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An almost-integral universal Vassiliev invariant of knots

Simon Willerton

Abstract A "total Chern class" invariant of knots is defined. This is a universal Vassiliev invariant which is integral "on the level of Lie algebras" but it is not expressible as an integer sum of diagrams. The construction is motivated by similarities between the Kontsevich integral and the topological Chern character.

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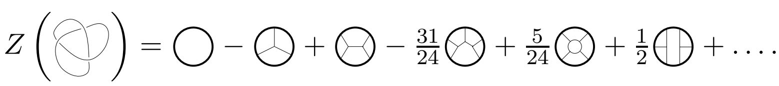
Introduction

The Kontsevich integral is a knot invariant with two related, but different universal properties. The first is that it is a universal rational Vassiliev invariant (see Section 1), and the second is that it is universal for perturbative quantum knot invariants coming from simple Lie algebras (see Section 3). It can be viewed as a map, $Z\colon \mathbb{Z}\mathbf{knots} \to \mathcal{A}$ from the abelian group of integer linear combinations of oriented knots to the rational algebra $\mathcal{A}$ which is generated by diagrams and is subject to the 1T and STU relations (see [4]). For example,



There is a lot of algebraic structure living in the algebra $\mathcal{A}$ (for a survey see [15]) some of which does reflect topological structure of knots. However there is no current topological interpretation of either the Kontsevich integral or the algebra $\mathcal{A}$.

The central observation of this paper is that several properties of the Kontsevich integral are remiscent of the topological Chern character. Some of these are summarized in Table 1 and details are given in Section 1.

Given this observation, an immediate question is "Is there a space $\mathbb{X}$ for which the Kontsevich integral *is* the Chern character?" A positive answer to this

Chern character	Kontsevich integral
Ring map	
$\mathrm{ch}\colon K(X) \to H^*(X;\mathbb{Q})$	$Z\colon \mathbb{Z}\mathbf{knots} \to \mathcal{A}$
Associated rational graded map is an isomorphism	
$(K_{2i}(X)/K_{2i+1}) \otimes \mathbb{Q} \cong H^{2i}(X;\mathbb{Q})$	$\big((\mathbb{Z}\mathbf{knots})_i/(\mathbb{Z}\mathbf{knots})_{i+1}\big) \otimes \mathbb{Q} \cong \mathcal{A}_i$
Integrality — when $H^{2i}(X,\mathbb{Z})$ (resp. $\mathcal{A}^{\mathbb{Z}}$) is torsion-free	
$x \in H^{2i}(X,\mathbb{Z}) \subset H^{2i}(X,\mathbb{Q}) \Leftrightarrow$ $\exists \eta \in K_{2i}(X)$ st $\mathrm{ch}(\eta) = x + \mathrm{hot}$	$D \in \mathcal{A}_i^{\mathbb{Z}} \subset A_i \Leftrightarrow$ $\exists k \in (\mathbb{Z}\mathbf{knots})_i$ st $Z(k) = D + \mathrm{hot}$
Riemann-Roch theorem v. cabling formula	
$\mathrm{ch}\big(f_K(\eta)\big) \cup \widehat{A}(Y)^{-1} =$ $f_H\Big(\mathrm{ch}(\eta) \cup \widehat{A}(X)^{-1} \cup \exp(c_1(f)/2)\Big)$	${}^{f}\!Z\left(\Psi^{m,p}(k)\right).\Omega =$ $\psi^m\left({}^{f}\!Z(k).\Omega.\exp\left(\frac{p}{2m}\bigcirc\right)\right)$

Table 1: Analogies between the topological Chern character and the Kontsevich integral for knots. Details are given in Section 1.

would certainly shed an interesting light on the Kontsevich integral, but to expect such a space seems a little too optimistic, for reasons given below. One way to test such an idea is to look at consequences of the existence of such a space. One consequence is the existence of a total Chern class. The topological total Chern class would be a group homomorphism $K(\mathbb{X}) \to H^*(\mathbb{X},\mathbb{Z})$ obtained by summing the Chern classes together. Note in particular that this takes values in *integer* cohomology, and the terms in it are closer to actual topological objects than the terms in the Chern character, although given the Chern character the total Chern class (modulo torsion) is easily obtained. This fact is used in Section 2.2 to define a "total Chern class for knots" $c\colon \mathbb{Z}\mathbf{knots} \to \mathcal{A}$. The obvious question to ask at this point is "Is this integer valued?" The answer to this is yes and no, as I will now explain.

This integrality question can be asked "on the level of simple Lie algebras" as is done in Section 3. This means looking at the total Chern class after applying a weight system coming from an irreducible representation of a simple Lie algebra. In this case we do get integer values:

Corollary 9 *The image of the total Chern class of knots is integral as far as irreducible representations of simple Lie algebras can detect, i.e. if ρ is such a* *representation, then* $w_\rho \circ c(\mathbb{Z}\mathbf{knots}) \subset \mathbb{Z}[[h]]$.

If the above integrality question is asked in the sense of "Is the total Chern class of knots expressible as a sum of diags with integer coefficients?" then the answer is no, as is shown in Section 4:

Theorem 10 *There is a degree four weight system taking integer values on all diagrams such that when evaluated on the total Chern class of the trefoil gives* $-5/4$, *thus the total Chern class is not expressible as an integer linear combination of diagrams.*

This would seem to preclude the existence of a "universal" space $\mathbb{X}$ for which the Kontsevich integral would be the Chern character. However, the previous result could be taken to indicate that it might be some sort of Chern character in individual cases. Thinking vaguely in the realm of topological quantum field theory, the invariant of a knot lives in the Verlinde algebra, that is the vector space associated to a torus. The Verlinde algebra seems to have two manifestations, as the K-group of representations of an algebra or as the invariant functionals on that algebra. The isomorphism from the former to the latter is the character map, that is the trace of the representation. In the case of a Lie algebra (or rather its universal enveloping algebra) this is essentially the universal Chern character for the corresponding group. Dually, the algebra $\mathcal{A}$ can be viewed as a "universal" source of characteristic classes, giving rise to characteristic classes of G-bundles for whatever G and for holomorphic bundles over holomorphic symplectic manifolds. In any case, there is certainly something character-esque going on.

Notation The monoid of oriented knots equipped with the connect sum operation is denoted **knots**, and $\mathbb{Z}$**knots** denotes the "monoid ring" i.e. the ring consisting of integer linear combinations of knots. By $\mathcal{A}$ is denoted the algebra of $\mathbb{Q}$-power series in connected unitrivalent diagrams modulo the STU and 1T relations, equipped with the connect sum product. The Kontsevich integral $Z\colon \mathbf{knots} \to \mathcal{A}$ is normalized so that $Z(\text{unknot}) = 1$, and hence is multiplicative. Multiplicative normalizations in general will be notated ungarnished — Z, J, w_ρ, *etc.* — tildes will be used to denote *quantum* normalizations — $\widetilde{Z}$, $\widetilde{J}$, $\widetilde{w}_\rho$, *etc.* So for instance, $\widetilde{Z}(k) = \widetilde{Z}(\text{unknot})Z(k)$.

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1 The Chern character v. the Kontsevich integral

In this section I give details of the analogous properties shared by the topological Chern character and the Kontsevich integral, which act as the motivation for the definition of the total Chern class of knots in Section 2.

1.1 The Chern character

I will give a reminder on the topological Chern character before giving some of its properties, more details can be found in, say, [8]. Suppose that X is a CW complex (eg. a manifold, if you prefer), then $K(X)$, the K-group of X, is the Grothendieck group of the monoid of vector bundles on X. A general element of $K(X)$ is a formal difference of two vector bundles, and the group operation comes from $\oplus$, the Whitney sum operation on vector bundles. The tensor product, $\otimes$, of vector bundles makes $K(X)$ into a ring. The ith Chern class $c_i(\eta) \in H^{2i}(X,\mathbb{Z})$ of a vector bundle η is an integral cohomology class on X which only depends on the class of η in the K-group of X. The Chern character map $\mathrm{ch}\colon K(X) \to H^{\mathrm{even}}(X,\mathbb{Q})$ is a special rational linear combination of Chern classes which has the following properties.

Multiplicativity The Chern character is a ring map from a ring to an algebra over the rationals, i.e. $\mathrm{ch}(\eta \oplus \zeta) = \mathrm{ch}(\eta) + \mathrm{ch}(\zeta)$ and $\mathrm{ch}(\eta \otimes \zeta) = \mathrm{ch}(\eta) \cup \mathrm{ch}(\zeta)$.

Rational associated graded map The K-group of a CW complex is naturally filtered via the skeletal filtration of the complex: i.e. if $X^i \hookrightarrow X$ is the inclusion of the ith skeleton into X, then the ith filtration, $K_i(X)$, can be defined as $K_i(X) := \ker\{K(X) \to K(X^i)\}$. The Chern character induces an isomorphism on the rationalized associated graded objects:

$$\mathrm{ch}\colon \big(K_{2i}(X)/K_{2i+1}(X)\big) \otimes \mathbb{Q} \xrightarrow{\cong} H^{2i}(X,\mathbb{Q}).$$

Integrality Assuming that $H^*(X,\mathbb{Z})$ is torsion-free, $H^*(X,\mathbb{Z})$ can be identified with its image under the inclusion $H^*(X,\mathbb{Z}) \hookrightarrow H^*(X,\mathbb{Q})$. In this case a rational class $x \in H^{2i}(X,\mathbb{Q})$ is integral if and only if there is an element $\eta \in K_{2i}(X)$ such that $\mathrm{ch}(\eta) = x + \text{higher order terms}$.

A result of Adams [1] (briefly) If $x \in H^{2i}(X,\mathbb{Z})$ and $\eta \in K_{2i}(X)$ are as above, write $\mathrm{ch}(\eta) = x + \sum_{r=1}^{\infty} x_{2i+2r}$ with $x_{2i+2r} \in H^{2i+2r}(X,\mathbb{Q})$. Now let p be prime and r be an integer not divisible by $p-1$ then $x_{2i+2r}p^{\lfloor r/(p-1)\rfloor}$ is integral modulo p, and modulo p it can be expressed in terms of the inverse Steenrod reduced power of x, thus modulo p it does not depend on the choice of η.

Atiyah-Hirzebruch-Grothendieck-Riemann-Roch Formula Supposing that $f\colon X \to Y$ is a map between oriented manifolds one can define the push forward on cohomology $f_H : H^*(X;\mathbb{Z}) \to H^{*+\dim Y - \dim X}(Y;\mathbb{Z})$ by using the push forward on homology together with Poincaré duality. In fact the orientability assumption can be weakened so that the push-forward can be defined provided that the *map* is orientable, ie. the first Steifel-Whitney class $w_1(f) := w_1(X) - f^*w_1(Y) \in H^1(X, Z_2)$ vanishes. We want to define an analogous push-forward for K-theory, this requires that the map is *orientable in K-theory*, or, in other words, Spin^c, which means the following. Let $w_2(X) \in H^2(X,\mathbb{Z}_2)$ denote the second Stiefel-Whitney class, and assume for simplicity that the dimensions of X and Y have the same parity, then the map f is Spin^c if there is an integral class $c_1(f) \in H^2(X,\mathbb{Z})$ such that $c_1(f) \equiv w_2(X) - f^*w_2(Y)$ mod 2. Given that f is Spin^c, one can define a push forward on K-theory: $f_K : K(X) \to K(Y)$. If X is a smooth manifold then $\widehat{A}(X) \in H^*(X;\mathbb{Q})$ is defined in terms of the Pontryagin classes of the tangent bundle of X and a sequence involving the power series $\sinh(x/2)/(x/2)$. The Chern character naturally commutes with the pull-backs on homology and K-theory, the following theorem shows how it behaves with respect to the push-forward maps. (For more details see, eg., [8, Chapter V.4].)

Theorem 1 [2] *If $f\colon X \to Y$ is a map between smooth oriented manifolds whose dimensions have the same parity and there exists a cohomology class $c_1(f) \in H^2(X,\mathbb{Z})$ which satisfies $c_1(f) \equiv w_2(X) - f^*w_2(Y) \pmod 2$ then, for all $\eta \in K(X)$,*

$$\mathrm{ch}\big(f_K(\eta)\big) \cup \widehat{A}(Y)^{-1} = f_H\big(\mathrm{ch}(\eta) \cup \widehat{A}(X)^{-1} \cup \exp(c_1(f)/2)\big).$$

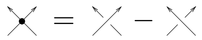
1.2 The Kontsevich integral

The Kontsevich integral, $Z\colon \mathbb{Z}\mathbf{knots} \to \mathcal{A}$, has analogous properties to those of the Chern character described above, as follows.

Multiplicativity The Kontsevich integral is a ring map from a ring to an algebra over the rationals, i.e. it satisfies $Z(k+l) = Z(k) + Z(l)$ and $Z(k\#l) = Z(k).Z(l)$ where k and l are integer linear combinations of knots. In fact it is actually a bialgebra map, that is it commutes with the coproduct: $\Delta \circ Z(k) = (Z \otimes Z) \circ \Delta(k)$.

Rational associated graded map The ring $\mathbb{Z}\mathbf{knots}$ is naturally graded by the Vassiliev filtration, this can be defined as

$$(\mathbb{Z}\mathbf{knots})_i = \{\text{linear combinations of knots with} \geq i \text{ double points}\},$$

knots with double points being considered as elements of $\mathbb{Z}\mathbf{knots}$ via the formal resolution . A fundamental property of the Kontsevich integral is that it induces an isomorphism on the rationalized associated graded objects:

$$Z\colon \big((\mathbb{Z}\mathbf{knots})_i/(\mathbb{Z}\mathbf{knots})_{i+1}\big) \otimes \mathbb{Q} \xrightarrow{\cong} \mathcal{A}_i.$$

This is essentially what is meant by it being a universal Vassiliev invariant.

Integrality Provided that $\mathcal{A}_{\mathbb{Z}}$, the analogue of $\mathcal{A}$ defined with integer coefficients, is torsion-free, an element $D \in \mathcal{A}_i$ can be expressed as an *integer* linear combination of diagrams if and only if there is some integer linear combination, $k \in (\mathbb{Z}\mathbf{knots})_i$, of i-singular knots such that $Z(k) = D + \text{higher order terms}$.

Congruences I do not know if any analogue of the result of Adams holds. However, the results on congruency between the terms in the Kontsevich integral in Proposition 12 seem to point in that direction.

Cabling formula The reader is directed to [15] for more details; here I will sketch briefly. For m and p coprime integers, there is the well defined notion of (m,p) cabling on framed knots. From a framed knot k, the (m,p)-cable $\Psi^{m,p}(k)$ is defined so that it twists m times longitudinally around the knot k and p times meridianally. The framed Kontsevich integral, ${}^f\!Z\colon \mathbb{Z}^f\mathbf{knots} \to {}^f\!\mathcal{A}$, can be defined to be a map from integer linear combinations of framed knots to rational linear combinations of diagrams modulo just the STU relations. One can consider ${}^f\!\mathcal{A}$ to be $\mathcal{A}$ with one generator adjoined in degree one: ${}^f\!\mathcal{A} \cong \mathcal{A}[[\bigcirc]]$. The framed Kontsevich integral can be defined for a framed knot k as ${}^f\!Z(k) := Z(k).\exp\left(F(k)\bigcirc/2\right)$, where $F(k)$ is the framing number of k and $Z(k)$ is the usual Kontsevich integral of the underlying unframed knot.

The wheels element, Ω, of $\mathcal{A}$ is defined in terms of "wheels diagrams" and a "disjoint union product", via a power series akin to the square root of the $\widehat{A}$ power series.

The following theorem is the statement of how the framed Kontsevich integral behaves under the cabling operation.

Theorem 2 (Le, see [14]) *If m and p are coprime integers and k is a framed knot, then*

$$^{f}\!Z\left(\Psi^{m,p}(k)\right).\Omega = \psi^m\left({}^{f}\!Z(k).\Omega.\exp\left(\tfrac{p}{2m}\bigcirc\right)\right).$$

The similarity with Proposition 1 is striking.

2 The total Chern class

In the first Section 2.1 some facts about the topological total Chern class and its relation to the Chern character are recalled. These are used in Section 2.2 to define a total Chern class for knots.

2.1 The classical case

The Chern classes of a vector bundle over a space X can be added together to give the total Chern class $c\colon \operatorname{Vect}(X) \to H^{\mathrm{ev}}(X,\mathbb{Z})$ with $c(\xi) = 1+\sum_{i=1}^{\infty} c_i(\xi)$. This is multiplicative under Whitney sum of vector bundles: $c(\xi\oplus\eta) = c(\xi)\cup c(\eta)$. So, defining $1+H^{\mathrm{ev}+}(X,\mathbb{Z})$ to be the group of multiplicative units in $H^{\mathrm{ev}}(X,\mathbb{Z})$, by the universal property of the K-group we get a group homomorphism

$$c\colon K(X) \to 1+H^{\mathrm{ev}+}(X,\mathbb{Z}); \qquad \xi \mapsto 1+\sum_{i=1}^{\infty} c_i(\xi), \qquad c_i(\xi) \in H^{2i}(X,\mathbb{Z}).$$

The information in the total Chern class is essentially the same as that in the Chern character, described in Section 1, and, modulo torsion, they can be obtained from each other in the fashion described below. Whereas the Chern character is very well behaved algebraically and arises naturally in Chern-Weil theory and index theory, it is the total Chern class that is somehow closer to actual topology. The Chern classes are defined integrally and have interpretations as obstructions, in terms of cells in Grassmanians and as being Poincaré dual to submanifolds. For instance, the zero set of a generic section of a rank n bundle ξ is a codimension $2n$ submanifold which is Poincaré dual to the top Chern class $c_n(\xi) \in H^{2n}(X,\mathbb{Z})$.

Switching between the total Chern class (modulo torsion) and the Chern character is entirely analogous to changing between elementary symmetric functions and power sums, and the terms in the two classes are related by Newton formulae (see eg. [8]). An equivalent approach is the following formula which follows easily from the splitting principal for vector bundles [7]. If $\xi \in K(X)$ then

$$c(\xi) \equiv \exp\big((-1)^{\deg/2-1}(\deg/2-1)!\,\mathrm{ch}(\xi)\big) \quad \text{mod torsion},$$

where $(-1)^{\deg/2-1}(\deg/2-1)!$ is the function on $H^{\mathrm{ev}}(X,\mathbb{Q})$ which multiplies a homogeneous element of degree $2n$ by the factor $(-1)^{n-1}(n-1)!$, with $(-1)!$ being interpreted as 0. Using this each Chern class (modulo torsion) is given by a polynomial in the Chern character classes. Eg. $c_2(\xi) = \frac{1}{2}\,\mathrm{ch}_1(\xi)^2 - \mathrm{ch}_2(\xi)$.

2.2 The total Chern class for knots

In this section we will use the relationship between the topological Chern character and the topological total Chern class given above, together with the analogy between the Kontsevich integral and the topological Chern character of Section 1 to define a "total Chern class for knots". The original point of doing this was to test the analogy and see if an "integer" invariant was obtained. As we will see in Section 3 this is indeed the case when a weight system coming from an irreducible representation of a simple Lie algebra is applied.

To define the total Chern class for knots, let $1+\mathcal{A}^+$ be the set of elements in $\mathcal{A}$ whose term in degree zero is precisely the diagram with no internal graph; this set forms a group under the connect sum operation. Now by analogy with the formula in Section 2.1, define the total Chern class of knots $c\colon \mathbb{Z}\mathbf{knots} \to 1+\mathcal{A}^+$ by

$$c(k) := \exp\big((-1)^{\deg-1}(\deg-1)!\,Z(k)\big).$$

Three remarks are in order. The first is that the grading used here is the conventional grading on $\mathcal{A}$, there are several arguments as to why the natural grading ought to be double that, but in this paper I will stick to the conventional grading: this is why the grading differs by a factor of two from that in the expression for the topological Chern class above. The second remark is that I am ignoring any questions of torsion here — it is not known whether there is any torsion in the analogue of $\mathcal{A}$ defined over $\mathbb{Z}$. The third remark is that if a basis of $\mathcal{A}$ is chosen which consists of the monomials in some set of connected diagrams (see [15]), then in the transition from the Kontsevich integral to the total Chern class, the coefficient of a connected diagram D just gets multiplied by $(-1)^{\deg(D)-1}(\deg(D)-1)!$, but the coefficients of non-connected diagrams get much more messed up in general.

We finish this section with two basic properties of the total Chern class of knots.

Proposition 3 *The total Chern class of knots is a group homomorphism:* $c(k+l) = c(k).c(l)$.

Proof This follows from the linearity of the Kontsevich integral, $Z(k+l) = Z(k)+Z(l)$, together with the fact that the exponential map satisfies $\exp(A+B) = \exp(A)\exp(B)$. □

Note here that the sum is the formal algebraic sum in $\mathbb{Z}$**knots**, and this means that c is unchanged by the formal addition of trivial knots, as $c(\text{unknot})$ is the unit in $1+\mathcal{A}^+$. This is the analogue of the fact that the topological Chern class does not detect trivial bundles of any rank.

Proposition 4 *The total Chern class of knots is a universal Vassiliev invariant in the following sense. If k is a knot with n double points and D is the underlying chord diagram with n chords, then the total Chern class satisfies*

$$c(k) = 1 + (-1)^{n-1}(n-1)!D + \textit{higher order terms.}$$

Proof The Kontsevich integral is a universal Vassiliev invariant in the sense that if k satisfies the hyphotheses then $Z(k) = D + \text{higher order terms}$. The result follows immediately from the definition of the total Chern class. □

3 Integrality on the level of Lie algebras

In this section it is shown that the total Chern class of a knot gives an integer power series when a weight system coming from an irreducible representation of a simple Lie algebra is evaluated on it. Recall the notational conventions from the introduction that a tilde means quantum normalization.

There is a standard construction in the theory of Vassiliev invariants, by which one obtains a weight system, $\widetilde{w}_\rho\colon \mathcal{A} \to \mathbb{Q}[[h]]$, from the data of a representation ρ of a Lie algebra $\mathfrak{g}$ which is equipped with an invariant, symmetric, bilinear form. The idea is that from a diagram D one constructs an element $w_\mathfrak{g}(D) \in ZU(\mathfrak{g})$ in the centre of the universal enveloping algebra of the Lie algebra, one then takes the trace in the representation ρ of this element, and multiplies by an appropriate power of h: i.e. $\widetilde{w}_\rho(D) := \mathrm{Tr}_\rho w_\mathfrak{g}(D)\cdot h^{\deg D}$.

Given the same Lie algebraic data, one can form the Reshetikhin-Turaev invariant $\widetilde{\tau}_\rho\colon \textbf{knots} \to \mathbb{Q}[[h]]$. Recalling that $\widetilde{Z}$ is the *quantum* normalization of the Kontsevich integral, it is a fundamental theorem [12, 9] of the theory of the

Kontsevich integral that the Reshetikhin-Turaev invariant $\widetilde{\tau}_\rho$ factors as $\widetilde{w}_\rho \circ \widetilde{Z}$, i.e. the following diagram of sets commutes.

$$\begin{array}{ccc} \mathbf{knots} & \xrightarrow{\widetilde{Z}} & \mathcal{A} \\ & \widetilde{\tau}_\rho \searrow & \downarrow \widetilde{w}_\rho \\ & & \mathbb{Q}[[h]]. \end{array} \tag{1}$$

Actually, the above constructions lead to *link* invariants and not just knot invariants. The modifications below will be made because I am thinking very specifically about the case of knots.

Consider the Jones polynomial: in its quantum normalization it is a link invariant $\widetilde{J}\colon \mathbf{links} \to \mathbb{Z}[q^{\pm 1/2}]$ which maps the empty link to 1 and the unknot to $q^{1/2}+q^{-1/2}$. If one substitutes $q = e^h$ and expands in powers of h, then one obtains precisely the Reshetikhin-Turaev invariant $\widetilde{\tau}_\rho$ where ρ is the fundamental representation of $\mathfrak{su}_2$ (equipped with the bilinear form coming from the trace in the fundamental representation). This is fine, but if one is thinking purely of knots then it makes sense to normalize so that the unknot gets mapped to 1; so to this end, set $J(k) := \widetilde{J}(k)\widetilde{J}(\text{unknot})^{-1}$. This normalization has two advantages: (i) $J(k)$ is a genuine Laurent polynomial in q with no odd powers of $q^{1/2}$; and (ii) it is multiplicative under connected sum, i.e. $J(k\#l) = J(k)J(l)$.

In general, for ρ an irreducible representation of a simple Lie algebra, one has the quantum link invariant $\widetilde{J}_\rho\colon \mathbf{links} \to \mathbb{Z}[q^{\pm 1/M}]$, which takes values in the ring of Laurent polynomials in some fractional power, $q^{1/M}$, of q, where M is some integer depending on the representation. This is related to the Reshetikhin-Turaev invariants via the following commutative diagram:

$$\begin{array}{ccc} \mathbf{links} & & \\ \widetilde{J}_\rho \downarrow & \searrow \widetilde{\tau}_\rho & \\ \mathbb{Z}[q^{\pm 1/M}] & \xrightarrow{q \mapsto e^h} & \mathbb{Q}[[h]]. \end{array} \tag{2}$$

If attention is restricted to knots and the invariant $\widetilde{J}_\rho$ is renormalized to J_ρ so that the unknot is mapped to 1 then one can ask what the image of the map J_ρ is. One might expect that, as in the case of the Jones polynomial above, the image is contained in $\mathbb{Z}[q^{\pm 1}]$. However, I was surprised to find nothing in the literature on this question, until the recent paper of Le [11]. Perhaps one reason for this lack is that the question is not terribly natural when the approach to the quantum invariants is skein theoretic, tangle theoretic, or via Markov traces. In the current context, the question is very natural. The following theorem is proved by a thorough understanding of Lusztig's work on quantum groups.

Theorem 5 (Le [11]) *If ρ is an irreducible representations of a simple Lie algebra, and J_ρ is normalized to take value 1 on the unknot, then it gives a multiplicative map $J_\rho\colon \mathbf{knots} \to \mathbb{Z}[q^{\pm 1}]$.*

For ρ an irreducible representations of simple Lie algebra $\mathfrak{g}$, define a weight system in the following manner. As the universal $\mathfrak{g}$ weight system $w_\mathfrak{g}$ takes values in the centre of the universal enveloping algebra and the latter acts by scalars in all irreducible representations, define w_ρ to be precisely the value of this scalar. As $\widetilde{w}_\rho$ was defined as the trace of this scalar operator, it is clear that $\widetilde{w}_\rho = (\dim \rho).w_\rho$. This normalization is algebraically well-behaved in the sense of the following theorem.

Theorem 6 *If ρ is an irreducible representation of a Lie algebra then the weight system $w_\rho\colon \mathcal{A} \to \mathbb{Q}[[h]]$ is an algebra map.*

Proof This follows immediately from the fact that $w_\mathfrak{g}$ is multiplicative, and the fact that ρ is multiplicative on the universal enveloping algebra. □

Now extend J_ρ linearly to $\mathbb{Z}\mathbf{knots}$, and define the ring map $\mathrm{ch}\colon \mathbb{Z}[q^{\pm 1}] \to \mathbb{Q}[[h]]$ by $q \mapsto e^h$; the notation will be explained after the following theorem.

Theorem 7 *If ρ is an irreducible representation of a simple Lie algebra, then the following is a commutative diagram of filtered rings:*

$$\begin{array}{ccc} \mathbb{Z}\mathbf{knots} & \xrightarrow{Z} & \mathcal{A} \\ J_\rho \downarrow & & \downarrow w_\rho \\ \mathbb{Z}[q^{\pm 1}] & \xrightarrow{\mathrm{ch}} & \mathbb{Q}[[h]]. \end{array}$$

Proof Let k be a knot. By the commutativity of diagrams (1) and (2) above, for the quantum normalizations, $\mathrm{ch}\big(\widetilde{J}_\rho(k)\big) = \widetilde{w}_\rho\big(\widetilde{Z}(k)\big)$. This can be written in the multiplicative normalizations as

$$\mathrm{ch}\big(\widetilde{J}_\rho(\text{unknot}).J_\rho(k)\big) = (\dim \rho).w_\rho\big(\widetilde{Z}(\text{unknot}).Z(k)\big).$$

By multiplicativity of ch and w_ρ,

$$\begin{aligned} \mathrm{ch}\big(\widetilde{J}_\rho(\text{unknot})\big).\,\mathrm{ch}\big(J_\rho(k)\big) &= (\dim \rho).w_\rho\big(\widetilde{Z}(\text{unknot})\big).w_\rho\big(Z(k)\big) \\ &= \widetilde{w}_\rho\big(\widetilde{Z}(\text{unknot})\big).w_\rho\big(Z(k)\big) \end{aligned}$$

But $\mathrm{ch}\big(\widetilde{J}_\rho(\text{unknot})\big) = \widetilde{w}_\rho\big(\widetilde{Z}(\text{unknot})\big)$ and this is an invertible element in $\mathbb{Q}[[h]]$, so $\mathrm{ch}\big(J_\rho(k)\big) = w_\rho\big(Z(k)\big)$ as required. □

As Z is being thought of as some kind of Chern character, one can ask what my interpretation of $\mathrm{ch}\colon \mathbb{Z}[q^{\pm 1}] \to \mathbb{Q}[[h]]$, $q \mapsto e^h$, is in this language. Thinking naïvely as a topologist, $\mathbb{Z}[q^{\pm 1}]$ can be thought of as the Grothendieck group of vector bundles over infinite projective space $\mathbb{C}P^\infty$, with q representing the canonical line bundle; similarly, $\mathbb{Q}[[h]]$ can be thought of as the (completed) rational, ordinary cohomology of $\mathbb{C}P^\infty$, with h being the canonical generator in degree 2 which is the first Chern class of the canonical line bundle q. In this framework $q \mapsto e^h$ corresponds *precisely* to the Chern character — this is the reason that I called it ch.

The natural question to ask next is: what corresponds to the total Chern class? This is easy. Let $1 + h\mathbb{Z}[[h]]$ be the group of multiplicative units of the ring of formal power series $\mathbb{Z}[[h]]$ and define $c\colon \mathbb{Z}[q^{\pm 1}] \to 1 + h\mathbb{Z}[[h]]$ to be the map of *groups* given by $q^n \mapsto 1 + nh$, i.e. $c(a+b) = c(a)c(b)$ for all $a, b \in \mathbb{Z}[q^{\pm 1}]$, so for instance, $2q^{-3} - q \mapsto (1-3q)^2(1 - h + h^2 - h^3 + \ldots)$.

Theorem 8 *For ρ an irreducible represention of a simple Lie algebra, the following is a commutative diagram of groups:*

$$\begin{array}{ccccc} \mathbb{Z}\mathbf{knots} & \xrightarrow{c} & & & 1 + \mathcal{A}^+ \\ J_\rho \downarrow & & & & \downarrow w_\rho \\ \mathbb{Z}[q^{\pm 1}] & \xrightarrow{c} & 1 + h\mathbb{Z}[[h]] & \hookrightarrow & 1 + h\mathbb{Q}[[h]]. \end{array}$$

Proof This follows from the multiplicativity of the weight system w_ρ and the fact that both the maps labelled c can be expressed in terms, respectively, of the maps Z and ch via the same formula which is polynomial in each degree. □

The following corollary is immediate and is the expression of the integrality of the total Chern class for knots on the level of Lie algebras.

Corollary 9 *The image of the total Chern class of knots is integral as far as irreducible representations of simple Lie algebras can detect, i.e. if ρ is such a representation, then $w_\rho \circ c(\mathbb{Z}\mathbf{knots}) \subset \mathbb{Z}[[h]]$.*

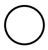
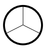
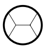
It is perhaps worth contemplating, in view of the above, whether the quantum invariants, such as the Jones polynomial, have any K-theoretic interpretation. Note that, thought of as a torsion, the Alexander polynomial takes values in something akin to a Whitehead group.

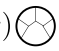
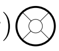
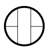
4 Non-integrality on the level of diagrams

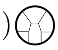
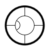
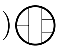
It will be shown here that the Chern class does not live in the integer lattice of $\mathcal{A}$, despite the integrality results of the previous section.

Theorem 10 *There is a degree four weight system taking integer values on all diagrams such that when evaluated on the total Chern class of the trefoil gives $-5/4$, thus the total Chern class is not expressible as an integer linear combination of diagrams.*

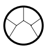
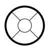
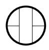
Proof In the appendix an expression for the Kontsevich integral is given in terms of integer valued knot invariants, B, C, D_1, D_2, E_1, E_2 and E_3. If the transformation to the total Chern class is made, then one finds:

$c(k) =$  $- B(k)$  $+ 2C(k)$ 

$- \frac{1}{4}\Big(D_1(k)$  $+ D_2(k)$  $+ 10B(k)^2$  $\Big)$

$+ 2E_1(k)$  $+ E_2(k)$  $+ E_3(k)$  $+ 22B(k)C(k)$  $+ \ldots.$

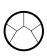
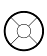
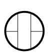
The problem here is with the degree four piece. One can check that the degree four weight system that maps

 $\mapsto 0;$  $\mapsto 1;$  $\mapsto 0,$

is integer valued on all degree four diagrams. Evaluating this on $c(k)$ one obtains the number $-D_2(k)/4$. From the example of the trefoil in the introduction, it is seen that $D_2(\text{trefoil}) = 5$ and hence the total Chern class of the trefoil is not in the integer lattice of $\mathcal{A}$. □

Note that there could be things that stop this from being a "nice" weight system; for example I do not know if this weight system extends to an integral multiplicative weight system, perhaps some power of the wheel with four legs is a multiple of some integer combination of other diagrams.

It is instructive here to consider the evaluation of the total Chern class under the weight systems coming from the irreducible representations of $\mathfrak{su}_2$; from the previous section this is known to give an integer power series. In the d-dimensional irreducible representation of $\mathfrak{su}_2$, the Casimir acts by $\lambda := (d^2 - 1)/2$, and, by [6], the corresponding weight system, w_d, behaves as follows:

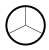
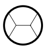
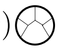
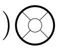
 $\mapsto 8\lambda h^4;$  $\mapsto 8\lambda^2 h^4;$  $\mapsto 4\lambda^2 h^4.$

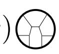
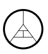
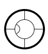
Then the coefficient of h^4 in $w_{\mathfrak{d}}c(k)$ is given by $-2\lambda((5B(k)^2 + D_2(k))\lambda + D_1(k))$. This is integral when λ is half-integral precisely because of the congruence $B(k) \equiv D_2(k) \pmod 2$ from Proposition 12 in the appendix. Note that the integrality result for the Alexander-Conway weight system corresponds to the same congruence.

Appendix: The Kontsevich integral up to degree five

In this appendix an expression for the Kontsevich integral up to degree five is given in terms of integral knot invariants. A corollary of the proof is that these invariants satisfy various congruences.

Theorem 11 *There exist integer valued knot invariants, B, C, D_1, D_2, E_1, E_2 and E_3 such that*

$Z(k) = \exp\Bigg(B(k)$  $+ C(k)$  $+ \frac{1}{24}\Big(D_1(k)$  $+ D_2(k)$  $\Big)$

$+ \frac{1}{24}\Big(2E_1(k)$  $+ E_2(k)$  $+ E_3(k)$  $\Big) + \dots \Bigg).$

Proof It follows from [15] that the Kontsevich integral has this form where the invariants are rational valued. Stanford [13] has calculated a basis for rational valued additive Vassiliev invariants up to degree six which consists of *integer* valued invariants. To express the above invariants in terms of Stanford's invariants it suffices to calculate the Kontsevich integral for four suitably chosen knots, compare the values with Stanford's table and then solve the requisite linear equations. I calculated the Kontsevich integral up to degree five of the knots 3_1, 5_1, 5_2, and 7_2 by using my formulae for torus knots and Bar-Natan's Mathematica program (see [5]). Denote Stanford's invariants II, III, IV_1, IV_2, V_1, V_2 and V_3. Define an alternative integral basis in degree five by $W_1 = (V_1 - 2V_2)/9$, $W_2 = V_2$ and $W_3 = (V_2 - V_3)/9$. Then one finds

$$
\begin{aligned}
B &= -II, & II &= -B,\\
C &= III, & III &= C,\\
D_1 &= 17II - 48IV_1 - 24IV_2, & IV_1 &= \tfrac{1}{12}(D_2 - 7B),\\
D_2 &= -7II + 12IV_1, & IV_2 &= \tfrac{1}{24}(11B - D_1 - 4D_2),\\
E_1 &= 11III - 140W_1 - 76W_2 + 340W_3, & W_1 &= \tfrac{1}{24}(22C + 2E_1 + 7E_2 + 3E_3),\\
E_2 &= III - 8W_1 - 4W_2 + 16W_3, & W_2 &= \tfrac{1}{12}(-23C - 2E_1 - 40E_2 - 5E_3),\\
E_3 &= -17III + 120W_1 + 60W_2 - 264W_3, & W_3 &= \tfrac{1}{24}(-2C - 15E_2 - E_3).
\end{aligned}
$$

The left-hand set of equations together with the integrality of Stanford's invariants proves the theorem. □

It is worth remarking that it is natural to consider the denominators of the Kontsevich integral, where an element $D \in \mathcal{A}$ has *denominator* dividing $m \in \mathbb{Z}$ if mD can be expressed as an integer linear combination of diagrams. Le [10] has considered this for the quantum normalization, $\widetilde{Z}$, and found that the denominator of the degree n piece divides $(2!3!...n!)^4(n+1)!$. I suspect that the denominators in the multiplicative normalization are better behaved. Note that if the degree n piece of $\ln Z$ has denominator dividing $n!$, then the degree n piece of Z has denominator dividing $n!$. From the above theorem, one might therefore be tempted to conjecture precisely that.

As a corollary of the above proof, one also obtains the following intriguing result.

Proposition 12 *The integral knot invariants B, C, D_1, D_2, E_1, E_2 and E_3 satisfy the following congruences:*

$$\begin{aligned} B \equiv D_1 \equiv D_2 \pmod 3; \qquad & D_1 \equiv -B \pmod 8; \\ 4C \equiv D_2 - D_1 \pmod 8; \qquad & D_2 \equiv -B \pmod 4; \\ E_1 \equiv E_2 + E_3 \pmod 3; \qquad & E_2 \equiv -E_3 \pmod 8; \\ C \equiv E_3 \pmod 3; \qquad & C \equiv -E_1 \equiv E_2 \pmod 4. \end{aligned}$$

Proof These congruences are consequences of the right-hand set of equations in the previous proof together with the fact that all of the invariants are integer valued. The third congruence also requires Stanford's observation that $IV_1 - III$ is always even. □

From these one can obtain for instance, $D_1 - D_2 \equiv 0 \pmod{12}$ and $C \equiv (D_1 - D_2)/4 \pmod 2$; but I have no idea what these mean. Perhaps it is useful to note that $B \pmod 2$ is the Arf invariant.

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Department of Pure Mathematics, University of Sheffield
The Hicks Building, Hounsfield Road, Sheffield, S3 7RH, UK

Email: `S.Willerton@sheffield.ac.uk`

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