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Non-triviality of the A-polynomial for knots in S^3

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Abstract The A-polynomial of a knot in S^3 defines a complex plane curve associated to the set of representations of the fundamental group of the knot exterior into $\mathrm{SL}_2\mathbb{C}$. Here, we show that a non-trivial knot in S^3 has a non-trivial A-polynomial. We deduce this from the gauge-theoretic work of Kronheimer and Mrowka on SU_2-representations of Dehn surgeries on knots in S^3. As a corollary, we show that if a conjecture connecting the colored Jones polynomials to the A-polynomial holds, then the colored Jones polynomials distinguish the unknot.

AMS Classification 57M25, 57M27; 57M50

Keywords Knot, A-polynomial, character variety, Jones polynomial

1 Introduction

Roughly speaking, the A-polynomial of a knot K in S^3 describes the $\mathrm{SL}_2\mathbb{C}$-representations of the knot complement, as viewed from the boundary. In a little more detail, let M be the exterior of K. The boundary of M is a torus, whose fundamental group $\pi_1(\partial M) = \mathbb{Z}^2$ comes with a natural meridian-longitude basis (μ, λ). Consider a representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ The restriction of ρ to $\pi_1(\partial M)$ has a simple form, since a pair of commuting 2-by-2 matrices are typically simultaneously diagonalizable, i.e. ρ can be conjugated so that:

$$\rho(\mu) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \qquad \text{and} \qquad \rho(\lambda) = \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix}.$$

The possible eigenvalues (M, L) of $(\rho(\mu), \rho(\lambda))$ as ρ varies form an complex algebraic subvariety of $\mathbb{C}^2$. The A-polynomial is the defining equation for the 1-dimensional part of this subvariety; that is, it describes a plane curve whose points correspond to the restrictions of representations to $\pi_1(\partial M)$.

The A-polynomial of a knot, which was introduced by Cooper et al. in [CCGLS], has deep connections to the topology and geometry of M As the group of isometries of hyperbolic 3-space is $\mathrm{PSL}_2\mathbb{C}$, the A-polynomial is connected to the study of deformations of (incomplete) hyperbolic structures on M. For example, the variation of the volume of hyperbolic structures on M depends only on their restriction to the boundary torus, and is controlled entirely by the A-polynomial. On the topological side, the sides of the Newton polygon of the A-polynomial give rise to incompressible surfaces in M.

Here, we address the basic question: can A-polynomial distinguish the unknot from all other knots in S^3? The A-polynomial of the unknot is simply $L-1$. The A-polynomial always contains a factor of $L-1$ coming from reducible representations; we say that the A-polynomial is non-trivial if it has an additional factor. Perhaps for some non-trivial knots, there are no other representations, or they don't deform in ways that change the holonomy on the boundary. Our main result shows that this does not happen, and hence the A-polynomial distinguishes the unknot:

1.1 Theorem *A non-trivial knot in S^3 has a non-trivial A-polynomial. Moreover, the A-polynomial is not a power of $L-1$.*

Steve Boyer and Xingru Zhang independently proved Theorem 1.1 using a similar approach [BZ].

We deduce Theorem 1.1 as a direct corollary of the following deep theorem of Kronheimer and Mrowka:

1.2 Theorem [KM] *Let K be a non-trivial knot in S^3. For $r \in \mathbb{Q}$, let M_r be the 3-manifold which is the r Dehn surgery on K. If $|r| \leq 2$, then there exists a homomorphism $\pi_1(M_r) \to \mathrm{SU}_2$ with non-cyclic image.*

Their proof uses gauge theory; in addition to their own major contributions, the proof relies on Gabai's theorem that the zero-surgery on knot has a taut-foliation, Eliashberg and Thurston's work connecting foliations to contact structures, Eliashberg's proof that contact 3-manifolds embed in symplectic 4-manifolds, Taubes' non-vanishing theorem for Seiberg-Witten invariants of symplectic 4-manifolds, and Feehan and Leness' work connecting the Seiberg-Witten and Donaldson invariants.

Theorem 1.1 was previously known for all non-satellite knots for simple geometric reasons, as we now describe. When M is hyperbolic, we have the holonomy representation $\pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ of the complete hyperbolic structure; Thurston showed in his Hyperbolic Dehn Surgery Theorem that this representation has a complex curve of deformations which change the holonomy along the boundary

[Th]. Thus, in this case, the A-polynomial is non-trivial. Non-hyperbolic knots are torus knots or satellites. For torus knots, a simple calculation shows they have non-trivial A-polynomial [CCGLS]. Satellite knots are those which have closed incompressible tori in their complements. One can look at the resulting geometric decomposition, and try to understand how the representations of each piece could glue together to give a representation of all of $\pi_1(M)$; however, this seems quite difficult to do in general.

Since our proof of Theorem 1.1 is based on the existence of SU_2 representations, we really show that if one looks only at representations $\rho\colon \pi_1(M) \to \mathrm{SU}_2$, then the eigenvalues (M, L) of $(\rho(\mu), \rho(\lambda))$ sweep out a real 1-dimensional subset of the unit torus in $\mathbb{C}^* \times \mathbb{C}^*$. This is interesting even in the case of hyperbolic knots.

1.3 Connection to the Jones polynomial

While the A-polynomial arose from the study of hyperbolic geometry, it turns out to have connections to seemingly disparate parts of low-dimensional topology, including the Jones polynomial. As we will now explain, the non-triviality of the A-polynomial of a knot has implications to the strength of the colored Jones function. The latter is essentially the sequence of Jones polynomials of a knot and its connected parallels. In [GL], it was proven that the colored Jones function of a knot is a sequence of Laurent polynomials which satisfy a q-difference equation. It was observed by the second author in [Ga] that one can choose the q-difference equation in a canonical manner. The corresponding operator to this q-difference equation is an element of the non-commutative ring

$$\mathbb{Z}[q^{\pm}]\langle Q^{\pm}, E^{\pm}\rangle/(EQ - qQE)$$

of Laurent polynomials in E and Q that satisfy the commutation relation $EQ = qQE$.

This operator defines the so-called non-commutative A-polynomial of a knot. In [Ga], the second author conjectured that specializing the non-commutative A-polynomial at $q = 1$ coincides with the A-polynomial of a knot after the change of variables $(E, Q) = (L, M^2)$ (there may also be changes in the multiplicities of factors and polynomials in Q). This is called the AJ Conjecture, and an immediate consequence of Theorem 1.1 is:

1.4 Corollary *If the AJ Conjecture holds, then the colored Jones function distinguishes the unknot.*

1.5 Connection to contact homology

Another surprise is that the A-polynomial is connected with contact geometry. Consider the unit conormal bundle to $\mathbb{R}^3$, denoted $ST^*(\mathbb{R}^3)$, which has a natural contact structure. If K is a knot in $\mathbb{R}^3$ then the unit conormal bundle to K is a Legendrian 2-torus L inside $ST^*(\mathbb{R}^3)$. Lenny Ng has constructed a homology theory for knots in S^3, the *framed knot contact homology*, which is strongly believed to be Eliashberg-Hofer of contact homology of the pair $(ST^*(\mathbb{R}^3), L)$ [Ng]. Ng has shown that the A-polynomial can be derived from the simplest piece of the framed knot contact homology. Combining this with Theorem 1.1, he proves:

1.6 Theorem [Ng, Prop. 5.9] *The framed knot contact homology distinguishes the unknot from any other knot in S^3.*

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2 Proofs

We begin by reviewing the definition of the A-polynomial for a compact 3-manifold M whose boundary is a torus (for details, see [Sh, CCGLS]). Let $R(M)$ denote the set of representations $\pi_1(M) \to \mathrm{SL}_2\mathbb{C}$, which is an affine algebraic variety over $\mathbb{C}$. It is natural to study such representations up to inner automorphisms of $\mathrm{SL}_2\mathbb{C}$, so consider the *character variety*, $X(M)$, which is the quotient of $R(M)$ under the action of $\mathrm{SL}_2\mathbb{C}$ by conjugation. Technically, one has to take the algebro-geometric quotient to deal with orbits of reducible representations which are not closed; in this way $X(M)$ is also an affine complex algebraic variety.

To define the A-polynomial, we first need to understand the character variety $X(\partial M)$ of the torus ∂M. The fundamental group of ∂M is just $\mathbb{Z} \times \mathbb{Z}$, and fix generators (μ, λ). Since $\pi_1(\partial M)$ is commutative, any representation $\rho\colon \pi_1(\partial M) \to \mathrm{SL}_2\mathbb{C}$ is reducible, that is, has a global fixed point for the Möbius action on $P^1(\mathbb{C})$. Moreover, if no element of $\rho(\pi_1(\partial M))$ is parabolic, ρ is conjugate to a diagonal representation with

$$\rho(\mu) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \qquad \text{and} \qquad \rho(\lambda) = \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix}.$$

As such, $X(\partial M)$ is approximately $\mathbb{C}^* \times \mathbb{C}^*$ with coordinates being the eigenvalues (M, L). This isn't quite right, as switching (M, L) with (M^{-1}, L^{-1}) gives a conjugate representation. In fact, $X(\partial M)$ is exactly the quotient of $\mathbb{C}^* \times \mathbb{C}^*$ under the involution $(M, L) \mapsto (M^{-1}, L^{-1})$.

Now the inclusion $i\colon \partial M \to M$ induces a regular map $i^*\colon X(M) \to X(\partial M)$ via restriction of representations from $\pi_1(M)$ to $\pi_1(\partial M)$. Let V be the (complex) 1-dimensional part of $i^*\big(X(M)\big)$. More precisely, take V to be the union of the 1-dimensional $i^*(X)$, where X is an irreducible component of $X(M)$. The curve V is used to define the A-polynomial. To simplify things, we look at the plane curve $\overline{V}(M)$ which is inverse image of V under the quotient map $\mathbb{C}^* \times \mathbb{C}^* \to X(\partial M)$ The A-polynomial is the defining equation for $\overline{V}(M)$; it is a polynomial in the variables M, L. Since all the maps involved are defined over $\mathbb{Q}$, the A-polynomial can be normalized to have integral coefficients.

In the definition of the A-polynomial, we looked only at those irreducible components where $i^*(X)$ is 1-dimensional. In the proof of Theorem 1.1, we will need the following standard lemma to show that we do not overshoot our goal of showing that the A-polynomial is non-trivial.

2.1 Lemma *Let X be an irreducible component of $X(M)$. Then $i^*(M)$ has dimension 0 or 1.*

Proof There are two proofs of this in the literature, and we include sketches of both to make this paper more self-contained.

To prove the lemma, we just need to rule out the possibility that $i^*(X)$ is 2-dimensional, and thus a Zariski-open subset of $X(\partial M)$. The approach in [CCGLS, §4.5] is to introduce the notion of the volume of a representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ (see also [Du, §2.5] and [Fr, §4] for a more complete definition of the volume). This gives a natural function $\mathrm{Vol}\colon X(M) \to \mathbb{R}$. Then Schläfli's formula for the change of volume of a family of polyhedra in $\mathbb{H}^3$ shows that the derivative of Vol depends only on the restriction of representations to $\pi_1(\partial M)$. This leads to a 1-form on $X(\partial M)$ which must be exact on $i^*(X(M))$. This form is not exact on any Zariski-open subset of $X(\partial M)$, and hence $i^*(X)$ is at most 1-dimensional.

The other argument is to observe that if $i^*(X)$ were 2-dimensional, it would let us construct ideal points of $X(M)$ where the associated surface has whatever boundary slope we want. This would contradict Hatcher's finiteness theorem for boundary slopes. In more detail, start with a slope $\alpha \in \pi_1(\partial M)$ and let β be a complementary slope. Choose a $c \in \mathbb{C}$ so that the curve Y in $X(\partial M)$ given by

$\mathrm{tr}_\alpha = c$ has $i^*(X) \cap Y$ dense in Y. Choose a curve $\widetilde{Y} \subset X$ whose image under i^* is dense in Y. As tr_β is non-constant on Y, there is an ideal point p of $\widetilde{Y}$ where tr_β has a pole. Since tr_α is constant on Y, an incompressible surface associated to the ideal point p must have boundary slope α. But Hatcher showed that there are only finitely many α which are boundary slopes of incompressible surfaces [H], a contradiction. □

When M is the exterior of a knot in S^3, then, up to orientation conventions, there is a canonical meridian-longitude basis (μ, λ) for $\pi_1(\partial M)$, and one uses this basis when writing the A-polynomial. Since we are interested in the non-triviality of the A-polynomial, we need to discuss the conventions for dealing with the reducible representations. When M is the exterior of a knot in S^3, one has $H_1(M, \mathbb{Z}) = \mathbb{Z}$, and so there are many reducible representations which factor via: $\pi_1(M) \to \mathbb{Z} \to \mathrm{SL}_2\mathbb{C}$. Irreducible components of $X(M)$ either consist solely of reducible representations, or have a Zariski-open subset of irreducible representations. In the case of the exterior of a knot in S^3, there is a single irreducible component of $X(M)$ consisting entirely of reducible representations. This component contributes a factor of $L-1$ to the A-polynomial. Some authors exclude this factor from the A-polynomial, and define the curve V above to be the image under i^* of the irreducible components of $X(M)$ which contain an irreducible representation. To say the A-polynomial is non-trivial, we mean that it does not just consist of the $L-1$ coming from the reducible representations. We will now show that the A-polynomial of a non-trivial knot in S^3 is non-trivial, and, moreover, is not just a power of $L-1$.

Proof of Theorem 1.1 Let M be the exterior of a non-trivial knot in S^3. Let $X'(M)$ denote $X(M)$ minus the component consisting of reducible representations, and let V' be the union of the 1-dimensional $i^*(X)$ where X is an irreducible component of $X'(M)$. The main part of the theorem is that V' is non-empty. To this end, we will show:

2.2 Claim *There exists an infinite collection of irreducible representations $\rho_n\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ whose restrictions to $\pi_1(\partial M)$ are all distinct in $X(\partial M)$.*

Before proving the claim, let us deduce $V' \neq \emptyset$ from it. Assuming the claim, then as a 0-dimensional algebraic variety consists of finitely many points, there must be some irreducible X in $X'(M)$ so that the dimension of $i^*(X)$ is at least 1. By Lemma 2.1, the dimension of $i^*(X)$ must be exactly one, and so $V' \neq \emptyset$.

To prove the claim, we use the SU_2 representations given by Theorem 1.2. Let $M_{1/n}$ denote the $1/n$-filling of M. By Theorem 1.2, for each non-zero $n \in \mathbb{Z}$ we have a representation $\rho_n\colon \pi_1(M_{1/n}) \to SU_2$ with non-cyclic image. First, we claim that the ρ_n are irreducible as representations into the larger group $SL_2\mathbb{C}$. Suppose ρ_n were reducible. Since $H_1(M_{1/n}, \mathbb{Z}) = 0$, the group $G = \pi_1(M_{1/n})$ satisfies $G = [G, G]$. As ρ_n is reducible, and commutators of elements of $SL_2\mathbb{C}$ with a common fixed point are parabolic with trace 2, it follows that $\mathrm{tr}(\rho_n(\gamma)) = 2$ for all $\gamma \in G$. But the only element of SU_2 with trace 2 is the identity, and so ρ_n would be trivial, a contradiction. So ρ_n is irreducible.

As $\pi_1(M_{1/n})$ is a quotient of $\pi_1(M)$, we will regard ρ_n as a representation of $\pi_1(M)$ into $SU_2 \leq SL_2\mathbb{C}$. To prove Claim 2.2, we need to show that the restrictions of the ρ_n to $\pi_1(\partial M)$ gives an infinite collection of points in $X(\partial M)$. Two representations of $\pi_1(\partial M)$ into SU_2 which correspond to the same point in $X(\partial M)$ are actually conjugate—this is because they both must be conjugate to diagonal representations (this isn't quite true for $SL_2\mathbb{C}$, where distinct parabolic representations get amalgamated). Because of this, to prove the Claim 2.2 it suffices to show that the kernels K_n of the ρ_n give an infinite collection of distinct subgroups of $\pi_1(\partial M) = \mathbb{Z}^2$.

For α a slope in ∂M, note that ρ_n extends to $\pi_1(M_\alpha)$ if and only if $\alpha \in K_n$. As ρ_n comes from $M_{1/n}$, we have $(1, n) \in K_n$ for each $n \neq 0$. As the $1/0$ filling gives S^3, which is simply connected, we have $(1, 0) \notin K_n$. Because of this, Claim 2.2 follows from directly from the following lemma with γ the line $x = 1$:

2.3 Lemma *Suppose γ is a line in $\mathbb{R}^2$ which contains infinitely many lattice points of $\mathbb{Z}^2$, and which does not contain 0. Consider a collection K_n of subgroups of $\mathbb{Z}^2$ whose union, K, contains all but finitely many of the lattice points on γ. Suppose, in addition, that there is a lattice point on γ which is not in K. Then there are infinitely many distinct K_n.*

Proof Assume that there are finitely many K_n. If K_n has rank less than 2, then K_n is contained in a line through the origin, and so K_n intersects γ in at most one point. So we can throw out all of the K_n of rank less than 2, and still have $\gamma - K$ finite.

So we can assume that $\mathbb{Z}^2/K_n$ is finite for each n. Let L be the intersection of K_n; as there are finitely many K_n, the subgroup L is also a finite-index subgroup of $\mathbb{Z}^2$. Now let γ' be the line parallel to γ which passes through the origin. As $\mathbb{Z}^2/L$ is finite, the subgroup $H = \gamma' \cap L$ is infinite. Let v_0 be the given point in $\gamma \setminus K$. Then if $h \in H$, we have that $v_0 + h$ is also in $\gamma \setminus K$ since if $v_0 + h$ is in some K_n, then so is $v_0 = (v_0 + h) - h$. But H is infinite, and thus so is $\{v_0 + h\}$, which contradicts that $\gamma \setminus K$ is finite. Thus we must have an infinite collection of distinct K_n. □

To complete the proof of Theorem 1.1, we need to show that the A-polynomial of M is not a power of $L - 1$. Assume the contrary Consider the point $p_n = (m_n, l_n) \in \mathbb{C}^* \times \mathbb{C}^*$ corresponding to the restriction of the representation ρ_n to $\pi_1(\partial M)$. As ρ_n comes from the $(1, n)$ filling of M, we have that $m_n l_n^n = 1$. By the above argument, all but finitely many of the pairs (m_n, l_n) satisfy the A-polynomial, and hence $l_n = 1$. Then for such n, the relation $m_n l_n^n = 1$ implies that $m_n = 1$. As ρ_n has image in SU_2, this implies that ρ_n is trivial when restricted to $\pi_1(\partial M)$. But then ρ_n factors over to the S^3 surgery, a contradiction. Thus the A-polynomial is not a power of $L - 1$. □

2.4 Remark Lemma 2.3 has other applications to studying Dehn filling. For instance, consider a non-trivial knot K in S^3 with exterior M. In relation to the Virtual Haken Conjecture, this lemma implies there is a infinite sequence n_k of non-zero integers so that the degree of the smallest non-trivial cover of M_{1/n_k} goes to infinity as $k \to \infty$.

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