ISSN 1472-2739 (on-line) 1472-2747 (printed) **1083**

Algebraic & Geometric Topology
Volume 4 (2004) 1083–1101
Published: 21 November 2004

Span of the Jones polynomial of an alternating virtual link

Naoko Kamada

Abstract For an oriented virtual link, L.H. Kauffman defined the f-polynomial (Jones polynomial). The supporting genus of a virtual link diagram is the minimal genus of a surface in which the diagram can be embedded. In this paper we show that the span of the f-polynomial of an alternating virtual link L is determined by the number of crossings of any alternating diagram of L and the supporting genus of the diagram. It is a generalization of Kauffman-Murasugi-Thistlethwaite's theorem. We also prove a similar result for a virtual link diagram that is obtained from an alternating virtual link diagram by virtualizing one real crossing. As a consequence, such a diagram is not equivalent to a classical link diagram.

AMS Classification 57M25; 57M27

Keywords Virtual knot theory, knot theory

1 Introduction

An (oriented) *virtual link diagram* is a closed (oriented) 1-manifold generically immersed in $\mathbf{R}^2$ such that each double point is labeled to be either (1) a *real* crossing which is indicated as usual in classical knot theory or (2) a *virtual* crossing which is indicated by a small circle around the double point. The moves of virtual link diagrams illustrated in Figure 1 are called *generalized Reidemeister moves*. Two virtual link diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves. A *virtual link* [2, 9] is the equivalence class of a virtual link diagram. Unless otherwise stated, we assume that a virtual link is oriented.

Kauffman defined the f-polynomial $f_D(A) \in \mathbf{Z}[A, A^{-1}]$ of a virtual link diagram D, which is preserved under generalized Reidemeister moves, and hence it is an invariant of a virtual link. It is also called the *normalized bracket polynomial* or the *Jones polynomial* [9]. For a virtual link L represented by a virtual link diagram D, we define the f-polynomial $f_L(A)$ of L by $f_D(A)$. The *span*

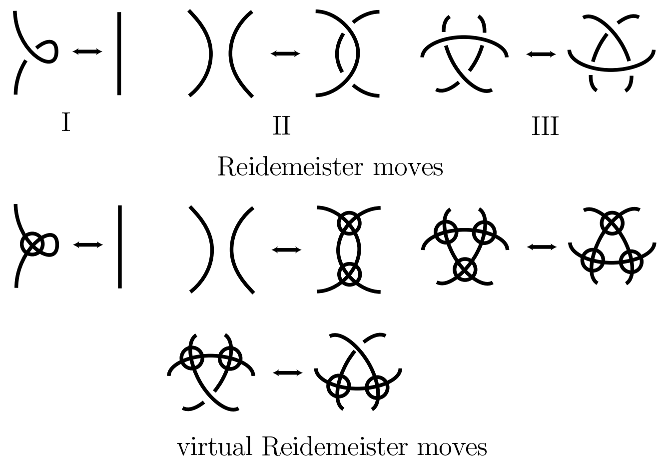


Figure 1

of $f_L(A)$ is the maximal degree of $f_L(A)$ minus the minimal. It is an invariant of a virtual link. We denote it by $\mathrm{span}(L)$ or $\mathrm{span}(D)$.

By $c(D)$, we mean the number of real crossings of D.

Theorem 1.1 (Kauffman [7], Murasugi [13], Thistlethwaite [14]) *Let L be an alternating link represented by a proper alternating connected link diagram D. Then we have*

$$\mathrm{span}(L) = 4c(D).$$

Any virtual link diagram D can be realized as a link diagram in a closed oriented surface [9]. The *supporting genus* $g(D)$ of D is the minimal genus of a closed oriented surface in which the diagram can be realized [5].

Note that $g(D)$ can be calculated. Consider a link diagram $\mathcal{D}$ in a closed oriented surface F that realizes D. If some regions of the complement of $\mathcal{D}$ in F are not open disks, replace them with open disks. Then we obtain a link diagram realizing D in a surface of genus $g(D)$. Alternatively we may also use a formula presented in Lemma 2.2.

Let D be a virtual link diagram. By forgetting crossing information, it is the union of immersed circles, say $C_1, \cdots, C_\mu$ (for some $\mu \in \mathbf{N}$). The restriction of D to each C_i is called a *component* of D, and D is also called a *μ-component* virtual link diagram. To state our results, we need the notion of a connected component of D: Consider an equivalence relation on $C_1, \cdots, C_\mu$ that is the transitive closure of binary relation $C_i \sim C_j$ where $C_i \sim C_j$ means that $C_i \cap C_j$

has at least one real crossing. Then, for an equivalence class $\{C'_1, \cdots, C'_\lambda\}$, the restriction of D to $C'_1 \cup \cdots \cup C'_\lambda$ is called a *connected component* of D. When D is a connected component of itself, we say that D is *connected*.

Theorem 1.2 *Let L be an alternating virtual link represented by a proper alternating virtual diagram D. Then we have*

$$\mathrm{span}(L) = 4(c(D) - g(D) + m - 1),$$

where m is the number of the connected components of D. In particular, if L is an alternating virtual link represented by a proper alternating connected virtual link diagram D. Then we have

$$\mathrm{span}(L) = 4(c(D) - g(D)).$$

Since the supporting genus of a classical link diagram is zero, Theorem 1.2 is a generalization of Theorem 1.1.

A similar result was proved in [3] for a link diagram in a closed oriented surface. Our argument is essentially the same with that in [3], whose basic idea is to use abstract link diagrams.

When a virtual link diagram D' is obtained from another diagram D by replacing a real crossing p of D with a virtual crossing, then we say that D' is obtained from D by *virtualizing* the crossing p.

A virtual link diagram D is said to be a *v-alternating* if D is obtained from a proper alternating virtual link diagram by virtualizing one real crossing.

Theorem 1.3 *Let D be a v-alternating virtual link diagram. Then we have*

$$\mathrm{span}(D) = 4(c(D) - g(D) + m - 1) + 2,$$

where m is the number of connected components of D. In particular, if D is a connected v-alternating virtual link diagram, then

$$\mathrm{span}(D) = 4(c(D) - g(D)) + 2.$$

T. Kishino [10] proved that $\mathrm{span}(D) = 4c(D) - 2$ when D is a connected v-alternating virtual link diagram which is obtained from a proper alternating classical link diagram by virtualizing a crossing. His result is a special case of Theorem 1.3, since $g(D) = 1$ for such a diagram D (Lemma 4.5).

Corollary 1.4 *Let D be a v-alternating virtual link diagram. Then D is not equivalent to a classical link diagram.*

Proof By Theorem 1.3, $\mathrm{span}(D)$ is not a multiple of four. On the other hand, the span of the f-polynomial of a classical link is a multiple of four [7, 13, 14]. Thus we have the result. □

2 Definitions

Let D be an unoriented virtual link diagram. The replacement of the diagram in a neighborhood of a real crossing as in Figure 2 are called *A-splice* and *B-splice*, respectively [7, 8].

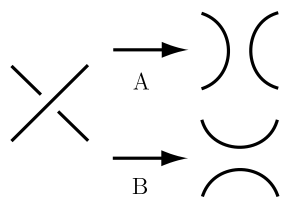


Figure 2

A *state* of D is a virtual link diagram obtained from D by doing A-splice or B-splice at each real crossing of D. The *Kauffman bracket polynomial* $\langle D\rangle$ of D is defined by

$$\langle D\rangle = \sum_{S} A^{\natural(S)}(-A^2 - A^{-2})^{\sharp(S)-1},$$

where S runs over all states of D, $\natural(S)$ is the number of A-splice minus that of B-splice used to obtain the state S, and $\sharp(S)$ is the number of loops of S.

For an oriented virtual link diagram D, the *writhe* $\omega(D)$ is the number of positive crossings minus that of negative crossings of D. The f*-polynomial* of D is defined by

$$f_D(A) = (-A^3)^{-\omega(D)}\langle D\rangle.$$

Theorem 2.1 [9] *The f-polynomial is an invariant of a virtual link.*

For a virtual link L represented by D, the f-polynomial $f_L(A)$ of L is defined by $f_D(A)$. When L is a classical link, the f-polynomial $f_L(A)$ is equal to the Jones polynomial $V_L(t)$ after substituting A^4 for t.

A pair $P = (\Sigma, \mathcal{D})$ of a compact oriented surface Σ and a link diagram $\mathcal{D}$ in Σ is called an *abstract link diagram* (ALD) if $|\mathcal{D}|$ is a deformation retract of Σ, where $|\mathcal{D}|$ is a graph obtained from $\mathcal{D}$ by replacing each crossing with a vertex. If $\mathcal{D}$ is an oriented link diagram, then P is said to be *oriented*. Unless otherwise stated, we assume that an ALD is oriented. If $|\mathcal{D}|$ is connected (or equivalently, Σ is connected), then P is said to be *connected*. Two examples of connected ALDs are illustrated in Figure 3 (a) and (b).

Let $P = (\Sigma, \mathcal{D})$ be an ALD. For a closed oriented surface F, if there exists an embedding $h\colon \Sigma \longrightarrow F$, then $h(\mathcal{D})$ is a link diagram in F. We call $h(\mathcal{D})$ a *link diagram realization* of $P = (\Sigma, \mathcal{D})$ in F. Figure 3 (c) and (d) are link diagram realizations of the ALDs illustrated in Figure 3 (a) and (b), respectively.

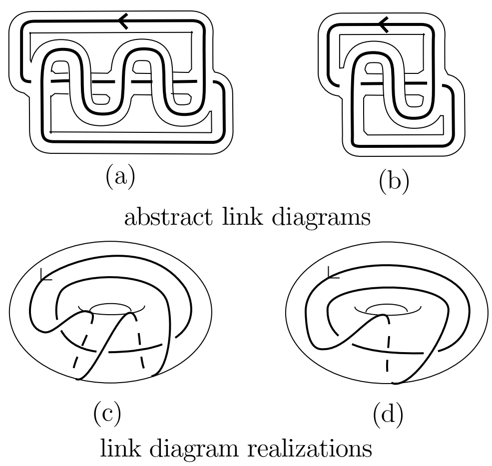


Figure 3

The *supporting genus* $g(P)$ of $P = (\Sigma, \mathcal{D})$ is the minimal genus of a closed oriented surface in which Σ can be embedded [5].

Lemma 2.2 *Let $P = (\Sigma, \mathcal{D})$ be an ALD, which is the disjoint union of m connected ALDs. Then*

$$g(P) = \frac{2m + c(\mathcal{D}) - \sharp\partial\Sigma}{2},$$

where $c(\mathcal{D})$ is the number of crossings of $\mathcal{D}$, $\partial\Sigma$ is the boundary of the surface Σ and $\sharp\partial\Sigma$ is the number of connected components of $\partial\Sigma$.

Proof of Lemma 2.2 Let F be a closed oriented surface which is obtained from Σ by attaching $\sharp\partial\Sigma$ disks to Σ along the boundary $\partial\Sigma$. Then $g(P) = g(F)$. Since F has m connected components, the Euler characteristic $\chi(F)$ is $2m - 2g(F)$. On the other hand, $\chi(F) = \chi(\Sigma) + \sharp\partial\Sigma = \chi(|\mathcal{D}|) + \sharp\partial\Sigma = -c(D) + \sharp\partial\Sigma$, since $\mathcal{D}$ is a 4-valent graph with $c(\mathcal{D})$ vertices (possibly with circle components). Thus we have the equality. $\square$

Let D be a virtual link diagram. Consider a link diagram realization $\mathcal{D}$ of D in a closed oriented surface F and take a regular neighborhood $N(\mathcal{D})$ of $\mathcal{D}$ in F.

Then $(N(\mathcal{D}), \mathcal{D})$ is an ALD. We call it the *ALD associated with* D, and denote it by $\phi(D)$. (Note that $\phi(D)$, up to homeomorphism, does not depend on the choice of F and the realization $\mathcal{D}$ in F.) An easy method to obtain $\phi(D)$ is illustrated in Figure 4 (see [5] for details). For example, the ALDs illustrated in Figure 3 (a) and (b) are the ALDs associated with the virtual link diagrams in Figure 5 (a) and (b), respectively.

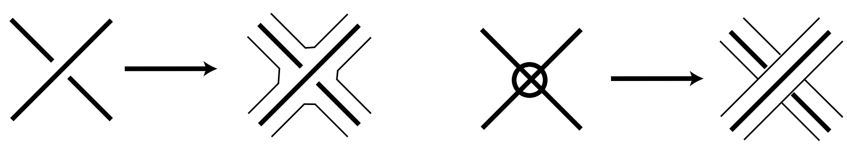
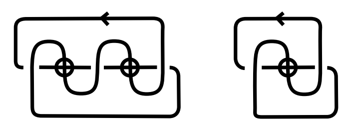


Figure 4



(a) (b)

Figure 5

Lemma 2.3 *Let D be a virtual link diagram and let $\phi(D) = P = (\Sigma, \mathcal{D})$ be the ALD associated with D. Then we have*

(1) $g(P) = g(D)$

(2) *P is connected if and only if D is connected.*

Proof It is obvious from the definition. □

Remark Let $P = (\Sigma, \mathcal{D})$ and $P' = (\Sigma', \mathcal{D}')$ be ALDs. We say that P' is obtained from P by an *abstract Reidemeister move* if there are embeddings $h\colon \Sigma \longrightarrow F$ and $h'\colon \Sigma' \longrightarrow F$ into a closed oriented surface F such that the link diagram $h(\mathcal{D}')$ is obtained from $h(\mathcal{D})$ by a Reidemeister move in F. Two ALDs $P = (\Sigma, \mathcal{D})$ and $P' = (\Sigma', \mathcal{D}')$ are *equivalent* if there exists a finite sequence of ALDs, $P_0, P_1, \cdots, P_u$, with $P_0 = P$ and $P_u = P'$ such that P_{i+1} is obtained from P_i by an abstract Reidemeister move. An *abstract link* is such an equivalence class (cf. [5]). It is proved in [5] that two virtual link diagrams D and D' are equivalent if and only if the associated ALDs, $\phi(D)$ and $\phi(D')$, are equivalent; namely, the map

$$\phi\colon \{\text{virtual link diagrams}\} \longrightarrow \{\text{abstract link diagrams}\}$$

induces a bijection

$$\{\text{virtual links}\} \longrightarrow \{\text{abstract links}\}.$$

Let $P = (\Sigma, \mathcal{D})$ be an ALD. A crossing p of $\mathcal{D}$ is *proper* if four connected components of $\partial\Sigma$ passing through the neighborhood of p are all distinct. See Figure 6. When every crossing of $\mathcal{D}$ is proper, we say that P is *proper*. Let D be a virtual link diagram and $\phi(D) = (\Sigma, \mathcal{D})$ the ALD associated with D. A real crossing of D is said to be *proper* if the corresponding crossing of $\mathcal{D}$ is proper. A virtual link diagram D is said to be *proper* if each crossing of D is proper (or equivalently if $\phi(D)$ is a proper ALD).

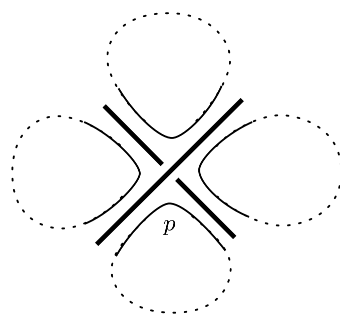
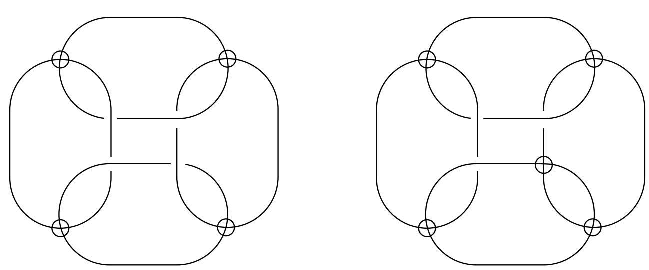


Figure 6

The left hand side of Figure 7 is a proper alternating virtual link diagram and the right hand side is a non-proper virtual link diagram. The right hand side is a v-alternating virtual link diagram obtained from the left diagram by virtualizing a real crossing.



proper alternating link diagram v-alternating link diagram

Figure 7

3 Checkerboard coloring

Let $P = (\Sigma, \mathcal{D})$ be an ALD. We say that P is *chekerboard colorable* if we can assign two colors (black and white) to the region of $\Sigma \setminus |\mathcal{D}|$ such that two adjacent regions with an arc of $|\mathcal{D}|$ have distinct colors, where $|\mathcal{D}|$ is the graph obtained from $\mathcal{D}$ by assuming each crossing to be a vertex of degree four. A *checkerboard coloring* of P is such an assignment of colors.

If P is an alternating ALD, then it has a checkerboard coloring such that for each crossing, the regions around each crossing are colored as in Figure 8. (This fact is seen as follows: Walk on any knot component of $\mathcal{D}$ and look at the right hand side. When we pass a crossing as an over-arc, or as an under-arc, the right is colored black, or white respectively. Since $\mathcal{D}$ is alternating, we have a coherent coloring.) We call such a coloring a *canonical checkerboard coloring* of an alternating ALD, which is unique unless P has a connected component without crossings.

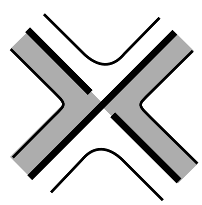


Figure 8

Let $P = (\Sigma, \mathcal{D})$ be an ALD and let $\mathcal{S}_A$ (or $\mathcal{S}_B$, resp.) be the state of $\mathcal{D}$ obtained from $\mathcal{D}$ by doing A-splice (resp. B-splice) for every crossing. (See Figure 9. The states on Σ are no longer ALDs.)

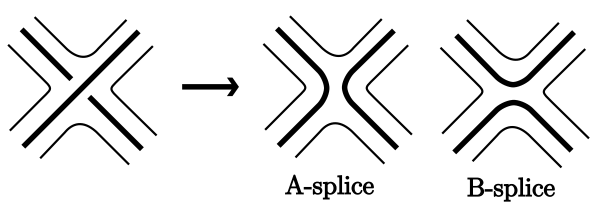


Figure 9

Suppose that $P = (\Sigma, \mathcal{D})$ be alternating, and consider a canonical checkerboard coloring of P. Then $(\Sigma, \mathcal{S}_A)$ and $(\Sigma, \mathcal{S}_B)$ inherit checkerboard colorings. See Figure 10. Black regions of $(\Sigma, \mathcal{S}_A)$ are annuli. Thus we have a one-to-one correspondence

$$\{\text{the loops of } \mathcal{S}_A\} \longrightarrow \{\text{the loops of } \partial\Sigma \text{ in black regions}\}$$

so that a loop of $\mathcal{S}_A$ and the corresponding loop of $\partial\Sigma$ bound an annulus colored black. Similarly, white regions of $(\Sigma, \mathcal{S}_B)$ are annuli. Thus we have a one-to-one correspondence

$$\{\text{the loops of } \mathcal{S}_B\} \longrightarrow \{\text{the loops of } \partial\Sigma \text{ in white regions}\}$$

so that a loop of $\mathcal{S}_B$ and the corresponding loop of $\partial\Sigma$ bound an annulus colored white. Thus we have the following.

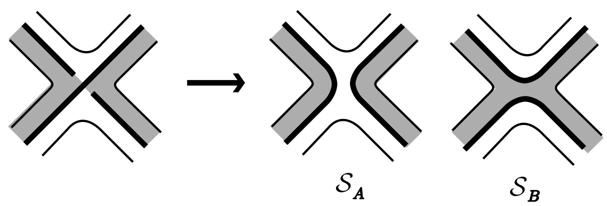


Figure 10

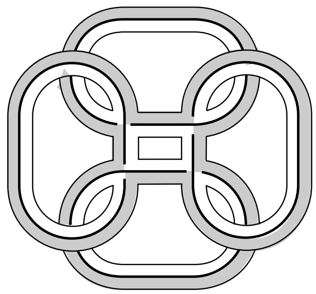
Lemma 3.1 *In the situation above, there is a bijection*

$$\{\text{the loops of } \mathcal{S}_A\} \cup \{\text{the loops of } \mathcal{S}_B\} \longrightarrow \{\text{the loops of } \partial\Sigma\}.$$

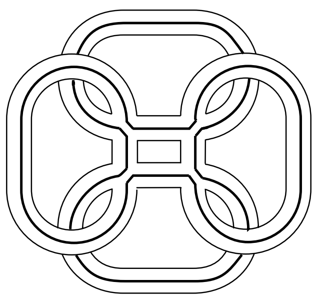
We have an example of an alternating ALD with a canonical checkerboard coloring and the states $\mathcal{S}_A$ and $\mathcal{S}_B$ in Figure 11.

Lemma 3.2 *Let $P = (\Sigma, \mathcal{D})$ be an alternating ALD, and let $\mathcal{S}_A$ (or $\mathcal{S}_B$, resp.) be the state of $\mathcal{D}$ obtained from $\mathcal{D}$ by doing A-splice (resp. B-splice) for every crossing. For a crossing p of $\mathcal{D}$, let $l_1(p)$ and $l_2(p)$ be the loops of $\mathcal{S}_A$ (or $l'_1(p)$ and $l'_2(p)$ be the loops of $\mathcal{S}_{\mathcal{B}}$) that pass through the neighborhood of p. If p is a proper crossing, then $l_1(p) \neq l_2(p)$ and $l'_1(p) \neq l'_2(p)$.*

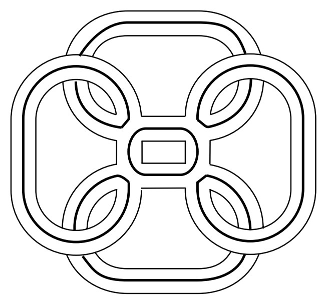
Proof Since p is a proper crossing, the four loops of $\partial\Sigma$ appearing around p arc all distinct. Since P is alternating, it has a canonical checkerboard coloring and there is a one-to-one correspondence as in Lemma 3.1. Then $l_1(p)$, $l_2(p)$, $l'_1(p)$ and $l'_2(p)$ correspond to the four distinct loops of $\partial\Sigma$ around p. Thus $l_1(p) \neq l_2(p)$ and $l'_1(p) \neq l'_2(p)$. □



an alternating ALD



$\mathcal{S}_A$



$\mathcal{S}_B$

Figure 11

4 Proofs of Theorems 1.2 and 1.3

We denote the maximal (or minimal, resp.) degree of a Laurent polynomial η by $\text{maxd}(\eta)$ (resp. $\text{mind}(\eta)$). For a state S of a virtual link diagram D, let $\langle S|D\rangle$ stand for $A^{\natural S}(-A^2 - A^{-2})^{\sharp S-1}$.

Proof of Theorem 1.2 Let D be a proper alternating virtual link diagram of m connected components, and let $P = (\Sigma, \mathcal{D})$ be the ALD associated with D. Let S_A (or S_B resp.) be the state of D obtained from D by doing A-splice (resp. B-splice) at each crossing of D, and let $\mathcal{S}_A$ (resp. $\mathcal{S}_B$) be the corresponding state of $\mathcal{D}$ in Σ.

Let $S_A(j)$ (or $S_B(j)$, resp.) be a state obtained from S_A (resp. S_B) by changing A-splices (resp. B-splices) to B-splices (resp. A-splices) at j crossings of D.

Claim 4.1 *$\sharp S_A(1) = \sharp S_A - 1$ and $\sharp S_B(1) = \sharp S_B - 1$.*

Proof Let $S_A(1)$ be obtained from S_A by changing A-splice to B-splice at a crossing point $\tilde{p}$ of D. Let $\mathcal{S}_A(1)$ be the corresponding state of $\mathcal{D}$, and let p be the crossing of $\mathcal{D}$ corresponding to $\tilde{p}$. Since D is proper, the crossing p is proper. We prove the former equality for the corresponding ALD version; namely, $\sharp\mathcal{S}_A(1) = \sharp\mathcal{S}_A - 1$. In the situation of Lemma 3.2, $l_1(p) \neq l_2(p)$. Since $\mathcal{S}_A(1)$ is obtained from $\mathcal{S}_A$ by changing A-splice with B-splice at p, two distinct loops $l_1(p)$ and $l_2(p)$ become a single loop. Hence $\sharp\mathcal{S}_A(1) = \mathcal{S}_A - 1$. Therefore we have $\sharp S_A(1) = \sharp S_A - 1$. Similarly, we have $\sharp S_B(1) = \sharp S_B - 1$. □

Claim 4.2 *$\sharp S_A(j) \leq \sharp S_A + j - 2$ and $\sharp S_B(j) \leq \sharp S_B + j - 2$ for $j = 1, \cdots, c(D)$.*

Proof Any $S_A(k)$, $k = 1, \cdots, c(D)$, is obtained from some $S_A(k-1)$ by changing A-splice to B-splice at a crossing. Then

$$\sharp S_A(k-1) - 1 \leq \sharp S_A(k) \leq \sharp S_A(k-1) + 1.$$

Thus $\sharp S_A(j) \leq \sharp S_A(1) + j - 1$. By Claim 4.1, we have $\sharp S_A(j) \leq \sharp S_A + j - 2$. Similarly, we have $\sharp S_B(j) \leq \sharp S_B + j - 2$. □

Now we continue the proof of Theorem 1.2. By definition,

$$\begin{aligned} \mathrm{maxd}(\langle S_A|D\rangle) &= \mathrm{maxd}(A^{c(D)}(-A^2 - A^{-2})^{\sharp S_A - 1}) \\ &= c(D) + 2\sharp S_A - 2 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \mathrm{mind}(\langle S_B|D\rangle) &= \mathrm{mind}(A^{-c(D)}(-A^2 - A^{-2})^{\sharp S_B - 1}) \\ &= -c(D) - 2\sharp S_B + 2. \end{aligned} \tag{2}$$

For a state $S_A(j)$ for $j = 1, \cdots, c(D)$, using Claim 4.2, we have

$$\begin{aligned} \mathrm{maxd}(\langle S_A(j)|D\rangle) &= \mathrm{maxd}(A^{c(D)-2j}(-A^2 - A^{-2})^{\sharp S_A(j) - 1}) \\ &= c(D) - 2j + 2\sharp S_A(j) - 2. \\ &\leq c(D) + 2\sharp S_A - 6. \end{aligned} \tag{3}$$

For a state $S_B(j)$ for $j = 1, \cdots, c(D)$, using Claim 4.2, we have

$$\begin{aligned} \mathrm{mind}(\langle S_B(j)|D\rangle) &= \mathrm{mind}(A^{-c(D)+2j}(-A^2 - A^{-2})^{\sharp S_B(j) - 1}) \\ &= -c(D) + 2j - 2\sharp S_B(j) + 2. \\ &\geq -c(D) - 2\sharp S_B + 6. \end{aligned} \tag{4}$$

From (1), (2), (3), (4) we have

$$\begin{cases} \mathrm{maxd}(\langle D\rangle) = c(D) + 2\sharp S_A - 2, \\ \mathrm{mind}(\langle D\rangle) = -c(D) - 2\sharp S_B + 2. \end{cases}$$

Thus
$$\mathrm{span}(D) = 2c(D) + 2(\sharp S_A + \sharp S_B) - 4.$$
By Lemma 3.1, we have $\sharp S_A + \sharp S_B = \sharp \mathcal{S}_A + \sharp \mathcal{S}_B = \sharp \partial \Sigma$. Therefore
$$\mathrm{span}(D) = 2c(D) + 2\sharp \partial \Sigma - 4.$$
By Lemma 2.2, we have the desired equality. □

Proof of Theorem 1.3 Let D' be a v-alternating virtual link diagram obtained from a proper alternating virtual link diagram D by virtualizing a real crossing p of D, and let $P' = (\Sigma', \mathcal{D}')$ be the ALD associated with D'. Note that $P' = (\Sigma', \mathcal{D}')$ is obtained from the ALD, $P = (\Sigma, \mathcal{D})$, associated with D by changing the neighborhood of the crossing which corresponds to p of D as in Figure 12.

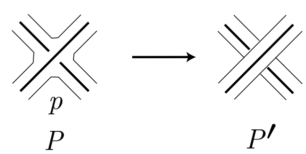


Figure 12

Let S_A (or S_B resp.) be the state of D obtained by doing A-splice (resp. B-splice) at each crossing, and let S'_A (resp. S'_B) be the state of D' obtained by doing A-splice (resp. B-splice) at each crossing. S'_A (or S'_B resp.) is obtained from S_A (resp. S_B) by connecting two connected components of S_A which pass through the neighborhood of p as in Figure 13.

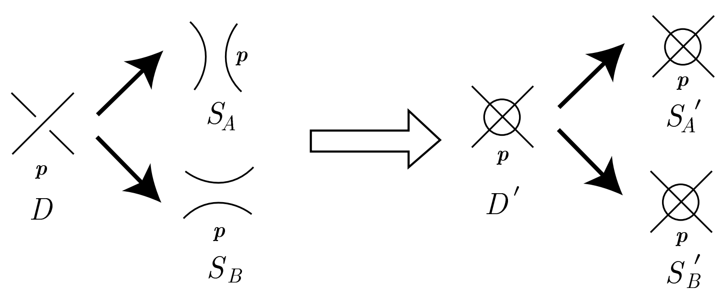


Figure 13

Let $S'_A(j)$ (or $S'_B(j)$, resp.) be a state obtained from S'_A (resp. S'_B) by changing A-splices (resp. B-splices) to B-splices (resp. A-splices) at j crossings of D'.

Claim 4.3 (1) $\sharp S'_A - 1 \leq \sharp S'_A(1) \leq \sharp S'_A$ *and* $\sharp S'_B - 1 \leq \sharp S'_B(1) \leq \sharp S'_B$.

(2) $\sharp S'_A(j) \leq \sharp S'_A + j - 1$ *and* $\sharp S'_B(j) \leq \sharp S'_B + j - 1$ *for* $j = 1, 2, \cdots, c(D')$.

Proof Any $S'_A(k)$, $k = 1, \cdots, c(D')$, is obtained from some $S'_A(k-1)$ by changing A-splice to B-splice at a crossing. Then

$$\sharp S'_A(k-1) - 1 \leq \sharp S'_A(k) \leq \sharp S'_A(k-1) + 1. \tag{5}$$

In particular, $\sharp S'_A - 1 \leq \sharp S'_A(1) \leq \sharp S'_A + 1$. If $\sharp S'_A(1) = \sharp S'_A + 1$, then $\sharp S_A(1) = \sharp S_A + 1$ (see Figure 14). It contradicts that D is proper (recall Claim 4.1). Thus we have $\sharp S'_A - 1 \leq \sharp S'_A(1) \leq \sharp S'_A$. By (5), $\sharp S'_A(j) \leq \sharp S'_A(1) + j - 1$. Hence $\sharp S'_A(j) \leq \sharp S'_A + j - 1$. Similarly we have $\sharp S'_B - 1 \leq \sharp S'_B(1) \sharp S'_B$ and $\sharp S'_B(j) \leq \sharp S'_B + j - 1$. □

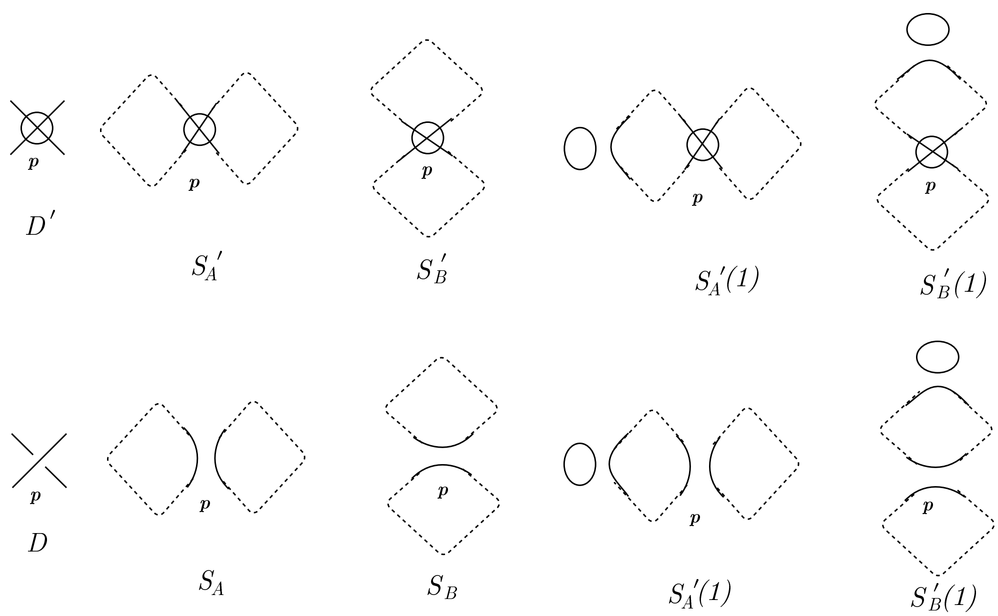


Figure 14

By definition, we have

$$\mathrm{maxd}(\langle S'_A | D' \rangle) = c(D') + 2\sharp S'_A - 2$$

and

$$\mathrm{mind}(\langle S'_B | D' \rangle) = -c(D') - 2\sharp S'_B + 2.$$

For a state $S'_A(j)$ and $S'_B(j)$, using Claim 4.3, we have

$$\begin{aligned}\mathrm{maxd}(\langle S'_A(j)|D'\rangle) &= c(D') - 2j + 2\sharp S'_A(j) - 2\\ &\leq c(D') + 2\sharp S'_A - 4\end{aligned}$$

and

$$\begin{aligned}\mathrm{mind}(\langle S'_B(j)|D'\rangle) &= -c(D') + 2j - 2\sharp S'_B(j) + 2\\ &\geq -c(D') - 2\sharp S'_B + 4.\end{aligned}$$

Therefore, we have

$$\begin{cases}\mathrm{maxd}\langle D'\rangle = c(D') + 2\sharp S'_A - 2\\ \mathrm{minxd}\langle D'\rangle = -c(D') - 2\sharp S'_B + 2\end{cases}$$

and

$$\mathrm{span}(D') = 2c(D') + 2(\sharp S'_A + \sharp S'_B) - 4.$$

Since p is proper, by Lemma 3.2, we see that $\sharp S'_A = \sharp S_A - 1$ and $\sharp S'_B = \sharp S_B - 1$. By Lemma 3.1, we have $\mathrm{span}(D') = 2c(D') + 2(\sharp S_A + \sharp S_B) - 8 = 2c(D') + 2\sharp\partial\Sigma - 8$.

Claim 4.4 $\sharp\partial\Sigma' = \sharp\partial\Sigma - 3$.

Proof Since p is a proper crossing, the four loops of $\partial\Sigma$ around p are all distinct. After changing $P = (\Sigma, \mathcal{D})$ to $P' = (\Sigma', \mathcal{D}')$ as in Figure 12, the four loops become a single loop of $\partial\Sigma'$ (see Figure 15). □

Thus $\mathrm{span}(D') = 2c(D') + 2\sharp\partial\Sigma' - 2$. By Lemma 2.2, we have $g(D') = (2m + c(D') - \sharp\partial\Sigma')/2$. Therefore $\mathrm{span}(D') = 4(c(D') - g(D') + m - 1) + 2$. This completes the proof of Theorem 1.3. □

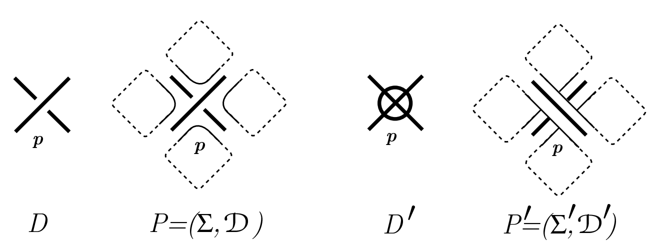


Figure 15

Lemma 4.5 *Suppose that a virtual link diagram D' is obtained from a virtual link diagram D by virtualizing a crossing p of D. If p is proper, then $g(D') = g(D) + 1$.*

Proof Let $P = (\Sigma, \mathcal{D})$ and $P' = (\Sigma', \mathcal{D}')$ be the ALDs associated with D and D'. Since p is proper, the numbers of connected components of Σ and Σ' must be the same, and as we saw in Claim 4.4 (Figure 15), $\sharp\partial\Sigma' = \sharp\partial\Sigma - 3$. Since $c(D') = c(D) - 1$, by Lemma 8, we seen that $g(\Sigma') = g(\Sigma) + 1$. Thus $g(D') = g(D) + 1$. □

5 2-braid virtual link

For non-zero integer $r_1, \cdots, r_s$, we denote by $K(r_1, \cdots, r_s)$ a virtual link diagram illustrated in Figure 16. The virtual link represented by this diagram is also denoted by $K(r_1, \cdots, r_s)$. M. Murai [12] proved that $K(r_1)$ and $K(r_1, r_2)$ are not classical links and that $K(r_1)$ and $K(r_2, r_3)$ are distinct virtual links.

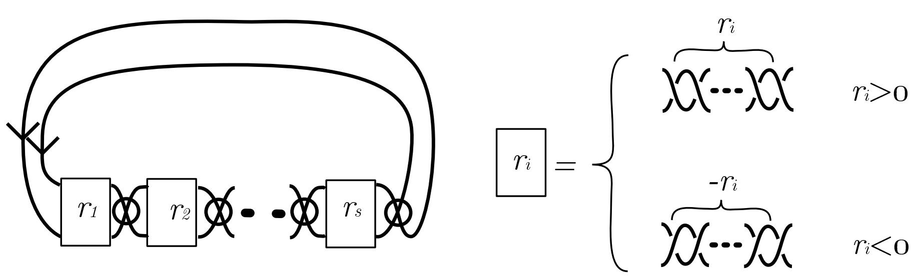


Figure 16

Kauffman [9] proved that the f-polynomial is invariant under the local move illustrated in Figure 17, which we call *Kauffman's twist* in this paper.

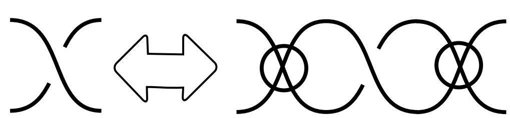


Figure 17

Using Kauffman's twists and generalized Reidemeister moves, we see that the f-polynomial of $K(r_1, \cdots, r_s)$ is equal to the f-polynomial of a virtual link illustrated in Figure 18, where $r = r_1 + \cdots + r_s$. If s is even, then it is a $(2, r)$-torus link or a trivial link. If s is odd and $r \neq 0$, then it is a v-alternating virtual link diagram satisfying the hypothesis of Corollary 1.4. Thus we have the following.

Corollary 5.1 (1) *If s is odd and $r_1 + \cdots + r_s \neq 0$, then $K(r_1, \cdots .r_s)$ is not a classical link.*

(2) *If s is odd, $r_1 + \cdots + r_s \neq 0$ and s' is even, then $K(r_1, \cdots .r_s)$ and $K(r'_1, \cdots .r'_{s'})$ are distinct virtual links.*

Remark When s is even, only from a calculation of the f-polynomials, we cannot conclude that $K(r_1, \cdots .r_s)$ is not a classical link. However this is true. It will be discussed in a forthcoming paper.

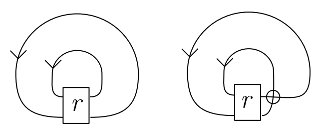


Figure 18

6 Remarks on supporting genera

Theorem 6.1 *For any positive integer n, there exists an infinite family of virtual link diagrams, $D(n,r)$ $(r = 0, 1, 2, \cdots)$, such that*

(1) *$D(n,r)$ is a proper alternating virtual link diagram,*

(2) *the supporting genus is n, and*

(3) *$c(D(n,r)) = 10n + r - 2$.*

Proof A diagram $D(n,r)$ illustrated in Figure 19 satisfies the conditions. In the figure, the boxed r stands for the r right half twists. The supporting genus is n, since it has a link diagram realization as in Figure 19(b) on a genus n surface such that the complementary region consists of open disks. □

Corollary 6.2 *For any positive integer N, there are proper alternating (1-component) virtual link diagrams $D_1, \cdots, D_N$ with the same crossing number and the supporting genus of D_k is k $(k = 1, \cdots, N)$.*

Proof Let D_k be the diagram $D(k, 10(N-k))$ introduced in Theorem 6.1. The crossing number of D_k is $10N - 2$. □

Corollary 6.3 *The span of the f-polynomial of an alternating (1-component) virtual link K is not determined only from the number $c(D)$ of real crossings of a proper alternating virtual link diagram D representing K.*

Proof Let $D_1, \cdots, D_N$ be the proper alternating 1-component virtual link diagrams in the proof of Corollary 6.2. Then $c(D_k) = 10N - 2$ and $g(D_k) = k$ for $k = 1, \cdots, N$. By Theorem 1.2, $\mathrm{span}(D_k) = 4(10N - 2 - k)$. Thus $D_1, \cdots, D_N$ have the same real crossing number but the spans of their f-polynomials are distinct. □

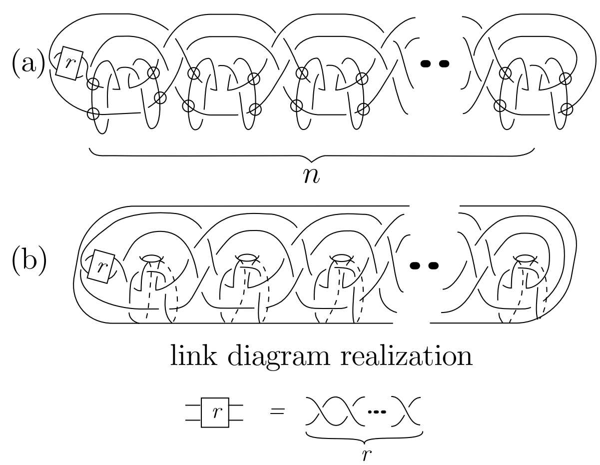


Figure 19

For a virtual link L, we define the minimal crossing number $c(L)$ and the supporting genus $g(L)$ of L by

$$c(L) = \min\{c(D) | D \text{ is a virtual link diagram representing } L\}$$

and

$$g(L) = \min\{g(D) | D \text{ is a virtual link diagram representing } L\}.$$

In the category of classical links, the following theorem holds.

Theorem 6.4 ([7], [13], [14]) *Let L be an alternating link represented by a proper alternating link diagram D. Then $c(L) = c(D)$.*

Question 6.5 Let L be an alternating virtual link represented by a proper alternating virtual link diagram D.

(1) Is $c(L)$ equal to $c(D)$?

(2) Is $g(L)$ equal to $g(D)$?

By Theorem 1.2, two assertions (1) and (2) are mutually equivalent.

As a related result, C. Adams et al. [1] and T. Kaneto [6] proved the following theorem. (C. Hayashi also informed the author the same result independently.)

Theorem 6.6 ([1], [6]) *Let D be a proper (or reduced) alternating link diagram in a closed oriented surface F. For any link diagram D' in F which is related to D by a finite sequence of Reidemeister moves in F, we have $c(D) \leq c(D')$.*

This theorem is a generalization of Theorem 6.4 when we consider that D represents a link in the thickened surface $F \times \mathrm{R}$; namely, for a link L in $F \times \mathrm{R}$ represented by a proper alternating link diagram D in F, we have $c(D) = c(L)$, where $c(L)$ is the minimal crossing number of L as a link in $F \times \boldsymbol{R}$. Note that Question 6.5 (1) is different from Theorem 6.6.

Remark V.O. Manturov [11] established another kind of generalization of Kauffman-Murasugi-Thistlethwaite's theorem (Theorem 6.4). He introduced the notion of quasi-alternating virtual link diagram and proved that any quasi-alternating virtual link diagram without nugatory crossing is minimal. A virtual link diagram is said to be *quasi-alternating* if it is obtained from a classical alternating link diagram by doing Kauffman's twists (Figure 17) at some crossings and virtual Reidemeister moves (in the second and third rows of Figure 1). Note that a quasi-alternating virtual link diagram is not an alternating virtual link diagram in our sense unless it is a classical alternating diagram or its consequences by virtual Reidemeister moves.

Acknowledgement This research is supported by the 21st COE program "Constitution of wide-angle mathematical basis focused on knots".

References

[1] **C Adams**, **T Fleming**, **M Levin**, **A Turner**, *Crossing number of alternating knots in $S \times I$*, Pacific J. Math. 203 (2002) MathReview 1–22

[2] **M Goussarov**, **M Polyak**, **O Viro**, *Finite-type invariants of classical and virtual knots*, Topology 39 (2000) 1045–1068 MathReview

[3] **N Kamada**, *The crossing number of alternating link diagrams of a surface*, Proceedings of the international conference "Knots 96"(Tokyo, 1996), World Scientific Publishing Co. (1997) 377–382 MathReview

[4] **N Kamada**, *On the Jones polynomials of checkerboard colorable virtual links*, Osaka J. Math. 39 (2002) 325–333 MathReview

[5] **N Kamada**, **S Kamada**, *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications 9 (2000) 93–106 MathReview

[6] **T Kaneto**, *Tait type theorems on alternating links in thickened surface*, Proceedings of the conference "Knot Theory"(Toronto, 1999), eds. Mitsuyoshi Kato et al. (2000) 148–156

[7] **L H Kauffman**, *State models and the Jones polynomial*, Topology 26 (1987) 395–407 MathReview

[8] **L H Kauffman**, Knots and physics, third edition, Series on Knots and Everything, 1, World Scientific Publishing Co. River Edge, NJ (2001) MathReview

[9] **L H Kauffman**, *Virtual knot theory*, Europ. J. Combinatorics 20 (1999) 663–690 MathReview

[10] **T Kishino**, *On classification of virtual links whose crossing numbers are equal to or less than* 6 (in Japanese), Master Thesis, Osaka City University (2000)

[11] **V O Manturov**, *Atoms and minimal diagrams of virtual links*, Dokl. Akad. Nauk 391 (2003) 166–168 (Dokl. Math. 63 (2003) 37–39) MathReview

[12] **M Murai**, *Classification of 2-braid virtual links whose virtual crossing numbers are* 2 (in Japanese), Master Thesis, Osaka City University (2000)

[13] **K Murasugi**, *Jones polynomials and classical conjectures in knot theory* Topology 26 (1987) 187–194 MathReview

[14] **M B Thistlethwaite**, *A spanning tree expansion of the Jones polynomial*, Topology 26 (1987) 297–309 MathReview

Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-ku
Osaka, 558-8585, Japan

Email: `naoko@sci.osaka-cu.ac.jp`

Received: 4 March 2004 Revised: 24 October 2004