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The compression theorem III: applications

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Abstract This is the third of three papers about the *Compression Theorem*: if M^m is embedded in $Q^q \times \mathbb{R}$ with a normal vector field and if $q-m \geq 1$, then the given vector field can be *straightened* (ie, made parallel to the given $\mathbb{R}$ direction) by an isotopy of M and normal field in $Q \times \mathbb{R}$.

The theorem can be deduced from Gromov's theorem on directed embeddings [5; 2.4.5 (C′)] and the first two parts gave proofs. Here we are concerned with applications.

We give short new (and constructive) proofs for immersion theory and for the loops–suspension theorem of James et al and a new approach to classifying embeddings of manifolds in codimension one or more, which leads to theoretical solutions.

We also consider the general problem of controlling the singularities of a smooth projection up to C^0–small isotopy and give a theoretical solution in the codimension ≥ 1 case.

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1 Introduction

We work throughout in the smooth (C^∞) category. The tangent bundle of a manifold W is denoted TW and the tangent space at $x \in W$ is denoted T_xW. Throughout the paper, "normal" means independent (as in the usual meaning of "normal bundle") and not necessarily perpendicular.

This is the third of a set of three papers about the following result:

Compression Theorem *Suppose that M^m is embedded in $Q^q \times \mathbb{R}$ with a normal vector field and suppose that $q - m \geq 1$. Then the vector field can be straightened (ie, made parallel to the given $\mathbb{R}$ direction) by an isotopy of M and normal field in $Q \times \mathbb{R}$.*

Thus the theorem moves M to a position where it projects by vertical projection (ie "compresses") to an immersion in Q.

The theorem can be deduced from Gromov's theorem on directed embeddings [5; 2.4.5 (C′)]. Proofs are given in parts I and II [14, 15]. Two immediate applications were given in [14; corollaries 1.1 and 1.2]; here we give more substantial applications.

Immersion theory [6, 18] implies the embedding is *regularly homotopic* to an immersion which covers an immersion in Q and using configuration space models of multiple-loops-suspension spaces [7, 11, 12, 17] it can be seen that the embedding is *bordant* (by a bordism mapping to M) to an embedding which covers an immersion in Q, see [9]. Thus the new information which the compression theorem provides is that the embedding is *isotopic* to an embedding covering an immersion in Q. Moreover we can apply the compression theorem to give short and constructive proofs for both immersion theory and configuration space theory.

The Compression Theorem can also be used to give a new approach to the embeddings and knot problems for manifolds in codimension one or more, which leads to theoretical solutions to both problems.

The theorem also sheds light on the following problem:

C^0**–Singularity Problem** *Given $M \subset W$ and $p\colon W \to Q$ a submersion, how much control do we have over the singularities of $p|M$ if we are allowed a C^0–small isotopy of M in W?*

The problem includes the problem of controlling the singularities of a map $f\colon M \to Q$ by a C^0–small homotopy. This is because we can always factor f as $f \times q\colon M \subset Q \times \mathbb{R}^t \xrightarrow{\text{proj}} Q$ where $q\colon M \to \mathbb{R}^t$ is an embedding.

The Compression Theorem gives a necessary and sufficient condition for desingularising the projection in the case that $\dim(W) = \dim(Q)+1$ and $\dim(M) < \dim(Q)$, namely that there should exist an appropriate normal line field, and it can be extended to give necessary and sufficient conditions for singularities of almost any pre-specified type, namely that there should exist a line field with those singularities. Both these results extend to the case with just the hypothesis $\dim(M) < \dim(Q)$ where "line field" is replaced by "plane field". They also have natural relative and parametrised versions.

It is worth contrasting these results with the C^∞–singularity problem ("C^0–small" is replaced by "C^∞–small") where there is the classical Thom–Boardman classification of C^∞–stable singularities [1]. For most topological purposes (eg for applications to homotopy theory) the C^0 classification is more natural than the classical C^∞ classification. Furthermore essential singularities (up to C^0 homotopy) have natural interpretations as generalised bordism characteristic classes similar to those investigated by Korschorke [8] (see comments at the end of section 5).

The application to Immersion Theory is in section 2 and the application to configuration space theory in section 3. Section 4 contains the new approach to the embedding and knot problems and section 5 is about the C^0–Singularity Theorem mentioned above.

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2 Immersion theory

In the compression theorem, the existence of the immersion of M in Q follows from immersion theory; however immersion theory gives us no explicit information about this immersion, which is only determined up to regular homotopy. By contrast the compression theorem gives us an explicit description of the immersion in terms of the given embedding and normal vector field in $Q \times \mathbb{R}$. Moreover the compression theorem can be used to give a new proof for immersion theory as we now show.

Let M be an n–manifold. We shall explicitly describe a way of *rotating the fibres* of the tangent bundle TM into M. Regard the zero section M as 'vertical' and the fibres as 'horizontal'. Consider TM as a smooth $2n$–manifold, then its tangent bundle restricted to M is the Whitney sum $TM \oplus TM$. The two copies of TM are the vertical copy parallel to M and the horizontal copy parallel to the fibres of TM. Each vector $v \in TM$ then determines two vectors v_v and v_h in $TM \oplus TM$ which span a plane. In this plane we can 'rotate' v_h to v_v. Since we are not at this moment considering a particular metric 'rotation' needs to be defined: to be precise we consider the family of linear transformations of this plane given by

2.1
$$v_h \mapsto cv_h + sv_v, \quad v_v \mapsto cv_v - sv_h \qquad \text{where}$$
$$c = \cos \tfrac{\pi}{2}t, \quad s = \sin \tfrac{\pi}{2}t, \quad 0 \le t \le 1.$$

This formula (applied to each such plane) determines a bundle isotopy (a 1–parameter family of bundle isomorphisms) which is the required rotation of the fibres of TM into M.

Immersion Theorem 2.2 *Suppose that we are given a bundle monomorphism $f\colon TM \to TQ$ (ie, a map $M \to Q$ covered by a vector space monomorphism on each fibre) and that either $q-m \geq 1$ or $q-m \geq 0$ and each component of M has relative boundary. Then the restriction $f|\colon M \to Q$ is homotopic to an immersion.*

Proof Composing f with an exponential map for TQ gives a map $g\colon TM \to Q$ which embeds fibres into Q. Choose an embedding $q\colon M \to \mathbb{R}^n$ for some n and also denote by q the map $TM \to \mathbb{R}^n$ given by projecting the bundle TM onto M (the usual bundle projection) and then composing with q. We then have the embedding $g \times q\colon TM \to Q \times \mathbb{R}^n$. The fibres of TM are embedded parallel to Q and the n directions parallel to the axes of $\mathbb{R}^n$ determine n independent vector fields at M normal to the fibres of TM.

Now choose a complement for $T(TM) \,|\, M \cong TM \oplus TM$ in $T(Q \times \mathbb{R}) \,|\, M$. Then the rotation of the fibres of TM into M (formula 2.1 above) extends (by the identity on the complement) to a bundle isotopy of $T(Q \times \mathbb{R}) \,|\, M$ which carries the these n fields normal to M to yield n independent normal fields. The result now follows from the multi-compression theorem in part I [14; corollary 4.5]. □

The proof of the immersion theorem just given is very explicit, which contrasts with the standard Hirsch–Smale approach [6, 18] or the proofs given by Gromov [5]. Given a particular bundle monomorphism $TM \to TQ$ the proof can be used to construct a homotopic immersion $M \to Q$. The only serious element of choice in the proof is the embedding of M in $\mathbb{R}^n$. It is worth remarking that Eliashberg and Gromov [3; Theorem 4.3.4] have also given a short proof of immersion theory which yields an explicit immersion.

Notice that, in the proof just given, there was an explicit bundle homotopy of the n independent normal fields to M to the vertical fields. Thus using the Normal Deformation Theorem [14; Theorem 4.7] in place of the multi-compression theorem, gives a parametrised version of the theorem. It is easy to deduce the the usual statement of immersion theory from this version.

The parametrised proof can also be made very explicit by examining carefully the above proof; for full details see [16; section 6, page 31].

3 Loops–suspension theorem

We next show that the compression theorem can be used to give a short new proof of the classical result of James [7] on the homotopy type of loops–suspension and of the generalisation due to May [11] and Segal [17] and implicit in Milgram [12]. In [13] the arguments in this section are extended to both the equivariant case and the disconnected case (where group completions are needed).

We denote the free topological monoid on a based space X by X_∞ and denote the loop space on and suspension of X by $\Omega(X)$, $S(X)$, respectively. We assume spaces are compactly generated and Hausdorff and we assume base points non-degenerate. In particular this means (up to homotopy) we can assume based spaces have *whiskers*, ie, the base point has a neighbourhood homeomorphic to $\{1\}$ in the interval $[0,1]$. It will be convenient to identify the whisker with $[0,1]$.

There is a map $k_X\colon X_\infty \to \Omega S(X)$ defined in [7]. Briefly, what k_X does is to map the word $x_1 \cdots x_m$ to a loop in $S(X)$ which comprises m vertical loops passing through $x_1, \dots, x_m$ respectively, with the time parameters carefully adjusted to make the time spent on a subloop go to zero as the corresponding point x_i moves to the basepoint of X.

Theorem 3.1 *Let X be path-connected. Then $k_X\colon X_\infty \to \Omega S(X)$ is a weak homotopy equivalence.*

Proof There is a well-known equivalent definition of X_∞ (up to homotopy type) as the configuration space $C_1(X)$ of points in $\mathbb{R}^1$ labelled in X, of which we briefly recall the definition. Consider finite subsets of $\mathbb{R}^1$ labelled by points of X. An equivalence relation on such subsets is generated by deleting points labelled by the basepoint. $C_1(X)$ is the set of equivalence classes. The topology is induced from the topologies of $\mathbb{R}^1$ and X.

We shall give a geometric description for the homomorphism $\pi_n(C_1X) \to \pi_n(\Omega SX)$ induced by k_X. The construction is similar to a construction in [9], which used transversality to the base of a Thom space. Here we use transversality to X in $X \times \mathbb{R}$ and to an interior point of the whisker in X.

We start by giving a geometric description for a map into $C_1(X)$. Let Q be a smooth manifold and $f\colon Q \to C_1(X)$ a map. Then f determines a subset of $Q \times \mathbb{R}^1$, continuously labelled by X, such that at points labelled in $X - \{*\}$ the projection on Q is a local homeomorphism. This subset is determined up to introduction and deletion of points labelled by $*$. By approximating the local map to $\mathbb{R}^1$ by a smooth map, we can assume, by a small homotopy of f, that the subset of points labelled in $X - \{*\}$ is a smooth submanifold of $Q \times \mathbb{R}$. Further, by using transversality of the labelling map to an interior point $\frac{1}{2}$ in the whisker in X, and composing with the stretch $[0,\frac{1}{2}] \to [0,1]$, we may assume that this labelled subset is in fact a smooth submanifold W

with boundary, such that the boundary is the subset labelled by $*$ and such that the projection on Q is a local embedding (of codimension 0). Conversely, such a subset determines a map $f\colon Q \to C_1(X)$. Notice that W is canonically framed in $Q \times \mathbb{R}^1$ by the $\mathbb{R}^1$–coordinate and also notice that if Q and f are based then W can be assumed to be empty over the basepoint of Q.

Next we interpret maps in ΩSX. A map $Q \to \Omega SX$ determines a map $f\colon Q \times \mathbb{R}^1 \to SX$. By making f transverse to the suspension line through $\frac{1}{2}$ in the whisker in X, and composing with the stretch $[0, \frac{1}{2}] \to [0,1]$, we may assume that $\overline{f^{-1}(SX - \{*\})}$ is a codimension 0 submanifold whose boundary maps to $*_{SX}$. By further making f transverse to $X \times \{0\}$ we may assume that $f^{-1}((X - \{*\}) \times \{0\})$ is a framed codimension 1 submanifold W with boundary, equipped with a map $l\colon W \to X$ such that $l^{-1}* = \partial W$ and that f maps framing lines to suspension lines and the rest is mapped to $*_{SX}$. Conversely such a framed submanifold determines a map $Q \to \Omega SX$. Further if Q and f are based then W is empty over the basepoint of Q.

With these geometric descriptions there is an obvious forgetful map $[Q, C_1X] \to [Q, \Omega SX]$ which may be seen to be induced by k_X. Now consider the case when $Q = S^n$ and consider a framed manifold W representing an element of $\pi_n(SX)$. If it has a closed component we can change the labelling function to map a small disc to $*_X$ by using a collar on the disc and mapping the collar lines to a path to the basepoint in X. Then the interior of the disc, which is now labelled by $*_X$, may be deleted from W. After eliminating all closed components in this way the compression theorem (the codimension 0 case) implies that $\pi_n(C_1X) \to \pi_n(\Omega SX)$ is surjective.

Injectivity is proved similarly by using the case $Q = S^n \times I$ and working relative to $Q \times \{0, 1\}$. □

Notice that only the Global Compression Theorem [14; 2.1] (and addendum (i)) were used for the proof of 3.1 so the complete new proof is short. However, for the full loops–suspension theorem below we need the multi-compression theorem. (For the shortest proof of the full theorem, use the short proof of the multi-compression theorem given in [15].)

Let $C_n(X)$ denote the configuration space of points in $\mathbb{R}^n$ labelled in X, with, as before, points labelled by $*$ removable. There is a map $q_X\colon C_n(X) \to \Omega^n S^n(X)$ defined in a similar way to k_X (as interpreted in the last proof) using little cubes around the points in $\mathbb{R}^n$ with axes parallel to the axes of $\mathbb{R}^n$.

Theorem 3.2 *Let X be a path-connected topological space with a non-degenerate basepoint then q_X is a weak homotopy equivalence.*

Proof The proof is similar to the proof for theorem 3.1. A map $Q \to C_n(X)$ can be see as partial cover of Q embedded in $Q \times \mathbb{R}^n$ and a map $Q \to \Omega^n S^n(X)$ can be see as a framed codimension n submanifold of $Q \times \mathbb{R}^n$.

There is then a function $\mathrm{Maps}(Q \to C_n(X))$ to $\mathrm{Maps}(Q \to \Omega^n S^n(X))$ given by taking the parallel framing on the partial cover. This can be seen to be given by composition with q_X. Closed components in the submanifold of $Q \times \mathbb{R}^n$ may be punctured as in the last proof. The codimension 0 case of the multi-compression theorem now implies that q_X induces a bijection between the sets of homotopy classes. □

4 Embeddings and knots

Two basic problems of differential topology are the *embedding problem*: given two manifolds M and Q, decide whether M embeds in Q, and the *knot problem*: classify embeddings of M in Q up to isotopy. The corresponding problems for immersions (replace embedding by immersion and isotopy by regular homotopy) are, in some sense, solved by immersion theory; ie, solved by reducing to vector bundle problems for which there are standard obstruction theories. Now if M embeds in Q then it certainly immerses in Q and if two embeddings are isotopic then they are certainly regularly homotopic. Thus it makes sense to consider the *relative embedding problem*: decide whether a given immersion of M in Q is regularly homotopic to an embedding, and the *relative knot problem*: given a regular homotopy between embeddings, decide if it can be deformed to an isotopy. Since a manifold often immerses in a considerably lower dimension than that in which it embeds, it makes sense to consider the following more general problems.

Embedding problem 4.1 *Suppose given an immersion $f\colon M \to Q$ and an integer $n \geq 0$. Decide whether $f \times 0\colon M \to Q \times \mathbb{R}^n$ is regularly homotopic to an embedding.*

Knot problem 4.2 *Classify, up to isotopy, embeddings of M in $Q \times \mathbb{R}^n$ within the regular homotopy class of $f \times 0\colon M \to Q \times \mathbb{R}^n$, where $f\colon M \to Q$ is a given immersion.*

The compression theorem gives substantial information on these problems. In particular we can give formal solutions which make it easy to define obstructions to the existence of such embeddings or isotopies.

The embedding theorem

Turning first to the embedding problem, we have the following application of the compression theorem.

Proposition 4.3 *Suppose we are given an immersion $f\colon M \to Q$ and that $c = q - m \geq 2$. Suppose the immersion $f \times 0\colon M \to Q \times \mathbb{R}^n$ is regularly homotopic to an embedding g. Then $f\colon M \to Q$ is regularly homotopic to an immersion f_1 covered by an embedding isotopic to g (ie an embedding $g_1\colon M \to Q \times \mathbb{R}^n$ such that $g_1(x) = (f_1(x), 0)$ for $x \in M$).*

Proof Let $F\colon M \times I \to Q \times \mathbb{R}^n \times I$ be determined by the regular homotopy of $f \times 0$ to g. Thus $F_0 = F\,|\,(M \times \{0\}) = f \times 0$ and $F_1 = F\,|\,(M \times \{1\}) = g$. The canonical n–frame on F_0 extends to an n–frame on F. Apply the multi-compression theorem to F_1 to yield an isotopy of F_1 to $F_1' = g_1$ which compresses to an immersion $f_1\colon M \to Q$. Extend the isotopy of F_1 to a regular homotopy of F to F' say, rel F_0, and apply the parametrised multi-compression theorem [14] again to compress F' to a regular homotopy between f and g_1. □

Recall from section 3 that $C_n(X)$ denotes the configuration space of points in $\mathbb{R}^n$ labelled in X, with points labelled by $*$ removable.

Suppose that we are given an immersion $f\colon M \to Q$ covered by an embedding $e\colon M \to Q \times \mathbb{R}^n$. Under these circumstances Koschorke and Sanderson [9] define a classifying map $\alpha_f^n\colon Q \to C_n(MO_c)$ where MO_c is the Thom space of the universal c–vector bundle γ^c/BO_c. The definition is roughly as follows. The normal (disc) bundle ξ on the immersion f immerses in Q and each point $q \in Q$ then lies in the image of a number of the fibres. This defines a configuration in $\mathbb{R}^n$ by considering the covering immersion. The labelling in MO_c comes from using the classifying map for ξ.

In detail the embedding e has a normal bundle which splits a trivial n–plane bundle say $\nu = \xi \oplus \epsilon^n$ and the embedding extends to an embedding $e'\colon D(\nu) \to Q \times \mathbb{R}^n$ so that $D(\xi) \subset D(\nu) \xrightarrow{proj} Q$ gives a tubular neighbourhood of M in Q. We can assume that, for each $q \in Q$, $\{q\} \times \mathbb{R}^n \cap e'(D(\xi))$ is finite. Let $h\colon D(\xi) \to D(\gamma^c)$ be a classifying (disc) bundle map, where γ^c/BO_c is a universal vector bundle. Let MO_c denote the Thom space $T(\gamma^c)$. Then we have a map $\alpha_f^n\colon Q \to C_n(MO_c)$ defined by $\alpha_f^n(q)$ is the configuration $p(e'(D(\xi)) \cap \{q\} \times \mathbb{R}^n)$, where p is the projection on $\mathbb{R}^n$, labelled by the map $D(\xi) \xrightarrow{h} D(\gamma^c) \to MO_c$.

Koschorke and Sanderson [9] use this construction to classify up to bordism embeddings $e\colon M \to Q \times \mathbb{R}^n$ which project to immersions in Q as the set of homotopy classes $[Q, C_n(MO_c)]$. To classify embeddings and knots we need to refine this result to give a classification up to isotopy. For this we need to define more general configuration spaces. Suppose $X \supset Y$ and the base point $* \in X$ is not in Y. Define $C_n(X,Y)$ to be based finite subsets of $\mathbb{R}^n$ with labels in X but with base point (of the configuration) labelled in Y. The space is topologised in the usual way — as a disjoint union of products with equivalence relation defined by ignoring points labelled by $*$. Now define a subspace $C_n^0(X)$ of $C_n(X)$ by restricting configurations to lie in $\mathbb{R}^n \setminus \{0\} \subset \mathbb{R}^n$. Then there is a homeomorphism $C_n(X,Y) \xrightarrow{p_1 \times p_2} C_n^0(X) \times Y$ where p_1 is determined by linear translation of the configuration so that the base point is translated to $\{0\}$ and p_2 is determined by the base point label of the configuration.

Returning to our immersion f covered by embedding e we have a map $\beta_f^n\colon M \to C_n(MO_c, BO_c)$ given by observing that, for $x \in M$, the labelled configuration $\alpha_f^n f(x) \in C_n(MO_c)$ has a distinguished point, given by $e(x)$, which is labelled in BO_c.

Notation We shall use the short notation $\mathcal{I}_n^c$ for $C_n(MO_c)$ and $\mathcal{E}_n^c$ for $C_n(MO_c, BO_c)$. This notation is intended to remind us that $\mathcal{I}_n^c$ classifies immersions (up to bordism) and that $\mathcal{E}_n^c$ corresponds to covering embeddings.

We now have a pull back square:

$$\gamma_f^n\colon \quad \begin{tikzcd} M \arrow[r, "\beta_f^n"] \arrow[d, "f"'] & \mathcal{E}_n^c \arrow[d, "\text{nat}"] \\ Q \arrow[r, "\alpha_f^n"'] & \mathcal{I}_n^c \end{tikzcd}$$

In particular for *any* immersion $f\colon M \to Q$ there is a pull back square:

$$\gamma_f^\infty\colon \quad \begin{tikzcd} M \arrow[r, "\beta_f^\infty"] \arrow[d, "f"'] & \mathcal{E}_\infty^c \arrow[d, "\text{nat}"] \\ Q \arrow[r, "\alpha_f^\infty"'] & \mathcal{I}_\infty^c \end{tikzcd}$$

since we can choose an embedding in $Q \times \mathbb{R}^\infty$ covering f. The choice is unique up to isotopy. Note also that a regular homotopy of f may be covered by an isotopy in $Q \times \mathbb{R}^\infty$. A simple application of standard obstruction theory shows that in codimension 2 a concordance between immersions can be replaced by a regular homotopy, hence using proposition 4.3 we have the following.

Embedding Theorem 4.4 *An immersion $f\colon M \to Q$ with $c = q - m \geq 2$ is regularly homotopic to an immersion which is covered by an embedding in $Q \times \mathbb{R}^n$ if and only if there is a pullback diagram*

$$\gamma_F^\infty\colon \quad \begin{tikzcd} M \times I \arrow[r, "\beta_F^\infty"] \arrow[d, "F"'] & \mathcal{E}_\infty^c \arrow[d, "\mathrm{nat}"] \\ Q \times I \arrow[r, "\alpha_F^\infty"'] & \mathcal{I}_\infty^c \end{tikzcd}$$

with $f(x) = F(x,0)$, for all $x \in M$, $\alpha_F(q,1) \in \mathcal{I}_n^c$, and $\beta_F(q,1) \in \mathcal{E}_n^c$, for all $q \in Q$. □

Remark The theorem extends to codimension 1, but only up to concordance. To be precise, regular homotopy should be replaced by immersed concordance in both proposition 4.3 and the Embedding Theorem. This is seen by using the unparametrised multi-compression theorem in the proof of 4.3.

The condition of pullback homotopy in the theorem is difficult to interpret homotopically, and does not immediately reduce the embedding problem to obstruction theory. For a covering embedding to exist the map of Q to $\mathcal{I}_\infty^c$ must lift to $\mathcal{I}_n^c$. By [9] this is the obstruction to finding a solution up to bordism. To find a solution up to isotopy the map of M into $C_\infty^0(MO_c)$ must lift to $C_n^0(MO_c)$ and this defines new obstructions. The algebraic topology corresponding to these new obstructions will be investigated in a future paper.

We now describe some of the geometry involved in these obstructions. We start by sharpening our description of the Koschorke–Sanderson map $Q \to \mathcal{I}_n^c$ induced by an immersion $f\colon M \to Q$. Suppose that this immersion is self-transverse [10]. Then we have k-tuple point manifolds M_k and Q_k and a k-fold covering $f_k\colon M_k \to Q_k$, for $k \geq 1$, defined as follows.

$$Q_k = \{X | X \subset M, |X| = k \text{ and } |fX| = 1\}, M_k = \{(X,x) | x \in X \text{ and } X \in Q_k\}.$$

Further there is a commutative diagram:

$$\gamma_f\colon \quad \begin{tikzcd} M \arrow[r, "f"] & Q \\ M_k \arrow[u, "g_k"] \arrow[r, "f_k"'] & Q_k \arrow[u, "h_k"'] \end{tikzcd}$$

where $h_k(X) = f(x)$, for any $x \in X$, $g_k(X,x) = x$, and $f_k(X,x) = X$.

If f is an embedding there is a map $M \to BO_c$ classifying its normal bundle and a Thom construction $Q \to MO_c$ where MO_c is the universal Thom space with $c = q - m$. The Koschorke–Sanderson map $Q \to \mathcal{I}^c_\infty$ induced by an immersion is a generalisation. The immersion h_k has a normal bundle classified by a map $Q_k \to E\Sigma_k \times_{\Sigma_k} BO_c^{[k]}$ where $[k]$ indicates k-fold Cartesian product and Σ_k denotes the symmetric group on k elements. The immersion h_k is an embedding away from $> k$ multiple points. There is a corresponding Thom construction. We can replace $E\Sigma_k$ by $C(k, \mathbb{R}^\infty)$ — configurations of k distinct (ordered) points in $\mathbb{R}^\infty$. A point in the space $C(k, \mathbb{R}^\infty) \times_{\Sigma_k} BO_c^{[k]}$ may be regarded as a set of k distinct points in $\mathbb{R}^\infty$ each with a label in BO_c. Now the configuration space $C_\infty(X)$ (of points of $\mathbb{R}^\infty$ labelled in X) is $\coprod_k C(k, \mathbb{R}^\infty) \times_{\Sigma_k} X^{[k]}/\sim$ where the equivalence relation is given by ignoring points labelled at the base point. The immersion f now determines a map $Q \to \mathcal{I}^c_\infty = C_\infty(MO_c)$, which 'puts together' the Thom constructions for each of the multiple point manifolds. It is well defined up to homotopy and is determined by the choice of points in $\mathbb{R}^n$ ie by a commutative diagram:

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[dr, "f"'] & Q \times \mathbb{R}^\infty \arrow[d, "\text{proj}"] \\ & Q \end{tikzcd}$$

where e is an embedding.

This map is 'transverse' to each of the bases of the various products of Thom complexes and the pull-backs are the multiple point manifolds. The process reverses. Any map $Q \to \mathcal{I}^c_\infty$ can be made transverse in this way and thus determines a self-transverse immersion.

There is a similar description for the Koschorke–Sanderson map $Q \to \mathcal{I}^c_n$ determined by an immersion covered by an embedding in $Q \times \mathbb{R}^n$.

The multiple point manifolds (with all the normal bundle information) can be regarded as higher 'Hopf invariants' [9]. There are analogous interpretations for the maps $M \to \mathcal{E}^c_n$ or $\mathcal{E}^c_\infty$ corresponding to the covering embeddings in embeddings in $Q \times \mathbb{R}^n$ or $Q \times \mathbb{R}^\infty$. We describe the $\mathcal{E}^c_n$ case; the $\mathcal{E}^c_\infty$ is similar.

For $1 \le n \le \infty$ there is a k-fold covering $p_k \colon C^\bullet(k, \mathbb{R}^n) \to C(k, \mathbb{R}^n)$ where $C^\bullet(k, \mathbb{R}^n)$ denotes based sets $A \subset \mathbb{R}^n$, with $|A| = k$. We can write the space $C^0_n(X)$ as $\coprod_k C^\bullet(k, \mathbb{R}^n) \times_{\Sigma_{k-1}} X^{[k-1]}/\sim$ and in case $(X, Y) = (MO_c, BO_c)$ the map $proj \circ \beta^n_f \colon M \to C_n(X, Y) = \mathcal{E}^c_n$ puts together the Thom constructions for the g_k. Using the homeomorphism $\mathcal{E}^c_n \cong C^0_n(MO_c) \times BO_c$ we can see that this map classifies (a) the self-transverse system of multiple point sets in M and (b) the normal bundle on M in Q. Thus part of the second obstruction (to lifting $M \to \mathcal{E}^c_\infty$ to $\mathcal{E}^c_n$) is given by higher Hopf invariants for the multiple point sets in M.

Classification of knots

There is a similar analysis for the knot problem. Rather than continuing to consider embeddings in $Q \times \mathbb{R}^n$ which cover immersions in Q, we shall consider embeddings equipped with n independent normal vector fields (for example framed embeddings) since by the compression theorem these are equivalent. Such an embedding f determines a pull-back square:

$$\rho_f\colon \qquad \begin{array}{ccc} M & \xrightarrow{\beta_{f_1}} & \mathcal{E}^c_n \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \mathrm{nat}} \\ Q & \xrightarrow[\alpha_{f_1}]{} & \mathcal{I}^c_n \end{array}$$

where $f_1 = \mathrm{proj} \circ f$.

The arguments used to prove theorem 4.3 now prove:

Theorem 4.5 *Suppose that $f, g\colon M \to Q \times \mathbb{R}^n$ are embeddings equipped with n independent normal vector fields and that $q - m \geq 2$. Then f is isotopic to g if and only if ρ_f is homotopic to ρ_g by a pull-back homotopy.* □

Corollary 4.6 (Classification of knots) *Suppose that $q - m \geq 2$. There is a bijection between isotopy classes of embeddings f of M in $Q \times \mathbb{R}^n$ equipped with n independent normal vector fields and pull-back homotopy classes of squares ρ_f.* □

The corollary gives many knot invariants, for example any homotopy invariant of $\mathcal{I}^c_n$, pulled back to Q, or of $\mathcal{E}^c_n$, pulled back to M. The former are invariants of the cobordism class of the knot and the latter are new invariants. These both contain higher Hopf invariants (suitably generalised) as outlined above for the embedding problem.

Remarks (1) The last two results also hold in codimension 1, but with regular homotopy replaced by regular concordance (see the remark below theorem 4.4).

(2) The invariants and obstructions discussed above have strong connections with many existing invariants. The case $c = 1, n = 1$ is studied by Fenn, Rourke and Sanderson see in particular [4; sections 2 and 4], where classifying spaces

related to $\mathcal{I}_1^1$ but depending of the fundamental rack, are also considered. There are combinatorial invariants defined in this case, for example the generalised James–Hopf invariants, which link with the higher Hopf invariants described above.

We shall give more details here and also explain connections with other known obstructions and invariants in a subsequent paper.

5 Controlling singularities of a projection

Definition A *weakly stratified set* is a set X with a flag of closed subsets

$$X = S_0 \supset S_1 \supset S_2 \ldots \supset S_t \supset S_{t+1} = \emptyset$$

such that, for each $i = 0, \ldots, t$, $S_i - S_{i+1}$ is a manifold.

We also say that S_i is a *weak stratification* of X and we call the manifolds $S_i - S_{i+1}$ the *strata*.

Remarks This is very much weaker than the usual notion of a stratified set — there is no condition on the neighbourhood of S_{i+1} in S_i or any relationship between the dimensions of the strata.

Definitions Suppose that X is a weakly stratified set and that $X \subset W$ (a manifold). Suppose that ξ^n is a plane field on W (ie a n–subbundle of TW) defined at X. We say ξ is *weakly normal* to X if ξ is normal to $S_i - S_{i+1}$ for each $i = 0, \ldots, p$.

Suppose that $M^m \subset W^w$ and that ξ^n is a plane field defined at M. We say that ξ has *regular singularities on M* if it is normal to a weak stratification of M.

Example 5.1 A plane field in general position has regular singularities. However so do many plane fields which are far from general position. Here is an explicit example with $n = 1$ constructed by Varley [19]. Let α be the x–axis in $\mathbb{R}^3$ and $C \subset \alpha$ a cantor set. Let π be the surface (a smooth plane) given by $z = y^3$ which contains α and has a line of inflection along α. Let ξ be the line field parallel to the y–axis and distort ξ to have a small negative z–component off C. Then ξ is tangent to π precisely at C and very far from general position. But it is has regular singularities by choosing α as one stratum and π as the next.

It is easy to construct plane fields with non-regular singularities: for an example with $n = 1$, choose any line field for which part of a flow line lies in M.

C^0–Singularity Theorem 5.2 *Suppose that $M^m \subset W^w$ and that ξ is an integrable n–plane field on TW defined on a neighbourhood U of M such that ξ has regular singularities on M and that $n+m < w$. Suppose given $\varepsilon > 0$ and a homotopy of ξ through integrable plane fields on TW defined on U finishing with the plane field ξ'. Then there is an ambient isotopy of M in W which moves points at most ε moving the pair $M, \xi|M$ to $M', \xi'|M'$.*

The theorem gives an answer to the C^0–Singularity Problem (stated in section 1) in the case that $\dim(M) < \dim(Q)$. Take ξ' to be the tangent bundle to the fibres of p then the singularities of $p|M$ can be made to coincide with those of any plane field homotopic to ξ' with regular singularities.

Some condition on the singularities is clearly necessary, for example, again with $n = 1$, suppose that part of a flow line of the normal line field lies in M and is not already vertical (thinking of ξ' as vertical) then no *small* isotopy can make this field vertical. The condition of regularity is very weak as can be seen by considering examples similar to 5.1. The condition that the plane fields are *integrable* is needed for our proof, but we do not have an example to show that it is necessary for the result. Note that example 5.1 is integrable and see [19] for more examples.

Proof We shall prove a more general result: M is replaced by any weakly stratified set X such that dimensions of strata are $< w - n$ and ξ is weakly normal to X. The idea is to apply the Normal Deformation Theorem to each stratum in turn starting with S_t and continuing with $S_{t-1} - S_t$ etc). After the $t-i^{\text{th}}$ move ξ concides with ξ' on S_i and with care this can also be assumed to be true near S_i and in particular in a neighbourhood of S_i in S_{i-1}. The next move is made relative to a smaller neighbourhood, and the result is proved in $t + 1$ steps.

So the only point that needs work is the point that ξ can be assumed to concide with ξ' *near* S_i. This is where integrability is needed. By integrability we can assume that ξ is the tangent bundle to a foliation $\mathcal{F}$ of W defined near S_i and similarly ξ' is the tangent bundle to $\mathcal{F}'$. Now $\mathcal{F}$ and $\mathcal{F}'$ coincide at S_i and by a C^∞–small isotopy $\mathcal{F}$ can be moved to $\mathcal{F}'$ near S_i and this carries ξ into coincidence with ξ' near S_i as required. □

Addenda The proof works (indeed was given) for a weakly stratified set instead of a submanifold and has a natural relative version directly from the proof. There is also a parametrised version which follows by combining the proof of the parametrised Normal Deformation Theorem [14, page 425] or [15, page 439] with the last proof:

Suppose that we have a family of embeddings of M_t in W where $t \in K$ (a parameter manifold) together with integrable plane fields ξ_t for $t \in K$ having regular singularities with M_t. Suppose further that the singularities are "locally constant" over K (ie the whole situation is locally trivial). Suppose given a K–parameter homotopy of ξ_t through integrable plane fields to ξ'_t. Then there is a K–parameter family of small ambient isotopies carrying M_t, ξ_t to M'_t, ξ'_t.

Comments

(1) Varley [19; chapter 3] gives a more general parametrised theorem in which the singularities are not assumed locally constant. He assumes that ξ has regular singularities with the parametrised family as a whole (giving a weak stratification of $M \times K$) and further that the tangent bundle along the fibres has regular singularities with respect to the each stratum of this stratification.

(2) The philosophy of our solution to the C^0–singularity problem is that the singularities of an arbitrary plane field are instrinsic to the embedding $M \subset W$ and invariant under isotopy of the situation. Notice that the theorem completely controls the type of singularity which can arise.

(3) It is also worth commenting that Eliashberg [2] has proved strong results about controlling singularities under more regular assumptions.

(4) There is a very nice description in the metastable range where there is a single singularity manifold (in general position). This can be regarded as a fine bordism class in the sense of Koschorke [8] and is an instrinsic embedded characteristic class. For more detail see [19; chapter 5].

(5) In future papers (to be written jointly with Varley) we intend to provide further applications of this theorem to give algebraic topological information about the existence of singularities of specified types.

References

[1] **J M Boardman**, *Singularities of differentiable maps*, Publ. IHES, 33 (1967) 21–57

[2] **Y M Eliashberg**, *Surgery of singularities of smooth mappings*, Izv. Akad. Nauk SSSR, Ser. Mat. Tom 36 (1972) no. 6, English translation: Math. USSR-Izv. 6 (1972), 1302–1326

[3] **Y M Eliashberg**, **M Gromov**, *Construction of non-singular isoperimetric films*, Proc. Steklov Math. Inst. 116 (1971) 13–28

[4] **R Fenn**, **C Rourke**, **B Sanderson**, *James bundles and applications*, `http://www.maths.warwick.ac.uk/~cpr/ftp/james.ps`

[5] **M Gromov**, *Partial differential relations*, Springer–Verlag (1986)

[6] **M Hirsch**, *Immersions of manifolds*, Trans. Amer. Math. Soc. 93 (1959) 242–276

[7] **I James**, *Reduced product spaces*, Annals of Math. 62 (1955) 170–197

[8] **U Koschorke**, *Vector fields and other vector bundle morphisms – a singularity approach*, Springer Lecture Notes Series, 847 (1981)

[9] **U Koschorke**, **B Sanderson**, *Self-intersections and higher Hopf invariants*, Topology, 17 (1978) 283-290

[10] **R Lashof**, **S.Smale**, *Self-intersections of immersed manifolds*, J. Math. Mech. 8 (1959) 143-157

[11] **J P May**, *The geometry of iterated loop spaces*, Springer Lectures Notes Series, 271 Springer-Verlag (1972)

[12] **R J Milgram**, *Iterated loop spaces*, Annals of Math. 84 (1966) 386–403

[13] **Colin Rourke**, **Brian Sanderson**, *Equivariant configuration spaces*, J. London Math. Soc. 62 (2000) 544-552, `arXiv:math.GT/9711216`

[14] **Colin Rourke**, **Brian Sanderson**, *The compression theorem I*, Geometry and Topology 5 (2001) 399–429, `arXiv:math.GT/9712235`

[15] **Colin Rourke**, **Brian Sanderson**, *The compression theorem II: directed embeddings*, Geometry and Topology 5 (2001) 431–440, `arXiv:math.GT/0003026`

[16] **Colin Rourke**, **Brian Sanderson**, *The compression theorem*, pre-publication version `arXiv:math.GT/9712235v2`

[17] **G Segal**, *Configuration-spaces and iterated loop-spaces*, Invent. Math. 21 (1973) 213–221

[18] **S Smale**, *The classification of immersions of spheres in Euclidean spaces*, Annals of Math. 69 (1959) 327–344

[19] **A Varley**, *Controlling tangencies from the global viewpoint*, PhD thesis, Univ. of Warwick (2003)

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