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Hopf algebras up to homotopy and the Bockstein spectral sequence

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Abstract Anick proved that every q-mild Hopf algebra up to homotopy is isomorphic to a primitively-generated chain Hopf algebra. We provide a new proof, that involves extensive use of the Bockstein spectral sequence.

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1 Introduction

Let R be a subring of the rationals containing $1/2$. A *Hopf algebra up to homotopy*, or *Hah*, is an augmented chain R-algebra A, equipped with an algebra morphism $\Delta : A \to A \otimes A$ that is homotopy-coassociative and homotopy-cocommutative. We will suppose that the augmentation provides a strict counit for Δ. A Hah morphism is an algebra morphism that commutes with the diagonals up to chain algebra homotopy.

The aim of this paper is to present an alternate proof of the following theorem of D. Anick.

Anick's Theorem [2, Theorem 5.6] *Let $\rho > 2$ be the least non-invertible prime in R, and let $q \geq 1$. If A is a Hah whose underlying algebra is free and generated by the range of dimensions q through $q\rho - 1$, inclusive, then A is isomorphic as a Hah to a primitively-generated chain Hopf algebra.*

Anick's theorem was one of the more important recent developments in rational homotopy theory. Applied to the Adams-Hilton model of a space X, it allows rational homotopy theorists to apply their techniques to decidedly non-rational problems. Indeed, as outlined in [4], the singular cochain complex $C^*(X;R)$ is then weakly equivalent to a commutative cochain algebra that plays the role of the algebra of polynomial differential forms, $A_{PL}(X)$, in rational homotopy theory.

The proof provided by Anick is quite technical, and involves constructing a "partial inverse" to the diagonal map. In the present paper, the author proposes a proof that uses nothing more than the Bockstein spectral sequence as detailed in Browder [3], and the results on Hopf algebras from Milnor and Moore [5].

The key to the new proof of Anick's theorem is Theorem 1.1 below, where we consider the result of adjoining a free algebra variable to a primitively-generated chain Hopf algebra. First we introduce some terminology. A *free monogenic extension* of a chain algebra A is a chain algebra $(A \amalg T(x), dx = b)$ where $\amalg$ denotes the free product of algebras, $b \in A_{\deg x - 1}$ is a cycle, and $T(x)$ is the tensor algebra on the graded module $R\{x\}$. If A is a cocommutative chain Hopf algebra, then a *free monogenic extension of Hopf algebras up to homotopy* is a free monogenic extension of A along with a homotopy-coassociative, homotopy-cocommutative choice of reduced diagonal $\Phi \in (A \otimes A)_{\deg x}$ for x. A good reference for the homotopy theory of chain algebras is Anick's paper [2]. Effectively, to say that Φ is a "homotopy-coassociative" and "homotopy-cocommutative" choice means that there exist elements $f \in A \otimes A \otimes A$ and $g \in A \otimes A$ such that $(\Delta \otimes 1 + 1 \otimes \Delta)\Phi = df$ and $(\tau - 1)\Phi = dg$. If $\Phi = 0$ then we call the extension *trivial*. An isomorphism of extensions

$$\theta : (A \amalg T(x), dx = b, \bar{\Delta}x = \Phi) \xrightarrow{\cong} (A \amalg T(x), dx = b', \bar{\Delta}x = \Phi')$$

is a chain algebra morphism that restricts to the identity on A, that satisfies $\theta(x) - x \in A$, and that commutes with diagonals up to a chain algebra homotopy vanishing on A.

Recall that a graded algebra A is called *q-reduced* if $A_n = 0$ for $0 < n < q$.

Theorem 1.1 *Let (A, ∂) be a connected, primitively generated, q-reduced, R-free chain Hopf algebra of finite type. Suppose $A \to A \amalg T(x)$ is a free extension of Hopf algebras up to homotopy, with ∂x primitive. If $\deg x < q\rho$, then the extension is isomorphic to the trivial one.*

Theorem 1.1 will be proved in Section 2.

Proof of Anick's Theorem Let $A = TV$ be a Hopf algebra up to homotopy. Let $\{v_j\}$ be a well-ordered basis of V, where we have chosen an ordering such that $\deg v_i < \deg v_j$ implies that $i < j$. Suppose inductively that $A_{(i-1)} = T(v_1, \ldots v_{i-1})$ is a primitively generated Hopf algebra. Adding the generator v_i yields the chain algebra extension

$$A_{(i-1)} \to A_{(i-1)} \amalg T(v_i)$$

where $\bar{\Delta}v_i = \Phi$. It is easy to see that $[\partial v_i]$ is primitive in $H(A_{(i-1)})$. By Lemma 2.2 below, $H_m(P(A_{(i-1)} \otimes \mathbf{Z}_p)) \cong PH_m(A_{(i-1)} \otimes \mathbf{Z}_p)$ if $m < q\rho$, for all primes p in R. Since $P(A_{(i-1)} \otimes \mathbf{Z}_p) = P(A_{(i-1)}) \otimes \mathbf{Z}_p$ in these degrees, we conclude that $H_m(PA_{(i-1)}) \cong PH_m(A_{(i-1)})$, and so we may choose ∂v_i to be primitive in $P(A_{(i-1)})$. Now we may apply Theorem 1.1. $\square$

Remark 1.2 Anick assumes a homotopy counit, but then goes on to show that every Hah is isomorphic to one with a strict counit [2, Lemma 5.4].

We use the usual definitions for chain algebras, coalgebras, Hopf algebras, and homotopies of chain algebra morphisms [2, Section 2]. Algebras are assumed to be connected and augmented to the ground ring. The linear dual of a chain complex V is denoted V^*.

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2 The Proof of Theorem 1.1

Let p be an odd prime, and denote by $\mathbf{Z}_p$ the field of integers modulo p. We prove a proposition about cocommutative extensions of Hopf algebras over $\mathbf{Z}_p$. We then prove several lemmas that allow us to manage the Bockstein spectral sequence, before proving the inductive step, Proposition 2.8. We finish the section with the proof of Theorem 1.1 proper.

We begin by quickly reviewing some facts about Hopf algebras, gleaned from the standard reference of Milnor and Moore [5]. Let C be a connected, associative, coassociative, graded Hopf algebra with multiplication $\mu : C \otimes C \to C$, diagonal $\Delta : C \to C \otimes C$ and augmentation $\varepsilon : C \to R$. Let $I(C) = \ker \varepsilon$. The *space of indecomposables* of C is the quotient $I(C)/\mu(I(C) \otimes I(C))$. Let $\bar{\Delta} : I(C) \to I(C) \otimes I(C)$ be the reduced diagonal, defined by $\bar{\Delta}c = \Delta c - c \otimes 1 - 1 \otimes c$. Then $P(C) = \ker \bar{\Delta}$ is the *subspace of primitive elements*. The natural map $I(C) \to Q(C)$ restricts to define a map $P(C) \to Q(C)$, natural with respect to morphisms of Hopf algebras.

Denote by $TC(V)$ the *tensor coalgebra* on V. As a graded module, $TC(V) = T(V) = \oplus_{n \geq 0} V^{\otimes n}$. Elements of $V^{\otimes n}$ are traditionally denoted $[v_1 | \cdots | v_n]$. The diagonal is defined by

$$\Delta[v_1|\cdots|v_n] = \sum_{j=0}^{n}[v_1|\cdots|v_j] \otimes [v_{j+1}|\cdots|v_n].$$

In particular, $P(TC(V)) = V$. Note that, if V is finite type and free as a graded R-module, then $(TV)^* = TC(V^*)$ as a graded coalgebra.

While the following proposition likely qualifies as folklore, we provide a proof nonetheless.

Proposition 2.1 *Let A be a connected, primitively-generated Hopf algebra of finite type over $\mathbf{Z}_p$. Suppose that*

$$A \to A \amalg T(x)$$

is a coassociative and cocommutative extension. If A is $(r-1)$-connected and $\deg(x) < rp$, then $A \amalg T(x)$ is primitively generated.

Proof Set $B = A \amalg T(x)$. Dualize the extension $A \to B$ to obtain a morphism of connected commutative Hopf algebras $f : B^* \to A^*$. It suffices to show that the natural morphism $P(B^*) \to Q(B^*)$ is injective. Let $\xi : B^* \to B^*$ be the Frobenius map defined by $\xi(x) = x^p$. Then ξB^* is a sub Hopf algebra of B^*. By [5, Proposition 4.21], there is an exact sequence

$$0 \to P(\xi B^*) \to P(B^*) \to Q(B^*)$$

so it suffices to show that $P(\xi B^*) = 0$.

Let $V = Q(A)$ and $W = Q(B)$, so that $W = V \oplus \mathbf{Z}_p\{x\}$. Let $\sigma : V \to A$ be a splitting of the natural projection $\pi : I(A) \to V$. Extend σ to a splitting $\tau : W \to B$ of $\rho : I(B) \to W$. Then σ and τ extend to algebra epimorphisms $TV \to A$ and $TW \to B$ such that the diagram of algebra morphisms

$$\begin{array}{ccc} TV & \rightarrowtail & TW \\ {\scriptstyle\sigma}\downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow{\scriptstyle\tau} \\ A & \rightarrowtail & B \end{array}$$

commutes. Dualize to obtain the commutative diagram of coalgebra morphisms,

$$\begin{array}{ccc} B^* & \xrightarrow{f} & A^* \\ {\scriptstyle\tau^*}\downarrow & & \downarrow{\scriptstyle\sigma^*} \\ TC(W^*) & \xrightarrow{g} & TC(V^*) \end{array}$$

wherein the vertical arrows are monomorphisms of coalgebras. Then the morphisms $P(\sigma^*) : P(A^*) \to P(TC(V^*)) = V^*$ and $P(\tau^*) : P(B^*) \to P(TC(W^*)) = W^*$ are also monic. Furthermore, $P(g) : W^* \to V^*$ is the canonical projection that kills x^*, the basis element dual to $x \in W$. Suppose $b^p \in P(\xi B^*) \subset P(B^*)$. Then $P(\tau^*)(b^p) = \alpha x^* + v^*$, according to the direct sum decomposition $W^* = \mathbf{Z}_p\{x^*\} \oplus V^*$. Then $0 = P(\sigma^*)P(f)(b^p) = P(g)P(\tau^*)(b^p) = P(g)(\alpha x^* + \beta v^*) = v^*$, so $v^* = 0$. Therefore $P(\tau^*)(b^p) = \alpha x^*$. But $\deg b^p \geq rp > \deg x^*$ so $\alpha = 0$. Since $P(\tau^*)$ is a monomorphism, $b^p = 0$ and the result follows. □

Let A be a chain Hopf algebra over $\mathbf{Z}_p$; that is, A is simultaneously a Hopf algebra and a chain complex, such that the differential ∂ is both a derivation and a coderivation. Then $H(A)$ is naturally a Hopf algebra. Let $i_A : P(A) \to A$ be the natural inclusion of the primitive subspace. Then $H(i_A) : H(PA) \to H(A)$ factors through the inclusion $i_{H(A)} : PH(A) \to H(A)$ to yield a morphism $j : H(PA) \to PH(A)$.

Let A be a chain Hopf algebra such that $\partial = 0$ on $A_{<n}$. If $a \in A_n$ is not a cycle, then a is indecomposable, and ∂a is primitive.

Let B be a connected, primitively-generated Hopf algebra of finite type over $\mathbf{Z}_p$. Since pth powers vanish in B^*, they also vanish in the commutative Hopf algebra $H(B^*) = H(B)^*$. By [5, Proposition 4.20], it follows that $H(B)$ is primitively generated.

Lemma 2.2 *Let A be a connected, primitively-generated chain Hopf algebra of finite type over $\mathbf{Z}_p$. If A is q-reduced, then $j : H(PA) \to PH(A)$ is an isomorphism in degrees $< qp$.*

Proof The primitive filtration $\{F_k\}$ on A leads to a first-quadrant spectral sequence of commutative, primitively-generated Hopf algebras that converges to $E^0H(A)$. By [5, Proposition 5.11], pth powers vanish outside of bidegree $(0,0)$, $P(E^0A) = E^0_{1,t}A = P(A)_{1+t}$, and $P(E^0A) \xrightarrow{\cong} Q(E^0A)$. The morphism $H(PA) \to PH(A)$ corresponds to the edge homomorphism $E^1_{1,t}A \to E^\infty_{1,t}A$.

Since $P(E^0A)$ is a sub chain complex of E^0A, we may write $P(E^0A) = X \oplus Y \oplus Z$, with $d^0 : X \xrightarrow{\cong} Y$ and $H(P(E^0A)) \cong Z$. It follows that E^0A is the tensor product of differential Hopf algebras of the following forms:

(1) $B_1 = (\wedge(z), 0)$,

(2) $B_2 = (\mathbf{Z}_p[z]/(z^p), 0)$,

(3) $B_3 = (\wedge(x) \otimes \mathbf{Z}_p[y]/(y^p), dx = y)$,

(4) $B_4 = (\mathbf{Z}_p[x]/(x^p) \otimes \wedge(y), dx = y)$,

where $\wedge(-)$ denotes the exterior algebra. The indecomposables all lie in $E^0_{1,*}A$. A calculation shows that $H(B_3) = \wedge(xy^{p-1})$ and $H(B_4) = \wedge(x^{p-1}y)$. It follows that $Q(E^1A)$ is concentrated in columns 1 and p, and $P(E^1A) \xrightarrow{\cong} Q(E^1A)$. Let d^r be the first non-vanishing differential in the spectral sequence, $r \geq 1$. Let u be an element of lowest total degree such that $d^r u \neq 0$. Then $u \in Q(E^r A)$ and $d^r u \in P(E^r A)$. Thus $u \in E^r_{p,t}A$ for some t, $d^r u \in E^r_{1,t+p-2}A$, and so $r = p-1$. An element of F_p with non-vanishing differential has total degree at least $qp+1$, so $t \geq qp+1-p$. Therefore $d^{p-1}u \in E^{p-1}_{1,qp-1}A$, so the edge homomorphism $E^1_{1,t}A \to E^\infty_{1,t}A$ is an isomorphism for $t < qp-1$. We conclude that $H(P(A)) \to P(H(A))$ is an isomorphism in degrees $< qp$. ◻

Example 2.3 Consider the commutative, primitively-generated Hopf algebra over $\mathbf{Z}_p$, $A = \mathbf{Z}_p[x] \otimes \wedge(y)$, with $\deg x = 2n$, and $n \geq 1$. The differential is the derivation determined by the rule $\partial x = y$. Then $PA = \mathbf{Z}_p\{y, x, x^p, x^{p^2}, \ldots\}$. It follows that $H(PA) = \mathbf{Z}_p\{x^p, x^{p^2}, \ldots\}$. On the other hand, $H(A) = \mathbf{Z}_p[x^p] \otimes \wedge(x^{p-1}y)$, so $PH(A) = \mathbf{Z}_p\{x^{p-1}y, x^p, x^{p^2}, \ldots\}$. So $j : H(PA) \to PH(A)$ fails to be surjective in degree $2np-1$.

Example 2.4 Let $B = \wedge(x) \otimes \mathbf{Z}_p[y]$, with $\deg x = 2n+1$. The differential is given by $\partial x = y$. Then $PB = \mathbf{Z}_p\{x, y, y^p, y^{p^2}, \ldots\}$, and so $H(PB) = \mathbf{Z}_p\{y^p, y^{p^2}, \ldots\}$. Meanwhile, $H(B) = 0$, so $PH(B) = 0$. Therefore $j : H(PB) \to PH(B)$ is not injective.

Let A be a connected primitively-generated Hopf algebra of finite type over $\mathbf{Z}_{(p)}$, the ring of integers localized at p. Recall from [3] that the *Bockstein spectral sequence* of A in homology modulo p, denoted (E^r, β^r), is a spectral sequence of primitively-generated Hopf algebras over $\mathbf{Z}_p$ that converges

$$E^1 = H(A; \mathbf{Z}_p) \Rightarrow \left(\frac{H(A)}{\text{torsion}}\right) \otimes \mathbf{Z}_p.$$

We now introduce some notation to avoid confusion when following elements up and down the terms of the spectral sequence. Let C be a chain complex, and denote by E^r its Bockstein spectral sequence modulo p. If $x \in C$ is a cycle modulo p that survives to the r th term of the spectral sequence, then we denote its equivalence class in E^r by $[x]_r$. In particular, $[x]_2 = [[x]_1]$, and so on.

The following elementary lemma clarifies what it means for one element to be the higher Bockstein of another.

Lemma 2.5 *Let C be a chain complex and denote by E^r the r th term of the Bockstein spectral sequence of C modulo p. Suppose $a_r, b_r \in E^r$ satisfy $\beta^r a_r = b_r$. Suppose further that $a, b \in C$ represent a_r and b_r, respectively. That is, a and b are cycles modulo p that survive to E^r, where $[a]_r = a_r$ and $[b]_r = b_r$. Then there exist $c, e \in C$ such that $d(a+pc) = p^r(b+pe)$.*

Proof We proceed by induction on r. For $r = 1$, if a is a cycle modulo p, then $da = px$ for some $x \in C$. By definition, $\beta^1[a]_1 = [x]_1$, so $[x]_1 = [b]_1$. Thus there exist elements $c, e \in C$ such that $x = b + pe - dc$. Therefore $da = px = pb + p^2e - d(pc)$ and the statement holds, grounding the induction.

Assume that the lemma is true for $r = k-1$ for some $k \geq 2$, and suppose that $\beta^k[a]_k = [b]_k$. Since a survives to the k th term of the spectral sequence, $\beta^{k-1}[a]_{k-1} = 0$. Applying the inductive hypothesis, we find that there exist elements $f, g \in C$ such that $d(a+pf) = p^kg$. By definition, $\beta^k[a]_k = [g]_k$. Thus $[g]_k = [b]_k$, so $[g-b]_k = 0$, that is, there exists some $y \in C$ that survives to E^{k-1}, where $[g-b]_{k-1} = \beta^{k-1}[y]_{k-1}$. By the inductive hypothesis, there exist elements $z, e \in C$ such that $d(y+pz) = p^{k-1}(g-b-pe)$. Therefore $d(a+pf) = p^kg = p^kb + p^{k+1}e + d(py+p^2z)$. Rearranging, we find that $d(a+p(f-y-pz)) = p^k(b+pe)$. Set $c = f-y-pz$ to complete the inductive step and the proof. □

Corollary 2.6 *If $[b']_r = [b']_r$ in E^r, then there exist elements $e, f \in C$ such that $p^{r-1}b' = p^{r-1}b'' + p^re + df$.*

Proof Since $[b']_r = [b'']_r$, there exists $a \in C$ such that $\beta^{r-1}[a]_{r-1} = [b' - b'']_{r-1}$. Apply Lemma 2.5. □

Lemma 2.7 *Let A be a connected, q-reduced, primitively generated chain Hopf algebra over $\mathbf{Z}_{(p)}$. Suppose that $\partial a = p^rb$, for some $a, b \in A$, and that $\partial w = p^{r-1}\bar{\Delta}b$ for some $w \in A \otimes A$. If $\deg b < qp$, then there exist elements $x, y \in A$ such that $x \otimes 1 \in P(A \otimes \mathbf{Z}_p)$ and $\partial(a - x - py) = 0$.*

Proof We claim that for each $i \geq 0$, there exist elements $b_i, y_i, z_i \in A$ and $\Psi_i \in A \otimes A$, with z_i primitive mod p, that satisfy

$$\partial(a - z_i - py_i) = p^{r+i}b_i \tag{1}$$

and

$$\partial(w - \bar{\Delta}y_i - \Psi_i) = p^{r+i-1}\bar{\Delta}b_i. \tag{2}$$

When $i = 0$, we take $z_0 = y_0 = 0$, $b_0 = b$, and $\Psi_0 = 0$.

Suppose for some $i \geq 0$ that we have the elements b_i, y_i, and z_i as above. If b_i does not survive to the $(r+i)$ th term, then there exist $v, b_{i+1} \in A$ such that $\partial v = p^s b_i - p^{s+1} b_{i+1}$ for some $s < r+i$. Then $\partial(a - z_i - py_i - p^{r+i-s} v) = p^{r+i+1} b_{i+1}$ and $\partial(w - \bar{\Delta}(y_i + p^{r-s-1} v) - \Psi_i) = p^{r+i} \bar{\Delta} b_{i+1}$. Set $z_{i+1} = 0$, $y_{i+1} = y_i + p^{r+i-s} v$ and $\Psi_{i+1} = \Psi_i$. We note that $\deg b_{i+1} = \deg b_i$.

Suppose on the other hand that b_i survives to E^{r+i}. Then $[b_i]_{r+i}$ is primitive by equation (2). Let $\{[b]_{r+i}\}$ denote the homology class of $[b]_{r+i}$ in $H(PE^{r+i})$. Since $\beta^{r+i}[a - z_i]_{r+i} = [b_i]_{r+i}$, it follows that $\{[b]_{r+i}\} \in \ker\{j : HPE^{r+i} \to PE^{r+i+1}\}$. By Lemma 2.2, there exists $[z]_{r+i} \in PE^r$ such that $[b]_{r+i} = \beta^{r+i}[z]_{r+i}$. By repeated application of Lemma 2.2, we may suppose that z is primitive modulo p; say, $\bar{\Delta} z = p\Psi$. By Lemma 2.5,

$$\partial(z + py) = p^{r+i}(b_i - pb_{i+1}) \tag{3}$$

for some elements $y, b_{i+1} \in A$. Substituting (1) in (3) and rearranging, we find that

$$\partial\left(a - (z_i + z) - p(y_i + y)\right) = p^{r+i+1} b_{i+1}.$$

Taking the reduced diagonal of each side of (3) and using (2), we obtain

$$\partial\left(w - \bar{\Delta}(y_i + y) - (\Psi_i + \Psi)\right) = p^{r+i} \bar{\Delta} b_{i+1}.$$

Set $y_{i+1} = y_i + y$, $z_{i+1} = z_i + z$ and $\Psi_{i+1} + \Psi$ to complete the induction.

Now, since $H_{\deg b}(A)$ is finitely generated, its torsion submodule has an exponent, say m, so that the maximum order of torsion is p^m. In particular, there exists a chain $u \in A$ such that $\partial u = p^m b_{m-r+1}$. By construction, $\partial(a - z_{m-r+1} - py_{m-r+1}) = p^{m+1} b_{m-r+1}$. Therefore $\partial(a - z_{m-r+1} - p(y_{m-r+1} + u)) = 0$, completing the proof. □

Proposition 2.8 *Let A be a connected, primitively generated, q-reduced, $\mathbf{Z}_{(p)}$-free chain Hopf algebra of finite type. Let $n < qp$. If $\Phi \in (A \otimes A)_n$, $f \in (A \otimes A \otimes A)_{n+1}$, and $g \in (A \otimes A)_{n+1}$ satisfy $\partial \Phi = 0$, $\partial f = (\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})\Phi$, and $\partial g = (\tau - 1)\Phi$, then for all $r \geq 1$, there exist cycles $a_r \in A_n$, $\Phi_r \in (A \otimes A)_n$, and a chain $\Omega_r \in (A \otimes A)_{n+1}$, such that*

$$\bar{\Delta} a_r = \Phi - p^r \Phi_r + \partial \Omega_r.$$

Proof We proceed by induction on r. For $r = 1$, we note that $\Phi \in H(A; \mathbf{Z}_p) \cong E^1$ is strictly coassociative and cocommutative. Since $\deg \Phi < qp$, Proposition 2.1 tells us that there exists $[a_1']_1 \in E^1$ such that $\bar{\Delta}[a_1']_1 = [\Phi]_1$. Thus there exist chains $\Psi_1', \Omega_1 \in A \otimes A$ such that

$$\bar{\Delta} a_1' = \Phi - p\Phi_1' + \partial \Omega_1. \tag{4}$$

Since a_1' is a cycle modulo p, $\partial a_1' = pb_1$ for some $b_1 \in A$. Differentiating (4), we find that $\bar{\Delta} b_1 = -\partial \Phi_1'$. Now apply Lemma 2.7, to obtain elements $x \in P(A \otimes \mathbf{Z}_p)$, $y \in A$, such that $\partial(a_1' - x - py) = 0$. Set $a_1 = a_1' - x - py$. Since x is primitive modulo p, $\bar{\Delta} x = p\Psi$ for some Ψ. Then $\bar{\Delta} a_1 = \Phi - p(\Phi_1' + \Psi + \bar{\Delta} y) + \partial \Omega_1$. Set $\Phi_1 = \Phi_1' + \Psi + \bar{\Delta} y$. Since $\partial \Phi_1 = -\bar{\Delta} \partial a_1 = 0$, the induction is grounded.

Suppose now that

$$\bar{\Delta} a_{r-1} = \Phi - p^{r-1} \Phi_{r-1} + \partial \Omega_{r-1}. \tag{5}$$

Apply $\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta}$ to (5) and rearrange to obtain

$$\partial(f + (\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})\Omega_{r-1}) = p^{r-1}(\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})\Phi_{r-1},$$

so

$$\beta^{r-1}[f]_{r-1} = (\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})[\Phi_{r-1}]_{r-1}.$$

Similarly, by applying $\tau - 1$ to (5) and rearranging, we obtain

$$\beta^{r-1}[g]_{r-1} = (\tau - 1)[\Phi_{r-1}]_{r-1},$$

so $[\Phi_{r-1}]_r$ is a candidate for a diagonal in E^r. By Proposition 2.1, there exists $\tilde{a}_r \in E^r$ such that $\bar{\Delta} \tilde{a}_r = [\Phi_{r-1}]_r$. There exists a representative $\tilde{a} \in A$ of $\tilde{a}_r$ and an element $\tilde{b} \in A$, such that $\partial \tilde{a} = p^r \tilde{b}$. By Corollary 2.6,

$$p^{r-1} \bar{\Delta} \tilde{a} = p^{r-1} \Phi_{r-1} - p^r \Phi_r' + \partial u \tag{6}$$

for some $\Phi_r', u \in A \otimes A$. Adding equations (5) and (6), we obtain

$$\bar{\Delta}(a_{r-1} + p^{r-1} \tilde{a}) = \Phi - p^r \Phi_r' + \partial(u + \Omega_{r-1}).$$

Taking boundaries, we find that $-\partial \Phi_r = p^{r-1} \bar{\Delta} \tilde{b}$. So by Lemma 2.7, there exist elements $z, y \in A$, with z primitive modulo p, such that $\partial(\tilde{a} - z - py) = 0$. Set $a_r = a_{r-1} + p^{r-1}(\tilde{a} - z - py)$. Write $\bar{\Delta} z = p\Psi$. Then

$$\bar{\Delta} a_r = \Phi - p^r(\Phi_r' + \Psi + \bar{\Delta} y) + \partial(u + \Omega_{r-1}).$$

Set $\Phi_r = \Phi_r' + \Psi + \bar{\Delta} y$ and $\Omega_r = u + \Omega_{r-1}$. Then $\partial \Phi_r = -\bar{\Delta} \partial a_r = 0$, so the inductive step is complete. □

Proof of Theorem 1.1 Consider $C = \Omega A / \Omega^{\geq 3} A$, where $\Omega(-)$ is the Adams cobar construction [1]. Since A is finite type, $H_{n-2}(C)$ is a finitely generated R-module. We note that R is a principal ideal domain, so there is a finite list of primes p for which $H_{n-2}(C)$ contains p-torsion. For each of those primes, we make the following argument.

Let $\Phi = \bar{\Delta}x$, and let f and g be the relevant homotopies. Since ∂x is primitive, it follows that $\partial\Phi = 0$. Apply Proposition 2.8. Then $[\Phi] \in H_{n-2}(C)$ and $[\Phi] = p^r[\Phi_r]$ for all r. Since $H_{n-2}(C)$ is finitely generated, it follows that $[\Phi] = 0$. Therefore there exist $\Psi \in A \otimes A$ and $a \in A$ such that $\bar{\Delta}a = \Phi + \partial\Psi$. Define

$$\theta : (A \amalg T(x), \bar{\Delta}x = \Phi) \to (A \amalg T(x), \bar{\Delta}x = 0)$$

by $\theta|_A = 1_A$ and $\theta(x) = x + a$. Then θ is an isomorphism of chain algebras that commutes with the diagonals up to homotopy. □

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