_1 v_1 A_1 v_1 A_1 A_0^{-1} A_1^{-1} y_1^{-1} y_2^{-1} y_3^{-1} z^{-1} y_3 y_2 y_1 A_1 A_0 A_1^{-1} v_1^{-1} A_1^{-1} v_1^{-1} y_1^{-1} y_2^{-1} \\
& y_3^{-1} y_2 y_1 v_1 A_1 v_1 A_1 A_0^{-1} A_1^{-1} y_1^{-1} y_2^{-1} A_0 y_1 A_1 A_0 A_1^{-1} v_1^{-1} A_1^{-1} v_1^{-1} y_1^{-1} A_0^{-1} \quad \text{(T1)} \\
= \ & v_1 \cdot z y_3 y_2 y_1 A_1 A_0^{-1} y_1^{-1} y_2^{-1} y_3^{-1} z^{-1} y_3 y_2 y_1 A_0 A_1^{-1} y_1^{-1} y_2^{-1} y_3^{-1} y_2 y_1 A_1 A_0^{-1} y_1^{-1} y_2^{-1} \\
& A_0 y_1 A_0 A_1^{-1} y_1^{-1} A_0^{-1} \cdot v_1^{-1} \quad \text{(by (T2), (T3), (T4))} \\
= \ & v_1 \cdot \Delta^2(x_1, y_1, y_2, y_3, z)\Delta^{-1}(x_0, x_1, y_1, y_2, y_3, z) \cdot v_1^{-1} \quad \text{(by Lemma 3.8)} \\
= \ & 1 \quad \text{(by (R8a))}.
\end{aligned}$$

Now, we assume that $i > \sup\{r, 1\}$. Then:

$$\begin{aligned}
& g(\Delta^2(x_{i+1}, y_1, y_2, y_3, z)\Delta^{-1}(x_i, x_{i+1}, y_1, y_2, y_3, z)\Delta(x_0, x_i, x_{i+1}, y_1) \\
& \Delta^{-2}(x_0, x_{i+1}, y_1)) \\
= \ & z y_3 y_2 y_1 A_{i-r+1} A_{i-r}^{-1} y_1^{-1} y_2^{-1} y_3^{-1} z^{-1} y_3 y_2 y_1 A_{i-r} A_{i-r+1}^{-1} y_1^{-1} y_2^{-1} y_3^{-1} y_2 y_1 A_{i-r+1} \\
& A_{i-r}^{-1} y_1^{-1} y_2^{-1} A_{i-r} y_1 A_{i-r} A_{i-r+1}^{-1} y_1^{-1} A_{i-r}^{-1} \cdot A_{i-r} y_1 A_{i-r}^{-1} A_{i-r+1} y_1^{-1} A_{i-r}^{-1} x_0 y_1 \\
& A_{i-r} A_{i-r+1}^{-1} y_1^{-1} x_0^{-1} \quad \text{(by Lemma 3.8)} \\
= \ & z y_3 y_2 y_1 v_{i-r} A_{i-r} v_{i-r} A_{i-r} A_{i-r-1}^{-1} A_{i-r}^{-1} y_1^{-1} y_2^{-1} y_3^{-1} z^{-1} y_3 y_2 y_1 A_{i-r} A_{i-r-1} A_{i-r}^{-1} \\
& v_{i-r}^{-1} A_{i-r}^{-1} v_{i-r}^{-1} y_1^{-1} y_2^{-1} y_3^{-1} y_2 y_1 v_{i-r} A_{i-r} v_{i-r} A_{i-r} A_{i-r-1}^{-1} A_{i-r}^{-1} y_1^{-1} y_2^{-1} x_0 y_1 A_{i-r} \\
& A_{i-r-1} A_{i-r}^{-1} v_{i-r}^{-1} A_{i-r}^{-1} v_{i-r}^{-1} y_1^{-1} x_0^{-1} \text{(by (T1))} \\
= \ & v_{i-r} \cdot z y_3 y_2 y_1 A_{i-r} A_{i-r-1}^{-1} y_1^{-1} y_2^{-1} y_3^{-1} z^{-1} y_3 y_2 y_1 A_{i-r-1} A_{i-r}^{-1} y_1^{-1} y_2^{-1} y_3^{-1} y_2 y_1 \\
& A_{i-r} A_{i-r-1}^{-1} y_1^{-1} y_2^{-1} x_0 y_1 A_{i-r-1} A_{i-r}^{-1} y_1^{-1} x_0^{-1} \cdot v_{i-r}^{-1} \quad \text{(by (T2), (T3), (T4))} \\
= \ & v_{i-r} \cdot g(\Delta^2(x_i, y_1, y_2, y_3, z)\Delta^{-1}(x_{i-1}, x_i, y_1, y_2, y_3, z)\Delta(x_0, x_{i-1}, x_i, y_1) \\
& \Delta^{-2}(x_0, x_i, y_1)) \cdot v_{i-r}^{-1} \quad \text{(by Lemma 3.8)} \\
= \ & 1 \quad \text{(by induction)}.
\end{aligned}$$

Let $V_1, \dots, V_{n-1}$ denote the natural generators of the Artin group $A(A_{n-1})$, numbered according to Figure 6. Applying Lemma 2.5 to the exact sequence

(2.1) of Subsection 2.2, one has that $\mathcal{M}(F_{g,r+1},\mathcal{P}_n)$ is isomorphic with the quotient of the free product $PG(g,r,n) * A(A_{n-1})$ by the following relations.

- Relations from Σ_n:

(T15) $V_i^2 = \Delta^2(x_{r+i-1},x_{r+i+1},y_1)\Delta^{-1}(x_{r+i-1},x_{r+i},x_{r+i+1},y_1)$ for $1 \le i \le n-1$.

- Relations from conjugation by the V_i's:

(T16) $V_i w V_i^{-1} = w$ for $1 \le i \le n-1$ and $w \in \{x_0,\dots,x_{r+i-1},x_{r+i+1},\dots,x_{r+n},y_1,\dots,y_{2g-1},z,u_1,\dots,u_r\}$,

(T17) $V_i x_{r+i} V_i^{-1} = y_1 x_{r+i-1} x_{r+i}^{-1} y_1^{-1} x_{r+i+1} y_1 x_{r+i} x_{r+i-1}^{-1} y_1^{-1}$ for $1 \le i \le n-1$.

We can easily prove using Proposition 2.12 that the relation (T15) "holds" in $\mathcal{M}(F_{g,r+1},\mathcal{P}_n)$. The relation (T16) is obvious, while the relation (T17) has to be verified by hand.

Now, the homomorphism $\bar g : PG(g,r,n) \to G(g,r,n)$ extends to a homomorphism $f : PG(g,r,n) * A(A_{n-1}) \to G(g,r,n)$ which sends V_i on v_i for all $i = 1,\dots,n-1$.

Assertion 4 *The homomorphism $f : PG(g,r,n) * A(A_{n-1}) \to G(g,r,n)$ induces a homomorphism $\bar f : \mathcal{M}(F_{g,r+1},\mathcal{P}_n) \to G(g,r,n)$.*

One can easily verify on the generators of $G(g,r,n)$ that $\bar f \circ \bar\rho$ is the identity of $G(g,r,n)$. So, Assertion 4 finishes the construction of $\bar f$ and the proof of Theorem 3.1.

Proof of Assertion 4 We have to show that: if $w_1 = w_2$ is one of the relations (T15), (T16), (T17), then $f(w_1) = f(w_2)$.

- Relation (T15):

$$\begin{aligned}
& f(\Delta^{-1}(x_{r+i-1},x_{r+i},x_{r+i+1},y_1)\Delta^2(x_{r+i-1},x_{r+i+1},y_1)) \cdot v_i^{-2} \\
= {} & A_i^{-1}y_1^{-1}A_{i-1}^{-1}A_{i+1}^{-1}y_1^{-1}A_i^{-1}y_1A_{i-1}A_{i+1}y_1A_{i-1}A_{i+1}v_i^{-2} \\
& \quad \text{(by Propositions 2.8 and 2.9)} \\
= {} & A_i^{-1}y_1^{-1}A_{i-1}^{-1}A_{i-1}v_i^{-1}A_i^{-1}v_i^{-1}A_i^{-1}y_1^{-1}A_i^{-1}y_1A_{i-1}A_{i-1}^{-1}A_iv_iA_iv_iy_1A_{i-1}A_{i-1}^{-1}A_iv_i \\
& A_iv_iv_i^{-2} \text{(by (T1))} \\
= {} & A_i^{-1}y_1^{-1}v_i^{-1}A_i^{-1}v_i^{-1}A_i^{-1}A_iy_1^{-1}A_i^{-1}A_iv_iA_iv_iy_1A_iv_iA_iv_i^{-1} \quad \text{(by (T2), (T3), (T4))} \\
= {} & A_i^{-1}y_1^{-1}v_i^{-1}y_1A_i^{-1}y_1^{-1}v_iy_1v_i^{-1}A_iv_iA_i \quad \text{(by (T1), \dots, (T4))} \\
= {} & 1 \quad \text{(by (T2), (T3), (T4))}.
\end{aligned}$$

- The relation (T16) is a direct consequence of the braid relations in $A(\Gamma_{g,r,n})$.

- Relation (T17):

$$
\begin{aligned}
& f(y_1 x_{r+i-1} x_{r+i}^{-1} y_1^{-1} x_{r+i+1} y_1 x_{r+i} x_{r+i-1}^{-1} y_1^{-1}) v_i f(x_{r+i}^{-1}) v_i^{-1} \\
= \ & y_1 A_{i-1} A_i^{-1} y_1^{-1} A_{i+1} y_1 A_i A_{i-1}^{-1} y_1^{-1} v_i A_i^{-1} v_i^{-1} \\
= \ & y_1 A_i^{-1} A_{i-1} y_1^{-1} A_{i-1}^{-1} A_i v_i A_i v_i y_1 A_i v_i A_{i-1}^{-1} y_1^{-1} A_i^{-1} v_i^{-1} \quad \text{(by (T1), (T2), (T3))} \\
= \ & y_1 A_i^{-1} y_1^{-1} A_{i-1}^{-1} y_1 A_i v_i A_i v_i y_1 A_i v_i A_{i-1}^{-1} y_1^{-1} A_i^{-1} v_i^{-1} \quad \text{(by (T4))} \\
= \ & A_i^{-1} y_1^{-1} A_i A_{i-1}^{-1} y_1 A_i v_i A_i v_i y_1 A_i v_i A_{i-1}^{-1} y_1^{-1} A_i^{-1} v_i^{-1} \quad \text{(by (T4))} \\
= \ & A_i^{-1} y_1^{-1} A_{i-1}^{-1} y_1 A_i v_i y_1 A_i v_i y_1 A_i v_i A_{i-1}^{-1} y_1^{-1} A_i^{-1} v_i^{-1} \quad \text{(by (T2), (T3), (T4))} \\
= \ & A_i^{-1} y_1^{-1} A_{i-1}^{-1} \cdot h_i(\Delta(x_1, x_2, x_3)) \cdot A_{i-1}^{-1} y_1^{-1} A_i^{-1} v_i^{-1} \quad \text{(by Proposition 2.8)} \\
= \ & A_i^{-1} y_1^{-1} A_{i-1}^{-1} A_{i-1} y_1 A_i v_i A_i y_1 A_{i-1} A_{i-1}^{-1} y_1^{-1} A_i^{-1} v_i^{-1} \text{(by (T5) Proposition 2.9)} \\
= \ & 1.
\end{aligned}
$$

□

3.2 Proof of Theorem 3.2

Let $c_1 : S^1 \to \partial F_{g,1}$ be the boundary curve of $F_{g,1}$. We regard $F_{g,0}$ as obtained from $F_{g,1}$ by gluing a disk D^2 along c_1, and we denote by $\varphi : \mathcal{M}(F_{g,1}, \mathcal{P}_n) \to \mathcal{M}(F_{g,0}, \mathcal{P}_n)$ the homomorphism induced by the inclusion of $F_{g,1}$ in $F_{g,0}$. The next proposition is the key of the proof of Theorem 3.2.

Proposition 3.9 (i) *Let $g \geq 2$, and let a_n, a'_n be the Dehn twists represented in Figure 22. Then φ is surjective and its kernel is the normal subgroup of $\mathcal{M}(F_{g,1}, \mathcal{P}_n)$ normaly generated by $\{a_n^{-1} a'_n\}$.*

(ii) *Let $g = 1$, and let e, e' be the Dehn twists represented in Figure 22. Then φ is surjective and its kernel is the normal subgroup of $\mathcal{M}(F_{1,1}, \mathcal{P}_n)$ normaly generated by $\{a_n^{-1} a_0, e^{-1} e'\}$.*

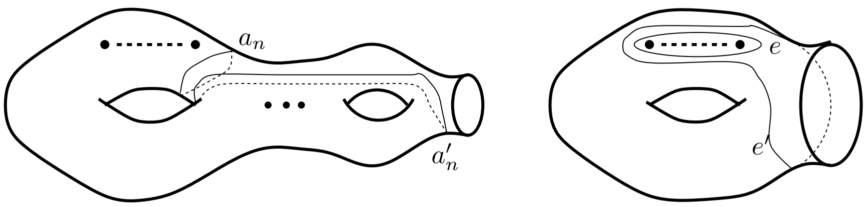

Figure 22: Relations in $\mathcal{M}(F_{g,0}, \mathcal{P}_n)$

Proof We choose a point Q in the interior of the disk D^2, and we denote by $\mathcal{M}_Q(F_{g,0}, \mathcal{P}_n \cup \{Q\})$ the subgroup of $\mathcal{M}(F_{g,0}, \mathcal{P}_n \cup \{Q\})$ of isotopy classes of elements of $\mathcal{H}(F_{g,0}, \mathcal{P}_n \cup \{Q\})$ that fix Q. An easy algebraic argument on the exact sequences (2.1), (2.2), and (2.3) of Subsection 2.2 shows that we have the following exact sequences.

$$(2.2.a) \qquad 1 \to \pi_1(F_{g,0} \setminus \mathcal{P}_n, Q) \to \mathcal{M}_Q(F_{g,0}, \mathcal{P}_n \cup \{Q\}) \xrightarrow{\varphi_1} \mathcal{M}(F_{g,0}, \mathcal{P}_n) \to 1,$$

$$(2.3.a) \qquad 1 \to \mathbf{Z} \to \mathcal{M}(F_{g,1}, \mathcal{P}_n) \xrightarrow{\varphi_2} \mathcal{M}_Q(F_{g,0}, \mathcal{P}_n \cup \{Q\}) \to 1.$$

Moreover, we have $\varphi = \varphi_1 \circ \varphi_2$.

A first consequence of these exact sequences is that φ is surjective. Now, we use them for finding a normal generating set of $\ker \varphi$.

The group $\pi_1(F_{g,0} \setminus \mathcal{P}_n, Q)$ is the free group freely generated by the loops $\bar\alpha_1, \dots, \bar\alpha_n$, $\bar\beta_1, \dots, \bar\beta_{2g-1}$ represented in Figure 23. One can easily verify by hand that the following equalities hold in $\mathcal{M}_Q(F_{g,0}, \mathcal{P}_n \cup \{Q\})$:

$$\begin{array}{rcll} \bar\alpha_i & = & \varphi_2(b_1 a_n' a_i b_1 a_n)^{-1} \cdot \bar\alpha_n^{-1} \cdot \varphi_2(b_1 a_n' a_i b_1 a_n) & \text{for } i = 1, \dots, n-1, \\ \bar\beta_1 & = & \varphi_2(b_1 a_n)^{-1} \cdot \bar\alpha_n \cdot \varphi_2(b_1 a_n), & \\ \bar\beta_j & = & \varphi_2(b_j b_{j-1})^{-1} \cdot \bar\beta_{j-1} \cdot \varphi_2(b_j b_{j-1}) & \text{for } j = 2, \dots, 2g-1. \end{array}$$

Moreover, by Lemma 2.6, we have:

$$\bar\alpha_n = \varphi_2(a_n^{-1} a_n').$$

On the other hand, by Lemma 2.7, the Dehn twists σ_1 along the boundary curve of $F_{g,1}$ generates the kernel of φ_2. So, the kernel of φ is the normal subgroup normaly generated by $\{a_n^{-1} a_n', \sigma_1\}$.

Now, assume $g \geq 2$. Let G' denote the quotient of $\mathcal{M}(F_{g,1}, \mathcal{P}_n)$ by the relation $a_n = a_n'$. Define a *spinning pair* of Dehn twists to be a pair (σ, σ') of Dehn twists conjugated to (a_n, a_n'), namely, a pair (σ, σ') of Dehn twists satisfying: there exists $\xi \in \mathcal{M}(F_{g,1}, \mathcal{P}_n)$ such that $\sigma = \xi a_n \xi^{-1}$ and $\sigma' = \xi a_n' \xi^{-1}$. Note that we have the equality $\sigma = \sigma'$ in G' if (σ, σ') is a spinning pair. Consider the Dehn twists $e_1, e_2, e_3, e_1', e_2', e_3'$ represented in Figure 24. The pairs (e_1, e_1'), (e_2, e_2'), (e_3, e_3') are spinning pairs, thus we have the equalities $e_1 = e_1'$, $e_2 = e_2'$, $e_3 = e_3'$ in G'. Moreover, the lantern relation of Lemma 2.4 implies:

$$e_1 e_2 e_3 \sigma_1 = e_1' e_2' e_3'.$$

Thus, the equality $\sigma_1 = 1$ holds in G'. This shows that the kernel of φ is the normal subgroup of $\mathcal{M}(F_{g,1}, \mathcal{P}_n)$ normaly generated by $\{a_n^{-1} a_n'\}$.

Now, we assume $g = 1$. Then $a_n' = a_0$. Let G' be the quotient of $\mathcal{M}(F_{1,1}, \mathcal{P}_n)$ by the relation $a_n = a_0$. By Proposition 2.12, we have the following equalities in G'.

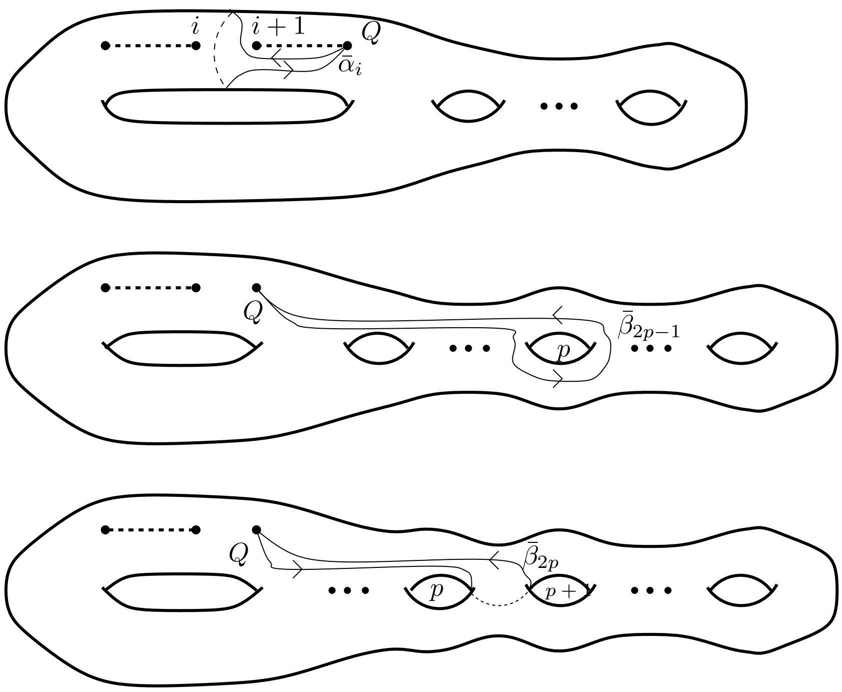

Figure 23: Generators of $\pi_1(F_{g,0} \setminus \mathcal{P}_n, Q)$

$$\begin{aligned} \sigma_1 e &= (a_0 b_1 a_n a_0 b_1 a_0)^2 = (a_0 b_1 a_0 a_0 b_1 a_0)^2, \\ e' &= (a_0 b_1 a_0)^4. \end{aligned}$$

Thus, we have the equality $\sigma_1 = e^{-1}e'$ in G'. So, the kernel of φ is the normal subgroup of $\mathcal{M}(F_{1,1}, \mathcal{P}_n)$ normaly generated by $\{a_n^{-1}a_0, e^{-1}e'\}$. $\square$

Proof of Theorem 3.2 Recall that $\Gamma_{g,0,n}$ denotes the Coxeter graph drawn in Figure 18, and that $G(g,0,n)$ denotes the quotient of $A(\Gamma_{g,0,n})$ by the relations (R1), (R2), (R7), (R8a). By Theorem 3.1, there is an isomorphism $\bar\rho : G(g,0,n) \to \mathcal{M}(F_{g,1}, \mathcal{P}_n)$ which sends x_i on a_i for $i = 0,1$, y_i on b_i for $i = 1, \dots, 2g-1$, z on c, and v_i on τ_i for $i = 1, \dots, n-1$.

First, assume $g \geq 2$. Let $G_0(g,n)$ denote the quotient of $G(g,0,n)$ by the relation (R9a). Proposition 2.12 implies:

$$\begin{aligned} a_n &= \bar\rho(x_0^{1-n}\Delta(x_1, v_1, \dots, v_{n-1})), \\ a_n' &= \bar\rho(x_0^{3-2g}\Delta(z, y_2, \dots, y_{2g-1})). \end{aligned}$$

Thus, by Proposition 3.9, $\bar\rho$ induces an isomorphism :

$$\bar\rho_0 : G_0(g,n) \to \mathcal{M}(F_{g,0}, \mathcal{P}_n).$$

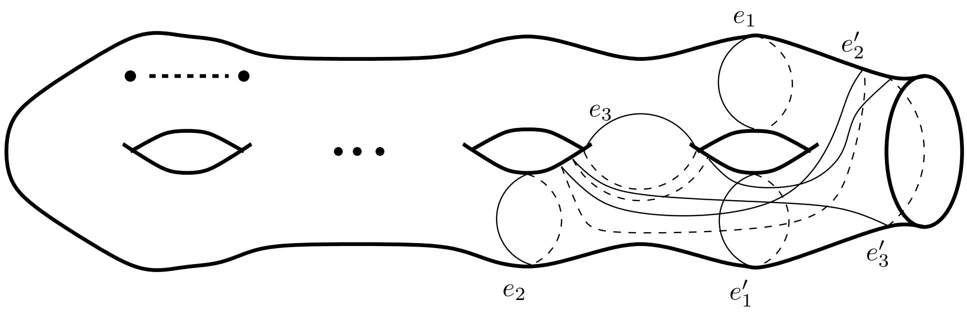

Figure 24: Lantern relation in $\mathcal{M}(F_{g,1}, \mathcal{P}_n)$

Now, assume $g = 1$. Let $G_0(1,n)$ denote the quotient of $G(1,0,n)$ by the relations (R9b), (R9c). Proposition 2.12 implies:

$$\begin{aligned} a_n &= \bar\rho(x_0^{1-n}\Delta(x_1, v_1, \dots, v_{n-1})), \\ e &= \bar\rho(\Delta^2(v_1, \dots, v_{n-1})), \\ e' &= \bar\rho(\Delta^4(x_0, y_1)). \end{aligned}$$

Thus, by Proposition 3.9, $\bar\rho$ induces an isomorphism :

$$\bar\rho_0 : G_0(1,n) \to \mathcal{M}(F_{1,0}, \mathcal{P}_n). \qquad \square$$

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