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Classical $6j${symbols and the tetrahedron

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Abstract

A classical $6j${symbol is a real number which can be associated to a labelling of the six edges of a tetrahedron by irreducible representations of $SU(2)$. This abstract association is traditionally used simply to express the symmetry of the $6j${symbol, which is a purely algebraic object; however, it has a deeper geometric signi cance. Ponzano and Regge, expanding on work of Wigner, gave a striking (but unproved) asymptotic formula relating the value of the $6j${ symbol, when the dimensions of the representations are large, to the volume of an honest Euclidean tetrahedron whose edge lengths are these dimensions. The goal of this paper is to prove and explain this formula by using geometric quantization. A surprising spin-o is that a generic Euclidean tetrahedron gives rise to a family of twelve scissors-congruent but non-congruent tetrahedra.

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1 Introduction

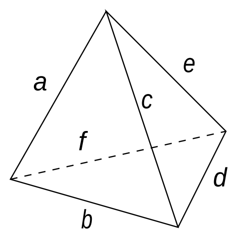
A classical $6j$–symbol is a real number which can be associated to a labelling of the six edges of a tetrahedron by irreducible representations of $SU(2)$, in other words by natural numbers. Its definition is roughly as follows.

Let V_a ($a = 0, 1, 2, \ldots$) denote the $(a+1)$–dimensional irreducible representation. The $SU(2)$–invariant part of the triple tensor product $V_a \otimes V_b \otimes V_c$ is non-zero if and only if

$$a \leq b + c \quad b \leq c + a \quad c \leq a + b \quad a + b + c \text{ is even} \tag{1}$$

in which case we may pick, almost canonically, a basis vector ϵ^{abc} (details are given below).

Suppose we have a tetrahedron, labelled so that the three labels around each face satisfy these conditions: we will call this an *admissible* labelling. Then we may associate to each face an ϵ–tensor, and contract these four tensors together to obtain a scalar, the $6j$–symbol, denoted by a picture or a bracket symbol as in figure 1.



$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$$

Figure 1: Pictorial representation

This tetrahedral picture is traditionally used simply to express the *symmetry* of the $6j$–symbol, which is naturally invariant under the full tetrahedral group S_4. However, it has a deeper *geometric* significance. To an admissibly-labelled tetrahedron we may associate a metric tetrahedron τ whose side lengths are the six numbers $a, b, \ldots, f$. Its individual faces may be realised in Euclidean 2–space, by the admissibility condition (1). As a whole, τ is either *Euclidean*, *Minkowskian* or *flat* (in other words has either a non-degenerate isometric embedding in Euclidean or Minkowskian 3–space, or has an isometric embedding in Euclidean 2–space), according to the sign of a certain polynomial in its edge-lengths. If τ is Euclidean, let $\theta_a, \theta_b, \ldots, \theta_f$ be its corresponding exterior dihedral angles and V be its volume.

Theorem 1 (Asymptotic formula) *Suppose a tetrahedron τ is admissibly labelled by the numbers a, b, c, d, e, f. Let k be a natural number. As $k \to \infty$, there is an asymptotic formula*

$$\begin{Bmatrix} ka & kb & kc \\ kd & ke & kf \end{Bmatrix} \sim \begin{cases} \sqrt{\dfrac{2}{3\pi V k^3}} \cos\left(\sum (ka+1)\dfrac{\theta_a}{2} + \dfrac{\pi}{4}\right) & \text{if } \tau \text{ is Euclidean} \\ \text{exponentially decaying} & \text{if } \tau \text{ is Minkowskian} \end{cases} \qquad (2)$$

(*where the sum is over the six edges of the tetrahedron*).

A (slightly different) version of this formula was conjectured in 1968 by the physicists Ponzano and Regge, building on heuristic work of Wigner; they produced much evidence to support it but did not prove it. It is the purpose of this paper to prove the above theorem using geometric quantization, and to explain the relation between $SU(2)$ representation theory and the geometry of $\mathbb{R}^3$.

The formula has a lovely and peculiar consequence in elementary geometry. It is well-known that a generic tetrahedron is not *congruent* (by an orientation-preserving isometry of $\mathbb{R}^3$) to its mirror-image, but is *scissors-congruent* to it (in other words, the two tetrahedra are finitely equidecomposable). Inspired by the additional algebraic *Regge symmetry* of $6j$–symbols and the asymptotic formula above, one may derive from a generic tetrahedron a family of *twelve* non-congruent but scissors-congruent tetrahedra!

Section 2 contains the algebraic and section 3 the differential-geometric preliminaries. Section 4 is a warm-up example, computing asymptotic rotation matrix elements for $SU(2)$ representations. It works in the same way as the eventual computation (in section 5) for the $6j$–symbol, but is much simpler and displays the method more clearly. Section 6 contains the geometric corollaries mentioned above and further notes on the Ponzano–Regge paper.

Throughout the paper, the symbol "$\sim$" denotes an asymptotic formula, whereas "$\approx$" denotes merely an approximation.

Acknowledgements After having the basic ideas for this paper, I spent some time collaborating with John Barrett, trying to find a good method of doing the actual calculations required. Neither of us had much success during this period, and the details presented here were worked out by me later. (I feel quite embarrassed at ending up the sole author in this way.) I am especially grateful to John for many lengthy, interesting and helpful discussions on the subject, and also to Jørgen Andersen, Johan Dupont, James Flude, Elmer Rees, Mike Singer and Vladimir Turaev for other valuable discussions.

2 Definition and interpretation of $6j$–symbols

2.1 Combinatorial definition

The simplest definition is via Penrose's spin network calculus, which is related to Kauffman bracket skein theory at $A = -1$. The details are in the book of Kauffman and Lins [6]. There is a topological invariant $\langle\ \rangle$ of planar links (systems of generically immersed curves) defined by sending a link L to

$$\langle L \rangle = (-2)^{\text{number of loops}(L)}(-1)^{\text{number of crossings}(L)}. \tag{3}$$

It extends to an invariant of suitably-labelled trivalent graphs in S^2, for example the Mercedes (tetrahedron) and theta symbols shown in figure 2. To define it,

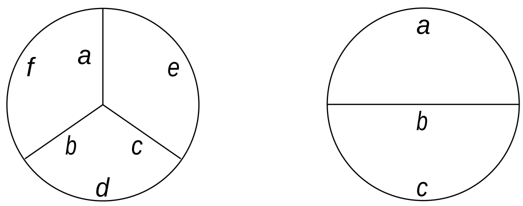


Figure 2: Mercedes and theta graphs

we replace each edge by a number of parallel strands equal to its label, and connect them up without crossings at the vertices (this imposes precisely the conditions (1) on the the three incident labels). Then we replace this diagram by the set of all planar links obtained by inserting a permutation of the strands near the middle of each edge. Finally, evaluate each of these using (3), add up their contributions, and divide by the number of such diagrams (the product of the factorials of the edge-labels). Explicit evaluations of these quantities are given in [6].

Definition 2 The $6j$–symbol shown in figure 1 is defined as the spin-network evaluation of the above admissibly-labelled Mercedes symbol, divided by the product of the square-roots of the absolute values of the four theta symbols associated with its vertices. It is manifestly S_4–invariant.

Remark It is important to note that the spin-network picture is *dual* to the one drawn in figure 1. There, the trilinear invariant spaces are associated with *faces* of the tetrahedron, whereas in the Mercedes symbol they are associated with *vertices*.

Although this definition is the simplest, we will need a more algebraic version where the $6j${symbol is exhibited as a hermitian pairing of two vectors.

2.2 Algebraic definition

Let V_1 be the fundamental 2{dimensional representation of $SU(2)$, which we will consider as the space of linear homogeneous polynomials in coordinate functions Z and W. Then the other irreducibles, the symmetric powers $V_a = S^a V$, $a = 0; 1; 2; \ldots$, are the spaces of homogeneous polynomials of degree a. The dimension of V_a is $a + 1$; when a is even, it is an irreducible representation of $SO(3)$.

Making Z, W orthonormal determines an invariant hermitian inner product $(-; -)$ on V_1, and induces inner products on the higher representations V_a, thought of as subspaces of the tensor powers of V_1. The fundamental representation has an invariant skew tensor $Z \quad W - W \quad Z$, which induces quaternionic or real structures on the V_a, according as a is odd or even.

The $SU(2)${invariant part of the tensor product of two irreducibles $V_a \quad V_b$ is zero unless $a = b$, when it is one-dimensional. Similarly, the invariant part of the triple tensor product $V_a \quad V_b \quad V_c$ of irreducibles is either empty or one-dimensional, according to the famous conditions (1). (The meaning of the parity condition is clear from the fact that the centre of $SU(2)$ is the cyclic group $\mathbb{Z}_2$. The other conditions, often written more compactly as $ja - bj \quad c \quad a + b$, are more surprising. Why the existence of a Euclidean triangle with the prescribed sides should have anything to do with this will be explained shortly.)

We want to pick well-defined basis vectors ^{aa} and ^{abc} for these spaces of bilinear and trilinear invariants. Since each such space has a hermitian form and a real structure, we could just pick real unit vectors, but this would still leave a sign ambiguity. To fix this we may as well just write down the invariants concerned. Consider the vectors corresponding to the polynomials

$$(Z_1W_2 - W_1Z_2)^a \qquad (Z_1W_2 - W_1Z_2)^k(Z_1W_3 - W_1Z_3)^j(Z_2W_3 - W_2Z_3)^i$$

on $\mathbb{C}^2 \quad \mathbb{C}^2$ and $\mathbb{C}^2 \quad \mathbb{C}^2 \quad \mathbb{C}^2$ respectively, where $i = (b + c - a)=2$, $j = (a + c - b)=2$, $k = (a + b - c)=2$. The required vectors are obtained from these by rescaling using positive real numbers, to obtain ^{aa} with norm $p\overline{a + 1}$ and ^{abc} with norm 1.

Definition 3 Given six irreducibles $V_a; V_b; \ldots ; V_f$, one can form $^{abc} \quad ^{cde} \quad ^{efa} \quad ^{fdb}$ (supposing these all exist) inside a 12{fold tensor product of irreducibles. One may always form $^{aa} \quad ^{bb} \quad \quad ^{ff}$, and permute the factors (without reversing the order of the paired factors) to match. Then the hermitian pairing of these two vectors (inside the 12–fold tensor product) defines the associated $6j$–symbol by

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{\sum a}\left(^{6}\ aa;\ ^{4}\ abc\right)$$

where $\sum a$ is simply the sum of the six labels. One should think of it as a function of six natural numbers $a, b, \ldots, f$, defined whenever the triples $(a,b,c), (c,d,e), (e,f,a), (f,d,b)$ satisfy the triangle and parity conditions (1), in other words when the associated tetrahedron labelling is admissible. It is standard to extend the definition to all such sextuples by setting the $6j$–symbol to zero elsewhere.

Lemma 4 *These two definitions agree.*

Proof (Sketch) The Mercedes spin network evaluation used in definition 2 can be reinterpreted as an explicit tensor contraction, using Penrose's diagrammatic tensor calculus (see [6]). The invariant $\langle L\rangle$ of a planar link may be evaluated by making the link Morse with respect to the vertical axis in $\mathbb{R}^2$, replacing cups and caps with i times the standard skew tensor ($Z\otimes W - W\otimes Z \in \mathbb{C}^2\otimes\mathbb{C}^2$) and its dual, crossings with the flip tensor, and composing these morphisms to obtain a scalar. If we draw the Mercedes graph as in figure 3 and use this recipe to compute it, we see that it is given as the composition of a vector in $V_1^{\otimes 2\sum a}$ (coming from the cups in the lower half of the diagram), a tensor product of twelve Young symmetrisers (coming from the flips associated to the crossings which are introduced where the labels are) and a vector in the dual of $V_1^{\otimes 2\sum a}$ (coming from the caps). This can be interpreted as a bilinear pairing between a vector in the tensor product of twelve irreps and one in its dual, if we use the symmetrisers to project to these. Reinterpreting it as a hermitian pairing and including the normalisation factors gives the purely algebraic formulation of definition 3. □

Remark This definition does not depend on the choice of hermitian form, coordinates, or real structure. It does depend on the sign conventions, but these can be seen to be sensible (in that the resulting $6j$–symbol is S_4–invariant) using the lemma.

Remark A third way of defining the $6j$–symbols is to build a basis for $(V_a\otimes V_b\otimes V_c\otimes V_d)^{SU(2)}$ out of the trilinear invariants using an isomorphism such as

$$(V_a\otimes V_b\otimes V_c\otimes V_d)^{SU(2)} = \bigoplus_e (V_a\otimes V_b\otimes V_e)^{SU(2)}\otimes(V_e\otimes V_c\otimes V_d)^{SU(2)}$$

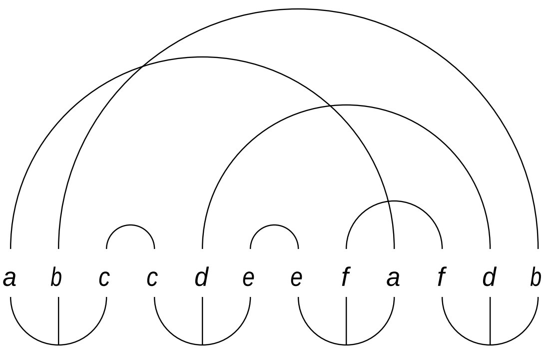


Figure 3: Writing the $6j$–symbol as a pairing

where e runs through all values such that (a, b, e) and (e, c, d) satisfy (1). There are three standard ways of doing this, corresponding to the three pairings of the four "things" a, b, c, d, and the change-of-basis matrix elements are (after mild renormalisation) the $6j$–symbols. Using this definition makes the Elliott–Biedenharn identity (pentagon identity) for $6j$–symbols very clear, but disguises their tetrahedral symmetry; therefore we will not consider this method here. See Varshalovich et al [15] for this approach. Their definition coincides with the two given here, and with the one in Ponzano and Regge (though in these physics-oriented papers, half-integer spins are used).

2.3 Heuristic interpretation

The representation theory of $SU(2)$ is well-known to physicists as the theory of quantized angular momentum. The fundamental 2–dimensional complex representation V_1 can be viewed as the space of states of spin of a spin–$\frac{1}{2}$ particle. The other irreducibles, the symmetric powers $\{V_a = S^aV; a \in \mathbb{N}\}$, are state spaces for particles of higher spin; indeed, physicists label them by their associated spins $j = \frac{1}{2}a$. Quantum and classical state-spaces are very different: the classical state of a spinning particle is described by an angular momentum vector in $\mathbb{R}^3$, whereas in the quantum theory, one should imagine the state vectors as wave-functions on $\mathbb{R}^3$, whose pointwise norms give *probability distributions* for the value of the angular momentum vector. However, when the spin is very large, the quantum and classical behaviour should begin to correspond. For example, the wave-functions representing states of a particle with large spin j should be concentrated near the sphere of radius j in $\mathbb{R}^3$.

Many representation-theoretic quantities, most obviously square-norms of matrix elements of representations, can be interpreted as probability amplitudes for quantum-mechanical observations. Wigner [17] explained the $6j$–symbol as follows. Suppose one has a system of four particles with spins $\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c, \frac{1}{2}d$ and total spin 0. Then the *square* of the $6j$–symbol is essentially the probability, given that the first two particles have total spin $\frac{1}{2}e$, that the first and third combined have total spin $\frac{1}{2}f$. (Compare with the third definition in the remark in subsection 2.2.) He reasoned that for large spins, because of the concentration of the wave-functions, one can treat this statement as dealing with addition of vectors in $\mathbb{R}^3$. Suppose one has four vectors of lengths $\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c, \frac{1}{2}d$ which form a closed quadrilateral. Then, given that one diagonal is $\frac{1}{2}e$, what is the probability that the other is $\frac{1}{2}f$? His analysis yielded the formula:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^2 \approx \frac{1}{3\pi V}$$

where V is the volume of the Euclidean tetrahedron whose edge-lengths are $a, b, \ldots, f$, supposing it exists. He emphasised that this is a dishonest approximation: the $6j$–symbols are wildly oscillatory functions of the dimensions, and his formula is only a local average over these oscillations, true in the same sense that one might write:

$$\text{For } \theta \neq 0,\ \cos^2(k\theta) \approx \frac{1}{2} \quad \text{as } k \to \infty.$$

Ponzano and Regge improved his formula to one very similar to (2), deducing the oscillating phase term from clever empirical analyses, and verified that as an approximation it is extremely accurate, even for small irreps.

3 Geometric quantization

3.1 Borel–Weil–Bott

To rigorize Wigner's arguments, we need a concrete geometric realisation of the representations V_a. This is provided by the Borel–Weil–Bott theorem (see for example Segal [1] or [3]): all finite-dimensional irreducible representations of semisimple Lie groups are realised as spaces of holomorphic sections of line bundles on compact complex manifolds, on which the groups act equivariantly. We only need the simplest case of this, namely that the irrep V_a of $SU(2)$ is the space of holomorphic sections of the ath tensor power of the hyperplane bundle L on the Riemann sphere $\mathbb{P}^1$. If one thinks of these as functions on the dual tautological bundle, which is really just $\mathbb{C}^2$ blown up at the origin, they can be identified with spaces of homogeneous polynomial functions on $\mathbb{C}^2$ (ie in two variables) with the obvious $SU(2)$ action (or possibly the dual of the obvious one, depending on quite how carefully you considered what "obvious" meant!)

Tensor products of such irreps are naturally spaces of holomorphic sections of the external tensor product of these line bundles over a product of Riemann spheres, for example

$$V_a \otimes V_b \otimes V_c = H^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; L^a \boxtimes L^b \boxtimes L^c)$$

with the diagonal action of $SU(2)$ on spheres and bundles.

The calculation we are going to perform is a stationary phase integration, for which we need local differential-geometric information about these holomorphic sections. We will take the primarily symplectic point of view of Guillemin and Sternberg [4], as well as using the main theorem of their paper (see below). Other relevant references are McDuff and Salamon [9] for general symplectic background and for symplectic reduction, Kirillov [7] for a concise explanation of geometric quantisation, and Mumford et al [12] for the wider context of geometric invariant theory.

3.2 Kähler geometry

Suppose M is a compact Kähler manifold of dimension $2n$. Thus, it has a complex structure J acting on the real tangent spaces T_pM, a symplectic form ω and a Riemannian metric B (we avoid the symbol g, which will denote a group element). The latter are J–invariant and compatible with each other according to the equation

$$B(X;Y) = \omega(X;JY);$$

this being a positive-definite inner product. The Liouville volume form $\Omega = \omega^n/n!$ equals the Riemannian volume form. The hermitian metric on T_pM (thought of as a complex space) is

$$H(X;Y) = B(X;Y) - i\,\omega(X;Y)$$

which is linear in the first factor and antilinear in the second (the convention used throughout the paper).

3.3 Hamiltonian group action

Let G be a compact group acting symplectically on M. We assume that the action is also Hamiltonian (ie that a moment map exists) and that it preserves the Kähler structure. This will certainly be true in the examples we will deal with.

We let $\mathfrak{g}$ be the Lie algebra of G. Then an element $\xi \in \mathfrak{g}$ defines a Hamiltonian vector field X_ξ, a Hamiltonian $\mu(\xi)$ and thus a moment map $\mu : M \to \mathfrak{g}^*$ according to the conventions

$$d\mu(\xi) = \iota_{X_\xi}\omega \quad (= \omega(X_\xi, -)).$$

I will later abuse notation slightly and write $\mu(X_\xi)$ instead of $\mu(\xi)$ when I want to emphasise the association of the moment map with a vector field corresponding to an infinitesimal action of G, rather than with an explicit Lie algebra element.

3.4 Equivariant hermitian holomorphic line bundle

If the symplectic form ω represents an integral cohomology class then there is a unique (we will assume M is simply-connected) hermitian holomorphic line bundle over M, with metric $\langle -, - \rangle$, whose associated compatible connection has curvature form $F = (-2\pi i)\omega$ (so that $[\omega]$ is its first Chern class).

We assume that G acts equivariantly on L, preserving its hermitian form. The space of holomorphic sections $V = H^0(M; L)$ is finite-dimensional and has a natural left G–action defined by

$$(gs)(p) = g.s(g^{-1}p).$$

V becomes a unitary representation of G when given the inner product

$$(s_1, s_2) = \int_M \langle s_1, s_2 \rangle \Omega.$$

The round bracket notation will be used to distinguish the *global* or *algebraic* hermitian forms from the *pointwise* form on the line bundle L, which will be written with angle brackets.

The infinitesimal action on sections is given by the formula

$$\xi s = \frac{d}{dt}(\exp(\xi t)s(\exp(-\xi t)p)) = (-\nabla_{X_\xi} + 2\pi i\mu(\xi))s \qquad (4)$$

This is the fundamental "quantization formula" of Kostant et al.

Remark One has to be very careful with signs here, especially as there is such variation of convention in the literature. This formula is *minus* the Lie derivative $L_X s$, because we are interested in the *left* action of G, and the Lie derivative is defined using the contravariant (right) action of G on sections via pullback. There is an identical problem if one looks at the derivative of the left action of G on vector fields, one has:

$$\frac{d}{dt}(\exp(\xi t)Y) = -L_X Y = -[X,Y]$$

provided one uses the standard conventions on Lie derivative and bracket:

$$[X,Y] = L_X Y, \qquad [X,Y]f = X(Yf) - Y(Xf)$$

For further comments on sign conventions see McDuff–Salamon [9], remark 3.3, though note that we do not here adopt their different Lie bracket convention. Anyway, it is a good exercise to check that the formula really does define a Lie algebra homomorphism: for this one also needs the standard conventions on curvature:

$$F(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

and on Poisson bracket:

$$\{f,g\} = -\omega(X_f, X_g)$$

3.5 Complexification

In the Borel–Weil–Bott setup, one actually has a complex group $G^{\mathbb{C}}$ acting on L and M. (Of course it does not preserve the hermitian structure on L, but its maximal compact part G does.) The action of $\mathfrak{g}^{\mathbb{C}}$ on sections of L is given by

$$(i\xi)s = i(\xi s) = (-i\nabla_X - 2\pi\mu(\xi))s = (-\nabla_{JX} - 2\pi\mu(\xi))s$$

the last identity coming because s is a holomorphic section, so is covariantly constant in the antiholomorphic directions in $TM \otimes \mathbb{C}$:

$$\nabla_{X+iJX}s = 0$$

The point of this identity is that it gives us information about the derivatives of an invariant section in directions orthogonal to the slice $\mu^{-1}(0)$.

3.6 Example

The $(k+1)$–dimensional irrep of $SU(2)$ is obtained by quantising S^2 with a round metric and with an equivariant hermitian line bundle $L^{\otimes k}$ of curvature $k\omega$, where ω is the standard form with area 1. We will always view S^2 as being the unit sphere in $\mathbb{R}^3$. In cylindrical coordinates, its unit area form is then ("Archimedes' theorem")

$$\omega = \frac{1}{4\pi} d\theta \wedge dz.$$

Let ξ be an element in the Lie algebra of the circle so that $e^{\xi} = 1$. The moment map for the 1–periodic rotation about the z–axis, generated by the vector field $X_\xi = 2\pi\,\partial/\partial\theta$ is $\mu : S^2 \to \mathbb{R}$ given by $\mu = \frac{1}{2}kz$. (Here we identify the dual of the Lie algebra with $\mathbb{R}$ by letting ξ be a unit basis vector.) Thus the image of the moment map is the interval $[-\frac{k}{2}, \frac{k}{2}]$, and in accordance with the Duistermaat–Heckman theorem, the length of the interval equals the area of the sphere.

3.7 Kähler quotients

Given as above a Kähler manifold M and the Kähler, Hamiltonian action of a compact group G, we may form the Kähler quotient M_G, which is just an enhanced version of the symplectic (Marsden–Weinstein) reduction.

Let M_0 denote the slice $\mu^{-1}(0)$. The G–equivariance of the moment map $\mu : M \to \mathfrak{g}^*$ (coadjoint action on the right) means that M_0 is G–invariant, and consequently that $\omega(X_\xi, Y) = d(\mu\cdot\xi)(Y) = 0$ for any $\xi \in \mathfrak{g}$ and $Y \in T_pM_0$. Let us suppose that G acts freely on M_0, since this will be enough for our purposes. We will use the symbol $\mathfrak{g}p \equiv \{X_\xi(p) : \xi \in \mathfrak{g}\}$ to denote the tangent space to the G–orbit at p and similarly, $i\mathfrak{g}p$ will denote $\{JX_\xi(p) : \xi \in \mathfrak{g}\}$. In fact $\mathfrak{g}p$ is the symplectic complement to T_pM_0 at p, and $i\mathfrak{g}p$ is the (Riemannian) orthogonal complement to T_pM_0, because $B(Y, JX_\xi) = -\omega(Y, X_\xi) = 0$ for any $Y \in T_pM_0$, and the dimensions add up.

As a manifold, M_G is just just the honest quotient M_0/G. It inherits an induced symplectic form ω_G whose pullback to M_0 is the restriction of that of M. Its tangent space $T_{[p]}M_G$ at a point $[p]$ (the orbit of a point $p \in M$) may be identified with its natural horizontal lift, namely the orthogonal complement of $\mathfrak{g}p \subset T_pM_0$ at any lift p of $[p]$. This space may also be described as the orthogonal complement of $\mathfrak{g}^{\mathbb{C}}p \subset T_pM$. As this subspace is complex, $T_{[p]}M_G$ inherits both a Riemannian metric and a complex structure by restriction. Hitchin proves in [5] that these induced structures make M_G into a Kähler manifold. Starting from L over M, we can also construct a hermitian holomorphic line bundle L_G over M_G with curvature $-2\ i\omega_G$ (in particular, the induced symplectic form is integral), as in [4]. The bundle and connection are such that their pullback to M_0 agrees with the restriction of L.

3.8 Reduction commutes with quantization

Let $Q(M)$ denote the *quantization* $H^0(M;L)$ associated to a Kähler manifold with equivariant hermitian line bundle L (which is suppressed in the notation). It is a representation of G, so we can consider the space of invariants $Q(M)^G$ $Q(M)$. (Whether we use G or $G^{\mathbb{C}}$ here is of course irrelevant.)

The main theorem in [4] is that there is an isomorphism $Q(M)^G = Q(M_G)$. There is obviously a restriction map from invariant sections over M to sections over M_G, more or less by de nition of M_G and L_G, so the task is to show injectivity and surjectivity.

A vital ingredient in their proof of surjectivity is fact that norms of invariant sections achieve their maxima (in fact decay exponentially outside of) the slice M_0. We will rely on this fact too. Suppose s is a holomorphic G–invariant section over M, and consider the real function $\|s\|^2$ on M. It is certainly G–invariant, but not $G^{\mathbb{C}}$–invariant. Following [4] we compute the derivative

$$(JX\ )\|s\|^2 = -4\ \ (\ )\|s\|^2$$

by using the quantization formula (4) and the compatibility of hermitian metric and connection

$$X\|s\|^2 = \langle\nabla_X s, s\rangle + \langle s, \nabla_X s\rangle.$$

Therefore, if $\gamma(t)$ is the flowline starting at $p \in M_0$ and generated by JX ,

$$\frac{d}{dt}\|s\|^2_{\gamma(t)} = -4\ \ (\ )\|s\|^2_{\gamma(t)}$$

and combining with

$$\frac{d}{dt}\ (\ )_{\gamma(t)} = B(X\ ;X\ ) > 0$$

we see that indeed the function $\|s\|^2_{\gamma(t)}$ has a single maximum at $t = 0$, ie on M_0.

3.9 Refinement of the Guillemin–Sternberg theorem

We need an addition to the “reduction commutes with quantization” theorem. Any space $Q(M)$ is in a natural way a Hilbert space, with inner product defined by

$$(s_1, s_2) = \int_M \langle s_1, s_2 \rangle \Omega$$

where $\langle -, - \rangle$ is its line bundle’s hermitian form. One might imagine that the restriction isomorphism $Q(M)^G \cong Q(M_G)$ is an isometry, but in fact it is not. However, asymptotically it becomes an isometry if one redefines the measure on M_G, as will be shown below.

First let us refine the observations about maxima of pointwise norms of sections given above. We can repeat the argument using the pointwise modulus of $\langle s_1, s_2 \rangle$, and establish that its maxima too are on M_0. Also, we can compute the second derivatives of the function $\langle s_1, s_2 \rangle$ in the JX_ξ directions, which span the orthogonal complement of TM_0. Redoing the above calcuation yields, at $p \in M_0$,

$$(JX_\xi . JX_\xi \langle s_1, s_2 \rangle)_p = -4\pi JX_\xi(\mu(\xi)) \langle s_1, s_2 \rangle_p = -4\pi B_p(X_\xi, X_\xi) \langle s_1, s_2 \rangle_p$$

because the moment map is zero at p. Parametrising a regular neighbourhood of M_0 as $M \times \exp(iU)$ for U some small disc about the origin in $\mathfrak{g}$, we see that to second order the function satisfies

$$\langle s_1, s_2 \rangle_{(p,\xi)} \approx \langle s_1, s_2 \rangle_p e^{-2\pi B(X_\xi, X_\xi)} \quad \text{for} \quad \xi \in U. \tag{5}$$

To understand the asymptotics, we must first understand what is varying! Let k be a natural number. Then one can consider M with the new symplectic form $k\omega$; its Liouville form scales by k^n (recall $\dim M = 2n$), the moment map for its G–action scales by k, its Riemannian metric scales by k, and there is a new equivariant hermitian holomorphic bundle $L^{\otimes k}$ over M whose Chern form is $k\omega$. If s is a G–invariant section of L then one can consider $s^k = s^{\otimes k}$, which is an invariant section of $L^{\otimes k}$. We will always write the k explicitly to indicate the scaling of forms, so that B, ω, Ω, X_ξ and so on retain their original definitions. The new pointwise hermitian form satisfies

$$\langle s_1^k, s_2^k \rangle = \langle s_1, s_2 \rangle^k.$$

Thus, in view of (5), as $k \to \infty$ the invariant section has pointwise norm concentrating more and more (like a Gaussian bump function) on the slice M_0. This localisation principle rigorises Wigner’s ideas and forms the basis for the proof of the Ponzano–Regge formula.

Theorem 5 *Let $\tilde{s}_1, \tilde{s}_2$ be G–invariant sections of L over M, and s_1, s_2 the induced sections of L_G over M_G. Let $\rho : M_G \to \mathbb{R}$ be the function which assigns to a point $[p]$ the Riemannian volume of the G–orbit in M represented by $[p]$, and let $d = \dim(G)$. Then, as $k \to \infty$, there is an asymptotic formula*

$$(\tilde{s}_1^k, \tilde{s}_2^k) = \int_M \langle \tilde{s}_1^k, \tilde{s}_2^k \rangle (k^n \Omega) \sim \left(\frac{k}{2}\right)^{d/2} \int_{M_G} \langle s_1^k, s_2^k \rangle (\rho\, k^{n-d} \Omega_G).$$

Proof From basic geometric invariant theory [4] we know $M_0/G = M_{ss}/G^{\mathbb{C}}$, where $M_{ss} = G^{\mathbb{C}} M_0$ is the set of semistable points, an open dense subset of M. We view the left hand integral as an integral over M_{ss} and then integrate over the fibres of $\pi : M_{ss} \to M_G$, which are $G^{\mathbb{C}}$–orbits. It helps to think of Ω as the Riemannian (coming from B) rather than the symplectic volume form.

Suppose $[p] \in M_G$ and $p \in M_0$. The fibre can be parametrised via the map $G \times i\mathfrak{g} \to \pi^{-1}([p])$ given by $(g, i\xi) \mapsto \exp(i\xi) gp$. This is a diffeomorphism because of the Cartan decomposition of $G^{\mathbb{C}}$. (Recall we are assuming that G acts freely on M_0.)

Let ϕ be the pullback to $G \times i\mathfrak{g}$ of the function $\log \langle \tilde{s}_1, \tilde{s}_2 \rangle$, so that $\log \langle \tilde{s}_1^k, \tilde{s}_2^k \rangle$ pulls back to $k\phi$. It is invariant in the G directions but has Hessian form in the $i\mathfrak{g}$ directions (at 0) given by

$$(i\xi . i\xi . k\phi)_0 = -4\pi k B_p(X_\xi, X_\xi).$$

The pullback Riemannian metric on $i\mathfrak{g}$ is given by $\beta(i\xi, i\xi) = B(X_\xi, X_\xi)_p$. Consequently the integral of ϕ_k over $i\mathfrak{g}$ is asymptotically given by

$$\langle \tilde{s}_1, \tilde{s}_2 \rangle_p^k \int_{i\mathfrak{g}} e^{-2\pi k \beta(i\xi, i\xi)} d\,\mathrm{vol}_\beta = \langle \tilde{s}_1, \tilde{s}_2 \rangle_p^k \left(\frac{1}{2k}\right)^{d/2}.$$

(The symbol vol_β denotes the measure induced by the same metric β as appears in the integrand; changing coordinates to an orthonormal basis, one obtains the a standard Gaussian integral, independent of β.)

Finally we integrate over the G orbit, picking up the factor $\mathrm{vol}(Gp) = \rho([p])$. Substituting into the original left-hand side and separating the powers of k in the correct way finishes the proof. □

3.10 Example

Let us return to example 3.6 and check these formulae. If we have area form $2k\omega$ then sections of L^{2k} are identified as homogeneous polynomials in Z, W

of degree $2k$, and under the circle action there is an invariant section which one can write as

$$s^k : (Z, W) \mapsto (Z^k W^k) \in \mathbb{C}.$$

In the complement of infinity, trivialise L^{2k} using the nowhere-vanishing holomorphic section b^{2k} corresponding to the homogeneous polynomial W^{2k}. Let $\zeta = Z/W$ be the coordinate on this chart. The pointwise norm of b^{2k} is $(1+|\zeta|^2)^{-k}$, because a unit element of the tautological bundle above ζ is $(\zeta, 1)/\sqrt{(1+|\zeta|^2)}$, which gets sent to $(1+|\zeta|^2)^{-k}$ by the section b^{2k}.) Thus the pointwise norm of s^k is $|\zeta|^k/(1+|\zeta|^2)^k$. Under stereographic projection from the unit sphere in $\mathbb{R}^3$

$$\zeta = \frac{x+iy}{1-z}$$

we get $1/(1+|\zeta|^2) = (1-z)/2$ and $|\zeta|^2/(1+|\zeta|^2) = (1+z)/2$. Thus the pointwise norm-squared of s^k is $((1-z^2)/4)^k$. Its global norm-square is

$$\begin{aligned} \|s^k\|^2 &= \int_{S^2} \left(\frac{1-z^2}{4}\right)^k 2k \frac{1}{4\pi} d\theta \wedge dz \\ &= 2k \int_{-1}^{1} \left(\frac{1-z^2}{4}\right)^k \frac{1}{2} dz \\ &= 2kB(k+1, k+1) \end{aligned}$$

by definition of the beta function B. Evaluating the beta function in terms of factorials gives

$$\|s^k\|^2 = \frac{2k}{2k+1} \binom{2k}{k}^{-1} \sim (\sqrt{\pi k}) 4^{-k}.$$

Compare this with the computation of the norm on the reduced space, which is a single point: one finds the norm-square of s^k at this point to be simply its value on the equator $z = 0$ of the sphere, namely 4^{-k}, so the ratio of the two is therefore $\sqrt{\pi k}$. If one applies theorem 5 with $\Omega = 2\omega$ then one gets the same ratio: the scaling factor is $\sqrt{k/2\pi}$ and the length of the equator (the factor in the formula) is $2\pi\sqrt{2/4\pi}$, because the usual spherical *area* form has been divided by 4π and multiplied by 2.

3.11 Orbit volumes

It is convenient here to note the formula for the volume of the G–orbit with respect to some basis (we will eventually have to roll up our sleeves and perform explicit calculations).

Lemma 6 *If $\{ _i\}$ is a basis for and is an invariant metric on $\mathfrak{g}$, then the volume of the G–orbit at $p \in M_0$ is*

$$\mathrm{vol}(Gp) = \frac{\mathrm{vol}_{B_p}\{X_{ _i}\}}{\mathrm{vol}\ \{ _i\}}\,\mathrm{vol}\ (G).$$

Proof Pulling back the metric via the diffeomorphism $G \to Gp$, $g \mapsto gp$ gives a metric on G of the form

$$(g\ ; g\ ) = B_{gp}(X\ ; X\ ) = B_p(X\ ; X\ )$$

whose associated volume form differs from that of by the given factor. □

It is also worth giving a formula for the orbit volume when G does not act freely on a space. Suppose as in the previous lemma that is an invariant metric, that the stabiliser at p is T, whose Lie algebra is $\mathfrak{t}$ $\mathfrak{g}$.

Lemma 7 *If $\{ _i\}$ is set of vectors in $\mathfrak{g}$ which, when projected onto into the orthogonal complement $\mathfrak{t}^{\perp}$ $\mathfrak{g}$, forms a basis $\{\hat{ }_i\}$ for that space,*

$$\mathrm{vol}(Gp) = \frac{\mathrm{vol}_{B_p}\{X_{ _i}\}}{\mathrm{vol}\ \{\hat{ }_i\}}\,\mathrm{vol}\ (G/T).$$

Proof Repeat the earlier proof with G/T mapping diffeomorphically to the orbit, and using the basis $\{\hat{ }_i\}$ for the tangent space of G/T. This gives a formula like the above except with $\mathrm{vol}_{B_p}\{X_{\hat{ }_i}\}$ on top. However, since the vectors $\hat{ }_i - \ _i$ are in $\mathfrak{t}$, they map to zero tangent vectors at p, and we can simply remove the hats. □

3.12 Stationary phase formulae

The standard stationary phase formula is as follows: on a manifold M^{2n} with volume form Ω, for a smooth real function f with isolated critical points $\{p\}$, one has

$$\int_M e^{ikf}\Omega \quad \left(\frac{2}{k}\right)^n \sum_p \frac{e^{ikf(p)}\cdot e^{\frac{i}{4}\,\mathrm{sgn}(\mathrm{Hess}_p(f))}}{\sqrt{\left|\mathrm{Hess}_p(f)\right|}}.$$

In our computation of we will actually have a *complex* function , so it is probably easier to rewrite/generalise the above formula to such a situation as

$$\int_M e^{k\ }\Omega \quad \left(\frac{2}{k}\right)^n \sum_p \frac{e^{k\ (p)}}{\sqrt{-\mathrm{Hess}_p(\ )}} \tag{6}$$

where the Hessian is now a complex number, and by the square root we mean the principal branch (the Hessian must not be real and positive). Of course, with $= if$, f real, this reduces to the previous version.

4 Warm-up example

In order to demonstrate more clearly the main points of the calculation to come in section 5, we will rst work out a simpler case.

Let V_{2k} be an irreducible representation of $SO(3)$, and let S^1_z be the circle subgroup xing the z{axis in $\mathbb{R}^3$. With respect to this subgroup, V_{2k} splits into one-dimensional weight spaces indexed by even weights from $-2k$ to $2k$. We may pick unit basis vectors inside these, uniquely up to a sign (by using the real structure of the representation). If g 2 $SO(3)$ is a rotation, we may compute the matrix elements of g with respect to such a basis by using the hermitian pairing. Most of these depend on the choices of sign, but the diagonal elements, those of the form $(v; gv)$, are independent. Below we will compute an asymptotic formula for such a matrix element, when v is the zero-weight (S^1{invariant) vector.

There are in fact explicit formulae for matrix elements given using Jacobi and Legendre polynomials, which are well-known in quantum mechanics. One can prove the theorem from these more easily, see for example Vilenkin and Klimyk [16]. Also, it could be computed explicitly from the example 3.10. But this method demonstrates how to do the calculation without such explicit knowledge of the sections.

If v is a weight vector for S^1_z then gv is a weight vector for $gS^1_z g^{-1} = S^1_{gz}$, the subgroup xing the rotated axis \"gz". The elements $(v; gw)$ can also be thought of as elements of the matrix expressing one weight basis in terms of the other.

Theorem 8 *As k ! 1 there is an asymptotic formula*

$$(v_0^{(k)}; gv_0^{(k)}) \qquad \sqrt{\frac{2}{k\sin}}\cos\quad(2k+1)\frac{}{2}+\frac{}{4}$$

where $v_0^{(k)}$ is a unit zero-weight vector in V_{2k} and > 0 the angle through which g rotates the z{axis.

Proof As in example 3.6, V_{2k} is the space of holomorphic sections of L^{2k} over S^2, with $SO(3)$ acting in the obvious way, symplectic form $2k\omega$, and the subgroup S^1_z acting with moment map kz. The zero-slice is a circle, and the reduced space a single point. Therefore there is a one-dimensional space of invariant sections, and such a section will have maximal (and constant) modulus on the slice $z = 0$.

We choose a section s of L^2 which is S^1_z–invariant and has peak modulus 1 where $z = 0$. (The phase doesn't matter, as noted above.) Then s^k is a section of L^{2k}, invariant under S^1_z, and also with peak modulus 1 at $z = 0$. It does not quite represent a choice of $v_0^{(k)}$ because its global norm is not 1 (we have fixed it locally instead).

We must compute an asymptotic expression for

$$(v_0^{(k)}, gv_0^{(k)}) = \frac{(s^k, gs^k)}{(s^k, s^k)}.$$

The denominator is asymptotically $\sqrt{\quad k}$, as we checked in example 3.10. So the main work is computing the integral

$$(s^k, gs^k) = \int_{S^2} \langle s^k, gs^k\rangle 2k\omega = k\int_{S^2} \langle s, gs\rangle^k (2\omega) = k\int_M e^{k} \; (2\omega)$$

where $\;= \log\langle s, gs\rangle$.

Now gs is an invariant section for S^1_{gz}, whose moment map is simply the "gz coordinate", and whose zero-slice "$gz = 0$" meets $z = 0$ in two antipodal points N and S. (They are both on the axis of g, and N is the one about which g is anticlockwise rotation.) Therefore, outside a neighbourhood of these two points, the modulus of the integrand is exponentially decaying, and the asymptotic contribution to the integral is just from N and S. In fact these two points will also turn out to be the critical points of , and we will evaluate the integral using the standard stationary phase procedure.

Let us denote by $\;,\;$ the moment maps for S^1_z and S^1_{gz} acting on the sphere with symplectic form 2ω. Let X, Y be the generating vector fields corresponding to these actions, as shown in figure 4.

The first derivatives of   can be calculated as follows: start by computing

$$X\langle s, gs\rangle = \langle \nabla_X s, gs\rangle + \langle s, \nabla_X gs\rangle.$$

The first term can be simplified to $2\;i\;\langle s, gs\rangle$ via the quantization formula (4), because s is invariant under the group corresponding to X. The second term is quite so easy, but becomes simpler if we write

$$X = pY + qJY$$

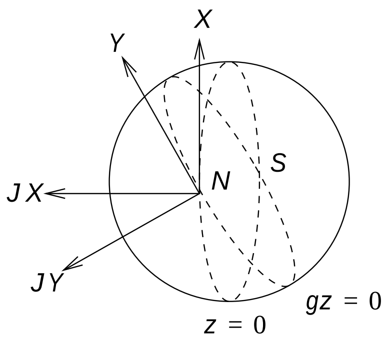


Figure 4: Localisation regions and generating vectors

for some scalar functions $p; q$ (generically $Y; JY$ span the tangent space), and then expand again:

$$X \quad = 2 \quad i \quad - 2 \quad ip \quad - 2 \quad q \; :$$

At either intersection point, the moment maps are both zero, so the whole thing vanishes. Similarly, the substitution

$$Y = p^0 X - q^0 JX$$

gives

$$Y \quad = 2 \quad ip^0 \quad + 2 \quad q^0 \quad - 2 \quad i \; :$$

Applying X and Y again to these formulae gives us the second derivatives at the critical points. The fact that ultimately we evaluate where the moment maps are zero shows we need only worry about the terms arising from the Leibniz rule in which the vector eld di erentiates them. These are evaluated using the following evaluations at N:

$$X \quad = 0 \qquad X \quad = 2!(Y; X) \qquad Y \quad = 2!(X; Y) \qquad Y \quad = 0.$$

In addition, at N we have $p = p^0 = \cos$ and $q = q^0 = -\sin$, by inspection. Thus, with respect to the basis $fX; Yg$, we have the matrix of second derivatives

$$(-2 \quad i)(-2!(X; Y)) \quad \begin{matrix} e^i & 1 \\ 1 & e^i \end{matrix} \quad :$$

To obtain the Hessian, we have to divide the determinant of this matrix by $(2!(X; Y))^2$, to account for the basis $fX; Yg$ not being unimodular with respect to the volume form $2!$. Therefore the Hessian at N is

$$(2 \quad i)^2 e^i \quad 2i \sin \quad = -8 \quad ^2 i \sin \quad e^i \; :$$

An easy check shows that the complex conjugate occurs at S.

The 0{order contributions $e^{k\ (N)}; e^{k\ (S)}$ must be computed. Both are of unit modulus, because of our convention on s. Their phases are easy to compute using the unitary equivariance of the bundle. Let h be the rotation through lying in S^1_z. It exchanges N and S, and preserves the section s. Thus

$$e^{\ (S)} = hs; gsi(S) = hh{:}h^{-1}s(hN); gh{:}h^{-1}s(hN)i = hhs(N); ghs(N)i$$
$$= hs(N); h^{-1}ghs(N)i{:}$$

Now $h^{-1}gh$ is clockwise rotation through at N, and therefore acts on the bre of $L\ ^2$, which is the tangent space at N, via e^{-i}. Hence

$$e^{\ (S)} = e^{i} \text{ and similarly } e^{\ (N)} = e^{-i}\ {:}$$

Finally we can put this all together:

$$k\int_{S^2} e^{k}\ (2!) = k\frac{2}{k}\left(\frac{e^{-ik}}{p\ 8\ ^2 i \sin\ e^{i}} + \frac{e^{ik}}{p\ -8\ ^2 i \sin\ e^{-i}} \right) {:}$$

Dividing by the normalising $(s; s) = \sqrt{k}$ gives

$$\sqrt{\frac{2}{k \sin}} \cos\ \ (2k + 1)\frac{}{2} + \frac{}{4}\ {:}$$

□

Remark The answer consists of a modulus, coming from the modulus of the Hessian's determinant and the normalisation factors for the original sections, and a phase, coming partially from the phase of the Hessian and partially from the global phase shift (0{order terms). The $=4$ is a standard stationary phase term. It will be possible to identify the terms in the $6j${symbol formula similarly.

Remark Other weight vectors behave similarly. If one repeats example 3.10, one nds beta integrals with more '$(1 - z)$'s than '$(1 + z)$'s or vice versa, and the peak of the integrand also shifts to the correct position: a circle of constant latitude whose constant z coordinate equals the weight divided by k. Similar formulae for general rotation matrix elements may be obtained.

5 Asymptotics of $6j$–symbols

5.1 Geometry of the sphere

The $6j$–symbol arises by pairing two $SU(2)$–invariant vectors in a 12–fold tensor product of irreducibles, according to definition 3. Let us first consider the geometry of single irreducibles.

Identify the Lie algebra of $SO(3)$ with $\mathbb{R}^3$ by using the standard vector product structure "$\times$", so that the standard basis vectors e_i generate infinitesimal rotations $v \mapsto e_i \times v$ in space. Define an invariant metric (the usual scalar product "$.$") by making them orthonormal. Using this scalar product we identify $\mathbb{R}^3$ with the coadjoint space also. This metric gives the flows generated by the 'e_i's period 2π, and so gives the circle T in $SO(3)$ generated by any of them length 2π. We can think of $SO(3)$ as half of a 3–sphere whose great circle has length 4π; such a sphere has radius $r = 2$, and therefore volume $2\pi^2 r^3 = 16\pi^2$. Therefore $SO(3)$ has volume $8\pi^2$, and the quotient sphere $SO(3)/T$ (with its induced metric) has volume 4π.

The irrep V_a is the space of holomorphic sections of the bundle L_a on S^2. To write down explicit formulae for the various structures on the sphere it is better to view it as the sphere S^2_a of radius a in $\mathbb{R}^3$, instead of as the unit sphere. The symplectic form with area a is given by

$$\omega_x(v, w) = \frac{1}{4\pi a^2}[x.v.w]$$

where x is a vector on the sphere, v, w are tangent vectors at x (orthogonal to it as vectors in $\mathbb{R}^3$) and the square brackets denote the triple product

$$[x.v.w] \equiv x.(v \times w).$$

The complex structure at x is the standard rotation

$$J_x(v) = \frac{1}{a} x \times v.$$

The final piece of Kähler structure, the Riemannian metric, is then

$$\begin{aligned} B_x(v, w) &= \omega_x(v, J_x w) = \frac{1}{4\pi a^3}(x \times v).(x \times w) = \frac{1}{4\pi a^3}(x^2(v.w) - (x.v)(x.w)) \\ &= \frac{1}{4\pi a} v.w. \end{aligned}$$

This formula agrees with the fact that with the *natural* induced metric given by $B_x(v, w) = v.w$, the area of S^2_a is $4\pi a^2$.

The group $G = SO(3)$ acts on the sphere, preserving its Kähler structure. This is a Hamiltonian action with moment map being simply inclusion $\mu_a \colon S^2_a \hookrightarrow \mathbb{R}^3$. If a is even then G also acts on the corresponding hermitian line bundle, but if a is odd one gets an action of the double cover $SU(2)$. (Since we need to work primarily with the geometric action, it is $SO(3)$ that wins the coveted title "G"!)

Products of Kähler manifolds have sum-of-pullback symplectic forms, and direct sums of complex structures. Their Liouville volume forms are wedge-products of pullbacks (there are no normalising factorials; this is one reason for the $n!$ in the definition of the Liouville form). The moment map for the diagonal action of G on $S^2_a \times S^2_b \times S^2_c$ is therefore $\mu(x_1, x_2, x_3) = x_1 + x_2 + x_3 \in \mathbb{R}^3$. We know from the discussion earlier that the pointwise norm of an invariant section over this space will attain its maximum on the set $\mu = 0$, which is in this case the "locus of triangles" $\{x_1 + x_2 + x_3 = 0\}$. G acts freely and transitively on this space, except in the exceptional cases when one of a, b, c is the sum of the other two. We can safely ignore this case, because it corresponds to a flat tetrahedron (about which the main theorem says nothing).

We need to pick a section s^{abc} corresponding to the invariant vector ^{abc}, whose normalisation was explained in section 2. There is a one-dimensional vector space of $SU(2)$–invariant sections of $L^{a} \boxtimes L^{b} \boxtimes L^{c}$ over $S^2_a \times S^2_b \times S^2_c$. First we define s^{abc} uniquely up to phase by setting its peak modulus to be 1. This section does *not* represent ^{abc} exactly, because we fixed the norm *locally*, instead of globally. However, it is more convenient for calculations, and we can renormalise afterwards. Similarly, let s^{aa} be a section over $S^2_a \times S^2_a$ with peak modulus 1. It will be concentrated near the zeroes of the moment map $\mu(x_1, x_2) = x_1 + x_2$, namely the anti-diagonal. Note that $SO(3)$ acts transitively on the antidiagonal with circle stabilisers everywhere. To represent ^{aa}, this section would have to be renormalised so that its global section norm was $\sqrt{a+1}$.

Fixing the phase of each section is slightly more subtle. The phase of the ' 's was fixed by using the spin-network normalisation. The analogue for sections is just the same, viewing the Riemann sphere as $\mathbb{P}^1$ with homogeneous coordinates Z and W, and defining the sections just as before. The choice will not actually matter until subsection 5.9.

5.2 A 24–dimensional manifold

Fix 6 natural numbers $a, b, \ldots, f$ satisfying the appropriate admissibility conditions for existence of the $6j$–symbol.

We will work on the following Kähler manifold M:

$$M = S^2_a \times S^2_b \times S^2_c \times S^2_c \times S^2_d \times S^2_e \times S^2_e \times S^2_f \times S^2_a \times S^2_f \times S^2_d \times S^2_b$$

This is taken to lie inside $(\mathbb{R}^3)^{12}$, and a point in it will be written as a vector $(x_1, x_2, \ldots, x_{12})$.

There are three useful actions on M. First there is the diagonal action of G on all 12 spheres. It has moment map $\mu : M \to \mathbb{R}^3$ given by

$$\mu(x_1, x_2, \ldots, x_{12}) = \sum_{1}^{12} x_i.$$

Algebraically, this generates the diagonal action of G on the corresponding tensor product of 12 irreducible representations

$$V_a \otimes V_b \otimes V_c \otimes V_c \otimes \cdots \otimes V_b.$$

Secondly, one has an action of $G^4 = G \times G \times G \times G$, the first copy acting diagonally on the first three spheres, the next on the next three, and so on. The moment map for this action is $\mu : M \to (\mathbb{R}^3)^4$,

$$(x_1, x_2, \ldots, x_{12}) \mapsto (x_1 + x_2 + x_3, x_4 + x_5 + x_6, x_7 + x_8 + x_9, x_{10} + x_{11} + x_{12}).$$

The section $s = s^{abc} \boxtimes s^{cde} \boxtimes s^{efa} \boxtimes s^{fdb}$ is a well-defined (given earlier conventions on s^{abc}) invariant section for this action with peak modulus 1.

Thirdly, we let G^6 act on M, each copy acting diagonally on a pair of spheres of the same radius. The moment map (which shows precisely how this works) is $\mu : M \to (\mathbb{R}^3)^6$,

$$\mu(x_1, x_2, \ldots, x_{12}) = (x_1 + x_9, x_2 + x_{12}, x_3 + x_4, x_5 + x_{11}, x_6 + x_7, x_8 + x_{10}).$$

Then $s' = s^{aa} \boxtimes s^{bb} \boxtimes \cdots \boxtimes s^{ff}$ (after a suitable permutation of its tensor factors, so that s^{aa} lives over the first and ninth spheres, for example) is an invariant section for this action with peak modulus 1.

5.3 Proof of theorem 1

Recall from 3 the definition of the $6j$–symbol as a hermitian pairing of two vectors, and their normalisations. The corresponding geometric formula, in terms of the sections s, s' just defined is

$$\left|\begin{matrix} ka & kb & kc \\ kd & ke & kf \end{matrix}\right| = (-1)^{\sum a} \frac{(s^k, s'^k)}{\|s^k\| \|s'^k\|} \prod (ka + 1) \qquad (7)$$

where the product on the right denotes simply $(ka+1)(kb+1)\cdots(kf+1)$.) Our convention on peak modulus 1 and phase of the sections mean that $(s^{abc})^{k} = s^{ka;kb;kc}$ and similarly $(s^{aa})^{k} = s^{ka;ka}$, so this formula is just the pairing of tensor products of these sections, with corrections for the global norms of s and s as explained at the end of subsection 5.1.

To extract the asymptotic formula for the 6j–symbol, we therefore need asymptotic formulae as $k \to \infty$ for the three integrals:

$$
\begin{aligned}
I &= (s^k; s^k) = \int_M \langle s^k; s^k \rangle k^{12}\Omega \\
I &= (s^k; s^k) = \int_M \langle s^k; s^k \rangle k^{12}\Omega \\
I &= (s^k; s^k) = \int_M \langle s^k; s^k \rangle k^{12}\Omega
\end{aligned}
$$

(Note the explicit inclusion of all factors of k; everything else is unscaled.) We can in fact immediately write down asymptotic formulae for the correction integrals I; I using theorem 5, because the reductions M_{G^4}, M_{G^6} are both single point spaces, and the sections have modulus 1 over these points.

$$I \sim \left(\frac{k}{2}\right)^{6} \mathrm{vol}(\ ^{-1}(0)) \tag{8}$$

$$I \sim \left(\frac{k}{2}\right)^{6} \mathrm{vol}(\ ^{-1}(0)) \tag{9}$$

(In the second case one must actually reconsider the proof of the theorem, because G^6 does not act freely on the set $\ ^{-1}(0)$, but there is no problem.)

The remaining integral I is evaluated by reduction to an integral over M_G followed by the method of stationary phase, which fills the rest of this section.

5.4 Localisation of the integral I

As $k \to \infty$, the integrand decays exponentially outside the region where both moment maps ; are zero, because it is dominated by the pointwise norms of the invariant sections s; s. What then is the set $\ ^{-1}(0) \cap \ ^{-1}(0)$?

At a point $(x_1; x_2; \ldots; x_{12})$, the condition $= 0$ requires that six of the x_i's are simply negatives of the other six, and then $= 0$ forces the six remaining ones, say $(x_1; x_2; x_3; x_5; x_6; x_8)$, to form a tetrahedron, shown *schematically* in figure 5. Recall that the lengths of the vectors are fixed integers $a; b; c; d; e; f$.

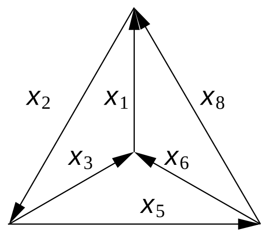


Figure 5: Schematic con guration of vectors

We have assumed that the numbers $a, b, \ldots, f$ satisfy the triangle inequalities in triples (otherwise the $6j$–symbol is simply zero), so the faces of this triangle can exist *individually* in $\mathbb{R}^3$. However, it is still quite possible that there is no Euclidean tetrahedron with sides $a, b, \ldots, f$. The sign of the Cayley polynomial $V^2(a^2, b^2, \ldots, f^2)$ (whose explicit form is irrelevant here) is the remaining piece of information needed to determine whether is Euclidean, flat or Minkowskian. In the last case, we see that $^{-1}(0) \cap {}^{-1}(0) = \emptyset$, and so have proved the second part of the main theorem: that if is Minkowskian then the $6j$–symbol is exponentially decaying as $k \to \infty$.

Suppose on the other hand that V^2 is positive. Then we *can* nd a set of six vectors a, b, c, d, e, f in $\mathbb{R}^3$ forming a tetrahedron oriented as shown in gure 6. Let denote both this tetrahedron and the corresponding point

$$(a, b, c, -c, d, e, -e, f, -a, -f, -d, -b) \in M$$

and $'$ be its mirror image (negate these 12 vectors). (Of course the whole tetrahedron is determined by just three of the vectors, say a, c, e.) It is clear that the localisation set $^{-1}(0) \cap {}^{-1}(0)$ will consist of exactly two G–orbits G , G'.

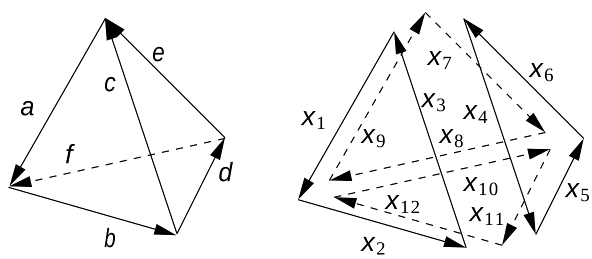


Figure 6: Actual con guration of vectors

Remark The symbols $a; b; c; d; e; f$ now denote *both* vectors and their integer lengths at the same time! This ought not be *too* confusing, as it should be clear from formulae what each symbol represents.

Returning to the integral I, since both $\tilde{s}^k; \tilde{s}^k$ are invariant under the diagonal action of G, we can apply theorem 5 with respect to the diagonal action of G, and obtain an integral over an 18{dimensional manifold M_G:

$$I = \int_M h\tilde{s}^k; \tilde{s}^k i k^{12}\Omega = k^9 \left(\frac{k}{2}\right)^{3=2} \int_{M_G} hs^k; s^k i \quad \Omega_G \qquad (10)$$

where in the right-hand integral, $s\ ; s$ are the descendents of $\tilde{s}\ ; \tilde{s}$, and the only thing depending on k is the integrand, which is the kth power of something independent of k. The function is the function on M_G giving the volume of the corresponding G{orbit in M, and we view it as part of the measure in the integral.

Let us de ne = log$hs\ ; s\ i$ on M_G and $\tilde{}$ = log$h\tilde{s}\ ; \tilde{s}\ i$ on M, so that the remaining problem is to compute

$$I^0 = \int_{M_G} e^k \quad \Omega_G:$$

Since the modulus of $e^{k\tilde{}}$ localises on the set $^{-1}(0) \setminus\ ^{-1}(0) = G\ [G\ ^0\ \ M$, the above integrand localises to the two *points* $[\ ]; [\ ^0]$.

5.5 Tangent spaces and stationary phase

The diagonal G{action does not commute with the other ones, so that M_G will not have any kind of induced actions of G^4 or G^6, but we don't need this for the localisation calculation to go through. We always work on the upstairs space M not M_G, precisely because the presence of the group actions de ning the invariant sections being paired is so useful. The tangent space $T\ M$ is 24{dimensional, and contains two 12{dimensional subspaces ker d and ker d which meet in the 3{dimensional space $\mathfrak{g}$. (This degree of transversality can be checked explicitly from formulae below, but it should be clear from the fact that there are just two isolated critical points in M_G.) Together they span the 21{dimensional $T\ M_0$, which is orthogonal to $i\mathfrak{g}$.

Projecting to 18{dimensional $T_{[\ ]}M_G$, we see two 9{dimensional subspaces we shall call W and W (the projections of ker d and ker d) meeting transversely at the origin.

We want to examine the behaviour of (its gradient and Hessian) at the point $[\;] \in M_G$. Let us choose orthonormal bases $\{X_1, X_2, \ldots, X_9\}, \{Y_1, Y_2, \ldots, Y_9\}$ for the transverse 9–dimensional tangent spaces W, W inside $T_{[\;]}M_G$. Then we can need to compute quantities such as X_i and $X_i Y_j$ (also at $[\;]$, of course!). These can be computed by choosing *arbitrary* lifts of the vectors to $T M$ and applying them to the G–invariant function $\tilde{\ } = \log\langle s, s\rangle$ on M. This is important, because it is very hard to write down any explicit *horizontal* lifts which would be needed to do computations directly in $T_{[\;]}M_G$.

So, to compute something like X_i one can choose any lift $\tilde{X}_i$ inside $\ker d$ in $T M$, and write

$$\begin{aligned} X_i &= \tilde{X}_i \tilde{\ } = \tilde{X}_i \log\langle s, s\rangle \\ &= \langle s, s\rangle^{-1}(\langle \nabla_{\tilde{X}_i} s, s\rangle + \langle s, \nabla_{\tilde{X}_i} s\rangle). \end{aligned}$$

Now $\tilde{X}_i$ is a generator of the G^4 action under which s is invariant, and therefore

$$\nabla_{\tilde{X}_i} s = 2 i (\tilde{X}_i) s$$

which vanishes at . (Here $(\tilde{X}_i)$ really denotes $(\ _i)$ for the Lie algebra element $_i$ corresponding to $\tilde{X}_i$.) For the second term we must first express X_i as a linear combination of the Y_j and JY_j (which span $T_{[\;]}M_G$), then we can lift and use the quantization formula to compute.

Therefore, introduce the 9×9 matrices P_{ij} and Q_{ij} according to

$$X_i = \sum P_{ik} Y_k + \sum Q_{ik} J Y_k.$$

Multiplying by J we get

$$J X_i = -\sum Q_{ik} Y_k + \sum P_{ik} J Y_k.$$

By applying $\omega_G(X_j, -)$ and similar operators to these equations one obtains

$$P_{ij} = B_G(X_i, Y_j) \qquad Q_{ij} = -\omega_G(X_i, Y_j).$$

These, together with the fact that the bases are orthonormal and span isotropic subspaces, determine completely matrices for B and ω on $T_{[\;]}(M_G)$. By a similar procedure one can invert the relations:

$$\begin{aligned} Y_i &= \sum P_{ki} X_k - \sum Q_{ki} J X_k \\ J Y_i &= \sum Q_{ki} X_k + \sum P_{ki} J X_k. \end{aligned}$$

A final point is that since $\{X_i, JX_i\}$ and $\{Y_i, JY_i\}$ are both complex-oriented orthonormal bases for $T_{[\ ]}(M_G)$, the change of basis matrix is special orthogonal, and hence

$$P^TP + Q^TQ = 1 \qquad QP^T = PQ^T \qquad Q^TP = P^TQ.$$

Now we may rewrite the tangent vectors appropriately, lift everything to $T\ M$ and then apply them to $\tilde{\ }$ via the fundamental formula (recall $\langle -, - \rangle$ is conjugate linear in the second factor). For example

$$\tilde{X}_i \tilde{\ } = 2\ i\ (\tilde{X}_i) - 2\ i \sum P_{ik}\ (\tilde{Y}_k) - 2 \sum Q_{ik}\ (\tilde{Y}_k).$$

This right hand side vanishes at ~, so indeed $X_i\ = 0$ there. The companion formula is

$$\tilde{Y}_i \tilde{\ } = 2\ i \sum P_{ki}\ (\tilde{X}_k) + 2 \sum Q_{ki}\ (\tilde{X}_k) - 2\ i\ (\tilde{Y}_i).$$

Together, these show that is stationary at $[\ ] \in M_G$, just as in the warm-up example.

5.6 Computation of the Hessian

Another application of the above formulae, remembering that

$$X\ (Y) = d\ (Y)(X) = \omega(Y, X)$$

will obtain formulae for second derivatives such as

$$\tilde{X}_j \tilde{X}_i \tilde{\ } = 2\ i\,\omega_G(X_i, X_j) - 2\ i \sum P_{ik}\,\omega_G(Y_k, X_j) - 2 \sum Q_{ik}\,\omega_G(Y_k, X_j)$$

where everything in this formula is to be evaluated at (for example, the first term now dies), and we have used the defining identity $\omega(\tilde{X}, \tilde{Y}) = \omega_G(X, Y)$ to replace ω by ω_G and remove the tildes from the right-hand side.

We can compute three similar formulae for the second derivatives and form the Hessian matrix for with respect to the basis $\{X_i, Y_i\}$ of $T_{[\ ]}M_G$:

$$(-2\ i) \begin{pmatrix} PQ^T - iQQ^T & Q \\ Q^T & P^TQ - iQ^TQ \end{pmatrix}$$

We can extract the matrix

$$\begin{pmatrix} Q^T & 0 \\ 0 & Q \end{pmatrix}$$

form the right, and expand

$$\det \begin{pmatrix} P - iQ & 1 \\ 1 & P^T - iQ^T \end{pmatrix}$$

as

$$\det((P - iQ)(P^T - iQ^T) - 1) = \det(-2QQ^T - 2iPQ^T)$$

using properties of P and Q discussed earlier. Hence this temporary "unnormalised Hessian" of   is:

$$(-2\ i)^{18} \cdot (-2i)^9 \cdot \det(P - iQ) \cdot \det(Q)^3$$

The reason for separating the last two parts is that $\det Q$ is real, whereas $P - iQ$ represents the change of basis between $\{X_i\}$ and $\{Y_j\}$ as bases of $T_{[\ ]}M_G$ as a 9–dimensional *complex* vector space (ie, it is the matrix of the hermitian form, $(P - iQ)_{ij} = \overline{H_G(X_i, Y_j)}$), so is unitary and contributes just a phase as determinant.

To normalise the Hessian we must compute the volume of the basis $\{X_i, Y_j\}$ with respect to the form $\omega_G^9/9!$ on $T_{[\ ]}M_G$. Expand, using the shu e product, the expression $(\omega_G^9/9!)(X_1, X_2, \ldots, X_9, Y_1, Y_2, \ldots, Y_9)$: since the spaces spanned by the X_i and by the Y_j are isotropic, the terms appearing are simply all possible orderings of all possible products of 9 terms of the form $\omega(X_i, Y_j)$ (the X before the Y). Reordering these cancels the denominator 9! and we obtain simply $\det(Q)$. So the determinant we actually computed was $(\mathrm{vol}(G\ )\det(Q))^2$ times what it should have been when computed in a unimodular basis. Therefore

$$\mathrm{Hess}_{[\ ]}(\ ) = (-2\ i)^{18} \cdot (-2i)^9 \cdot \det(P - iQ) \cdot \det(Q) \cdot \mathrm{vol}(G\ )^{-2} \cdot$$

This is the end of the general nonsense. To go any further we have to choose explicit bases, although not for $T_{[\ ]}M_G$, because of the di culties already mentioned in writing down *any* vectors there. In the next two sections we will write down nice vectors "upstairs" in $T\ M$ and show how to lift the computations of $\det(P - iQ), \det(Q)$ into this space.

5.7 The modulus of the Hessian

We need to compute $\det(Q)$, where $Q_{ij} = -\omega_G(X_i, Y_j)$, and the X_i, Y_j are orthonormal bases as chosen above.

Let us start by introducing some useful vectors in $T\ M$, with which to compute "upstairs". We make an explicit choice of basis for each of the 12–dimensional spaces $\ker d\ ;\ker d$ inside $T\ M$. Recalling that they intersect in the space $\mathfrak{g}$, we arrange for a suitable basis of this space to be easily obtained from each.

Let T_v^l be the in nitesimal rotation about the vector v, acting on the lth triple of vectors from $(x_1; x_2; \ldots; x_{12})$. For example, at any point $(x_1; x_2; \ldots; x_{12})$, we have

$$T_v^1 = (v \quad x_1; v \quad x_2; v \quad x_3; 0; 0; 0; 0; 0; 0; 0; 0; 0):$$

This vector clearly preserves the condition $x_1 + x_2 + x_3 = 0$, as well as the other three $\mathbb{R}^3${coordinate parts of .

Recall that $a; c; e$ are three vectors de ning the tetrahedron . Since $a; c; e$ are linearly independent, the vectors $T_a^1; T_c^1; T_e^1$ span the tangent space to $fx_1 + x_2 + x_3 = 0g$ inside the product of the rst three spheres of M. Combining four such sets of vectors, we see that the 12 vectors T_v^l, where $l = 1; 2; 3; 4$ and v is one of the three vectors $a; c$ or e, span the space $\ker d$ at . For convenience these vectors will also be numbered

$$T_1; T_2; \ldots; T_{12} = T_a^1; T_c^1; T_e^1; T_a^2; \ldots; T_e^4:$$

Note that although the formula de nes a vector eld everywhere on M, we only need the tangent vectors at two speci c points, namely and ^{0}.

We can easily obtain a basis for the in nitesimal diagonal action of G from these:

$$R_a = T_a^1 + T_a^2 + T_a^3 + T_a^4$$

is the in nitesimal rotation of all 12 coordinates about a, and similarly we may de ne $R_c; R_e$, each a sum of four 'T's, which together span $\mathfrak{g}$. We will also denote these by

$$R_1; R_2; R_3 = R_a; R_c; R_e:$$

Let u denote an edge of the tetrahedron , one of the vectors $a; b; c; d; e; f$. Let U_w^u be the in nitesimal rotation about w acting on the pair of spheres corresponding to u. For example, if $u = a$ then we have at $(x_1; x_2; \ldots; x_{12})$

$$U_w^a = (w \quad x_1; 0; 0; 0; 0; 0; 0; 0; w \quad (-x_1); 0; 0; 0);$$

This vector preserves $x_1 + x_9 = 0$ and hence , and so do the other U_w^u. We want just two vectors $w_1; w_2$ such that $U_{w_1}^a; U_{w_2}^a$ span the tangent space to the orbit of G acting on the rst and ninth spheres in M at (compare the previous case with the three 'T's.) Projecting into the rst and ninth spheres, becomes $(a; -a)$ and U_w^a becomes the tangent vector $(w \quad a; w \quad (-a))$. So all we need to do is pick $w_1; w_2$ such that $a; w_1; w_2$ are linearly independent. In this way we can construct 12 vectors spanning ker d at .

Unfortunately there isn't a totally systematic way of deciding which two values of w we should use, given u. We can at least choose them always to be two of the three vectors $a; c; e$, which forces for example the use of $U^a_c; U^a_e$ among our 12 vectors (because $U^a_a = 0$).

The twelve explicit choices are as follows:

$$U_1; U_2; ::: ; U_{12} = U^a_c; U^a_e; \quad U^b_a; U^b_e; \quad U^c_e; U^c_a; \quad U^d_c; U^d_a; \quad U^e_a; U^e_c; \quad U^f_c; U^f_e$$

The same three diagonal generators $R_a; R_c; R_e$ can be expressed in terms of these vectors by observing that

$$R_a = U^a_a + U^b_a + U^c_a + U^d_a + U^e_a + U^f_a$$

that $U^a_a = 0$ and that U^f_a (which is the only other not among our chosen $U_1; U_2; ::: ; U_{12}$) satis es $U^f_a = -U^f_e$, because the fact that the three sides of the tetrahedron satisfy $-e + f - a = 0$ implies

$$U^f_e + U^f_a = U^f_e - U^f_f + U^f_a = U^f_{e-f+a} = U^f_0 = 0:$$

Similarly, one obtains $U^b_c = -U^b_a$ and $U^d_e = +U^d_c$, and hence:

$$\begin{aligned} R_a &= U_3 + U_6 + U_8 + U_9 - U_{12} \\ R_c &= U_1 - U_3 + U_7 + U_{10} + U_{11} \\ R_e &= U_2 + U_4 + U_5 + U_7 + U_{12} \end{aligned}$$

In the following calculation, a symbol such as det $!(fX_ig; fY_ig)$, where fX_ig and fY_ig are some sets of vectors, will mean the determinant of the matrix whose entries are all evaluations of $!$ on pairs consisting of an element from the rst set followed by one from the second set (arranged in the obvious way). In the case where the two sets of vectors are both bases of some xed vector space, the symbol det$(fX_ig=fY_ig)$ will be the determinant of the linear map taking Y_i 7! X_i. The grossly-abused subscript i below stands for the complete list of such vectors (there are twelve 'T's, three 'R's, etc.) We regard all vectors as living in $T\ M$, in particular the original orthonormal bases are lifted horizontally into it. Let $fe_1; e_2; e_3g$ be some orthonormal basis of $\mathfrak{g}$.

By extending the orthonormal sets of vectors and then changing bases inside the spaces ker d and ker d to bring in the 'T's and 'U's , we have

$$\begin{aligned} \det(Q) &= -\det !(fX_ig; fY_ig) \\ &= -\det !(fX_i; e_i; Je_ig; fY_i; e_i; Je_ig) \\ &= -\det(fT_ig=fX_i; e_ig)^{-1} \det(fU_ig=fY_i; e_ig)^{-1} \\ &\qquad \det !(fT_i; Je_ig; fU_i; Je_ig) \end{aligned}$$

The remaining det ω term can be simplified further. Replace three of the 'T's (T_{10}, T_{11}, T_{12}) and three 'U's (U_8, U_{11}, U_4) by R_1, R_2, R_3. According to the earlier expressions for the R_i, each replacement is unimodular, and there is no sign picked up in reordering the 'U's to put U_8, U_{11}, U_4 (in that order) last. Then we change the 'R's back to 'e's and remove them. This gives:

$$\begin{aligned}
&\det \omega(\{T_i, Je_i\}, \{U_i, Je_i\}) \\
&= \det \omega(\{T_1, \ldots, T_9, R_i, Je_i\}, \{U_1, U_2, U_3, U_5, U_6, U_7, U_9, U_{10}, U_{12}, R_i, Je_i\}) \\
&= \det(\{R_i\}/\{e_i\})^2 \\
&\qquad \det \omega(\{T_1, \ldots, T_9, e_i, Je_i\}, \{U_1, U_2, U_3, U_5, U_6, U_7, U_9, U_{10}, U_{12}, e_i, Je_i\}) \\
&= \det(\{R_i\}/\{e_i\})^2 \det \omega(\{T_1, \ldots, T_9\}, \{U_1, U_2, U_3, U_5, U_6, U_7, U_9, U_{10}, U_{12}\})
\end{aligned}$$

This remaining 9×9 determinant has to be done by explicit calculation. Fortunately the good choice of vectors helps enormously. The 'T's have only three non-zero coordinates (out of 12), the 'U's have only two, and these must overlap if there is to be a non-zero matrix entry. So a representative non-zero matrix element is something like

$$\omega(T_v^l, U_w^u) = \frac{1}{4x^2}[x.(v \times x).(w \times x)] = \frac{1}{4}[x.v.w]$$

where x is whichever of the 'x_i's corresponds to the overlap. (It will be plus or minus one of a, b, c, d, e, f, depending on whether the overlap of coordinates happens in the first or second of the two non-zero slots of the 'U'–vector, respectively.)

Writing down the matrix with rows corresponding to $T_1, T_2, \ldots, T_9$ and columns corresponding to $U_c^a, U_e^a, U_a^b, U_e^c, U_a^c, U_c^d, U_a^e, U_c^e, U_e^f$ gives

$$\frac{1}{4}\begin{pmatrix}
0 & 0 & 0 & [cae] & 0 & 0 & 0 & 0 & 0 \\
0 & [ace] & [bca] & 0 & 0 & 0 & 0 & 0 & 0 \\
[aec] & 0 & [bea] & 0 & [cea] & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -[cae] & 0 & [dac] & 0 & [eac] & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & [eca] & 0 & 0 \\
0 & 0 & 0 & 0 & -[cea] & [dec] & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -[eac] & [fae] \\
0 & -[ace] & 0 & 0 & 0 & 0 & -[eca] & 0 & [fce] \\
-[aec] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

where $[ace]$ is just a shorthand for the vector triple product $[a.c.e]$. Substituting the relations $b = -a - c, d = c - e, f = a + e$ and extracting the factor of $[ace]$

gives

$$\frac{[ace]}{4}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix has determinant $[ace]^9/(4\ )^9$.

The various change-of-basis determinants may be evaluated in terms of orbit volumes. If we denote by $\mathrm{sgn}(\{X_i\}/\{Y_i\})$ the sign of the determinant of the appropriate transformation then

$$\det(\{T_i\}/\{X_i, e_i\}) = \mathrm{sgn}(\{T_i\}/\{X_i, e_i\})\,\mathrm{vol}_B\ \{T_i\}$$

because $\{X_i, e_i\}$ is B –orthonormal. The volume is given by

$$\mathrm{vol}_B\ \{T_i\} = \frac{\mathrm{vol}(G^4\ \ )}{\mathrm{vol}\ (G)^4}[ace]^4$$

by lemma 6 and the fact that the 12 'T's separate into four orthogonal triplets coming from the Lie algebra elements $a, c, e \in \mathbb{R}^3$. Similarly we have for the 'U' case:

$$\det(\{U_i\}/\{Y_i, e_i\}) = \mathrm{sgn}(\{U_i\}/\{Y_i, e_i\})\,\mathrm{vol}_B\ \{U_i\}$$

The volume term may be expressed via lemma 7. This will involve a product of six terms of the form vol $\{\Lambda_i\}$: for each choice of u, we have to calculate the area spanned by the vectors w_1 and w_2 once projected into the orthogonal complement of u. This is just $|[w_1.w_2.u]/u|$. Substituting the explicit choices we made gives

$$\mathrm{vol}_B\ \{U_i\} = \frac{\mathrm{vol}(G^6\ \ )}{\mathrm{vol}\ (G/T)^6}\frac{[ace]^6}{\prod a}$$

where the product on the right hand is simply $abcdef$. Yet another application of lemma 6 yields:

$$\det(\{R_i\}/\{e_i\})^2 = \left(\frac{\mathrm{vol}(G\ \ )}{\mathrm{vol}\ (G)}\right)^2 [ace]^2$$

The sign terms here depend on the original choice of orthonormal bases, which we did not specify. So we do not yet know the actual sign of $\det(Q)$. Similar terms will appear in the computation of $\det(P - iQ)$, however, so that the product of the two terms does not depend on the original choice. So far we have

$$\det(Q) = -\operatorname{sgn}(\{T_i\}/\{X_i; e_i\})\operatorname{sgn}(\{U_i\}/\{Y_i; e_i\})\left(\frac{[ace]}{4}\right)^9 \left(\frac{\mathrm{vol}(G^4\ )}{\mathrm{vol}\ (G)^4}[ace]^4\right)^{-1}\left(\frac{\mathrm{vol}(G^6\ )}{\mathrm{vol}\ (G/T)^6}\frac{[ace]^6}{\prod a}\right)^{-1}\left(\frac{\mathrm{vol}(G\ )[ace]}{\mathrm{vol}\ (G)}\right)^2 \tag{11}$$

5.8 Phase of the Hessian

We work out $\det(P - iQ)$ (de ned by $((P - iQ)_{ij} = \overline{H_G(X_i; Y_j)})$) using similar techniques. We choose slightly di erent bases in $T\ M$ this time.

For each face of the tetrahedron , numbered by l as earlier, choose three in nitesimal rotation vectors T^l_v by letting v be an exterior unit normal vector v_l to the face or one of the two edges x_{3l-2}, x_{3l-1} of that face. These occur in clockwise order, so that

$$x_{3l-2} \quad x_{3l-1} = A_l v_l$$

with A_l twice the area of the lth face. (Of course at the point , we know that each x_i is just one of the vectors a, b, c, d, e, f or their negatives. However, it is easier to calculate without substituting these yet.) Pick a set of 12 vectors U^u_w rather as before, except that given an edge u, we allow w to be the exterior unit normal to either of the two faces incident at u. Order the two choices so that the rst cross the second points along u (in fact this corresponds to v_i coming before v_j i $i < j$). Figure 7 shows these where these vectors are in $\mathbb{R}^3$, given the tetrahedron . We will refer to the chosen vectors as $U'_1, \ldots, U'_{12}$ and $T'_1, T'_2, \ldots, T'_{12}$.

By a familiar change of basis procedure

$$\begin{aligned}\det(P - iQ) &= \det H_G(\{Y_i\}, \{X_i\}) \\ &= \det H(\{Y_i; e_i\}, \{X_i; e_i\}) \\ &= \det(\{T'_i\}/\{X_i; e_i\})^{-1}\det(\{U'_i\}/\{Y_i; e_i\})^{-1}\det H(\{U'_i\}, \{T'_i\}).\end{aligned}$$

Since we are know the determinant is actually just a phase, we can throw away any positive real factors appearing during the computation. For example, the correcting determinants above may immediately be replace by correcting signs,

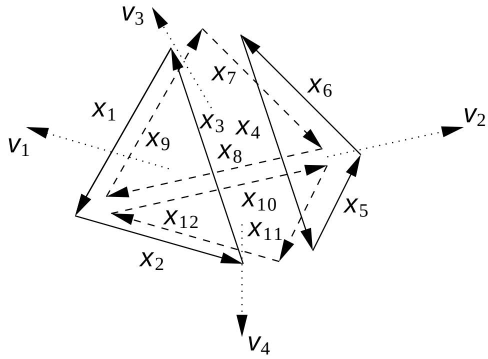


Figure 7: The relevant vectors

because the difference (a volume) is positive. This principle also facilitates the direct computation of $\det H(\{T_i^0\},\{U_i^0\})$ too.

Let us compute sample non-zero elements (once again, most of the matrix elements will be zero):

$$
\begin{aligned}
H(U_w^u, T_v^l) &= \omega(T_v^l, JU_w^u + iU_w^u) \\
&= \frac{1}{4\pi|x|^3}[x\cdot(v\times x)\cdot(x\times(w\times x))] + \frac{i}{4\pi|x|^2}[x\cdot(v\times x)\cdot(w\times x)] \\
&= \frac{1}{4\pi}(|x|(v\cdot w) + i[xvw])
\end{aligned}
$$

using the earlier notation for the triple product, and with x being whichever of the 'x_i's corresponds to the overlap of the non-zero coordinates of U_w^u, T_v^l. There has been some simplification because $w\cdot x = 0$.

Let us immediately forget about the 4π factors. If the ith and jth faces meet in an edge u, oriented along the direction of $v_i \times v_j$, then the exterior dihedral angle, written θ_u or θ_{ij}, is defined (in the range $(0,\pi)$) by

$$u\sin(\theta_{ij}) = [u\cdot v_i\cdot v_j]$$

$$\cos(\theta_{ij}) = v_i\cdot v_j.$$

In performing the computation we run across three kinds of non-zero matrix elements. If $v = v_l$ is normal to the lth face, and $w = v_k$ then we obtain $|x|\theta_{kl} \quad |x|e^{i\theta_{kl}}$. (If $k = l$ it is simply $|x|$.) If instead v is a vector lying in the lth face, and $w = v_k$, then we distinguish according to whether $k = l$ or not. In the case of equality, we get case we get 0, iA_l or $-iA_l$ according to whether v is x, its successor, or predecessor in the anticlockwise cyclic ordering around the face. If $k \neq l$ then we get 0, $iA_{l\ kl}$ or $iA_{l\ kl}$, according to the same conditions.

We can throw out the area factors and the eight powers of i coming from these second and third cases, and end up with a matrix:

$$\begin{pmatrix}
a & a_{13} & b & b_{14} & c & c_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -{}_{14} & 1 & {}_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & {}_{13} & 0 & 0 & -1 & -{}_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{12} & c & d & d_{24} & e & e_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -{}_{24} & 1 & {}_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & {}_{12} & 1 & 0 & 0 & -1 & -{}_{23} & 0 & 0 \\
a_{13} & a & 0 & 0 & 0 & 0 & 0 & 0 & e_{23} & e & f & f_{34} \\
{}_{13} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -{}_{34} \\
-{}_{13} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & {}_{23} & 1 & 0 & 0 \\
0 & 0 & b_{14} & b & 0 & 0 & d_{24} & d & 0 & 0 & f_{34} & f \\
0 & 0 & {}_{14} & 1 & 0 & 0 & -{}_{24} & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -{}_{14} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & {}_{34} & 1
\end{pmatrix}$$

Easy row operations and then permutation of the rows reduces the determinant to minus that of the direct sum of the six 2 2 blocks

$$\begin{matrix} 1 & {}_{ij} \\ {}_{ij} & 1 \end{matrix} :$$

The determinant of such a block is

$$- {}_{ij} . 2i \sin({}_{ij}).$$

Discarding the sines, which are positive and so only a ect the modulus of the determinant, we nd that its phase is

$$e^{i \sum} {}^{a} .$$

Let us collect up $\det(P - iQ) \det(Q)$ nally. The annoying sign terms may be combined into

$$\mathrm{sgn}(fT_i g / fT_i^0 g)\,\mathrm{sgn}(fU_i g / fU_i^0 g).$$

The rst term is a product of four signs arising from orientations of a three-dimensional vector space, and the second a product of six arising from two-dimensional spaces. All these signs are positive (an easy check).

Thus we have for the Hessian:

$$\mathrm{Hess}_{[\ ]}(\ ) = -i(2\ )^{18} 2^9 e^{i\sum\ _a} \left(\frac{[ace]}{4}\right)^9$$

$$\left(\frac{\mathrm{vol}(G^4\ )}{\mathrm{vol}\ (G)^4}[ace]^4\right)^{-1} \left(\frac{\mathrm{vol}(G^6\ )}{\mathrm{vol}\ (G/T)^6}\frac{[ace]^6}{\prod_a}\right)^{-1} \left(\frac{[ace]}{\mathrm{vol}\ (G)}\right)^2$$

$$= -ie^{i\sum\ _a}(2\ )^9[ace] \prod_a \frac{\mathrm{vol}\ (G)^2\,\mathrm{vol}\ (G/T)^6}{\mathrm{vol}(G^4\ )\,\mathrm{vol}(G^6\ )} \qquad (12)$$

Looking at the argument again, it is easy to see that the Hessian at $[\ ^0]$ is the complex conjugate of this one.

5.9 The overall phase of the integrand

We need to account for the 0{order contributions $([\ ^0]);\ ([\ ])$ of the integrand at the two critical points. Since the two sections being paired were xed to have norm 1 along their critical regions, these 0{order contributions also have modulus 1. We start by calculating the phase *di erence*.

The chosen lifts $;\ ^0$ of these points lie on the slice $= 0$ inside M. Since this slice is just a product of four \spaces of triangles", each of which is a principal G{space, ($G = SO(3)$) there is a unique element of G^4 which translates to ^{0}. In fact it is easy to describe such an element $g = (g_1; g_2; g_3; g_4)$ explicitly. The element g_1 must rotate the triangle $(a; b; c)$ (the projection of into the rst three sphere factors of M) to its negative $(-a; -b; -c)$. Therefore it is the rotation of about the normal to the triangle's plane. The other g_i are similarly half-turns normal to their respective triangular faces.

We must compare the values of the pointwise pairing $s\ ; s$ at $;\ ^0$. De ne for each face a lift g_i of g_i into $SU(2)$ by lifting the path of anticlockwise rotations from 0 to . Together these form $g\ 2\ SU(2)^4$. Since s is $SU(2)^4${invariant:

$$s\ (\ ^0) = s\ (g\ ) = gg^{-1}(s\ (g\ )) = g((g^{-1}s\ )(\ )) = g(s\ (\ ))$$

By contrast, s is not $SU(2)^4${invariant, though it is $SU(2)^6${invariant. We can write an equation like the above but we need to know what $(g^{-1}s\ )$ is to perform the last step. Now s is a sextuple tensor product, and we can study the action of g^{-1} on it by looking at the action on the six factors individually:

$$g^{-1}s\ = (g_1^{-1}; g_3^{-1})s^{aa}\quad (g_1^{-1}; g_4^{-1})s^{bb}\qquad (g_3^{-1}; g_4^{-1})s^{ff}$$

Using the diagonal invariance of each section of the form s^{aa}, we have identities like

$$(\tilde g_1^{-1}, \tilde g_3^{-1})s^{aa} = (1, \tilde g_3^{-1}\tilde g_1)s^{aa}.$$

Now $\tilde g_1, \tilde g_3$ are lifts of rotations through π about directions normal to the two faces of the tetrahedron meeting at side a, so the composite $\tilde g_3^{-1}\tilde g_1$ is a lift of an anticlockwise rotation through an angle equal to twice the exterior dihedral angle θ_a about the vector a, but it is slightly tricky to decide *which* lift. Let us denote by $\tilde r_a$ the lift of the path of anticlockwise rotations from 0 to $2\theta_a$, and write $\tilde g_3^{-1}\tilde g_1 = \epsilon_a \tilde r_a$, for some $\epsilon_a = \pm 1$. Then

$$\begin{aligned}((\tilde g_1^{-1}, \tilde g_3^{-1})s^{aa})(a, -a) &= (1, \epsilon_a \tilde r_a)(s^{aa}((1, \epsilon_a \tilde r_a^{-1})(a, -a))) \\ &= (1, \epsilon_a \tilde r_a)(s^{aa}(a, -a)).\end{aligned}$$

The action of $\tilde r_a$ on the fibre of the bundle $L^{\otimes a} \to \mathbb{P}^1$ at $-a$ is multiplication by $e^{-ia\theta_a}$ (remember that it acts as $e^{i\theta_a}$ on the fibre of the tangent bundle $L^{\otimes 2}$ at a). The sign ϵ_a acts as its ath power ϵ_a^a. So, using the invariance of the hermitian form,

$$\begin{aligned}\langle \tilde s(\tau'), \tilde s(\tau')\rangle &= \langle \tilde g(\tilde s(\tau)), \tilde g(\prod \epsilon_a^a\, e^{-i\sum a\theta_a} \tilde s(\tau))\rangle \\ &= \prod \epsilon_a^a\, e^{i\sum a\theta_a} \langle \tilde s(\tau), \tilde s(\tau)\rangle.\end{aligned}$$

In fact this identity is independent of the choice of lifts $\tilde g_i$. For example, changing the lift of $\tilde g_1$ negates $\epsilon_a, \epsilon_b, \epsilon_c$, changing the right-hand side by $(-1)^{a+b+c}$, which is $+1$ because of the parity condition on a, b, c. Further, if we imagine varying the dihedral angles of the tetrahedron in the range $(0, \pi)$, all our chosen lifts are continuous and so we may evaluate the sign $(\prod \epsilon_a^a)$ by deformation to a flat one, for example where $\theta_a, \theta_b, \theta_c$ are π and the others 0. In this case, $\tilde g_2 = \tilde g_3 = \tilde g_4 = -\tilde g_1$, and so all the ϵ_a turn out positive. Hence $\Phi([\tau']) = e^{ik\sum a\theta_a}\Phi([\tau])$. Because the $6j$–symbol itself is *real*, the two values of Φ must be conjugate. Therefore $\Phi([\tau']) = \pm e^{\frac{1}{2}ik\sum a\theta_a}$, but we still have an annoying sign ambiguity.

There are really three separate sign problems: how the sign depends on k (for fixed $a, b, \dots, f$); how it varies as we alter $a, b, \dots, f$; and one overall choice of sign. One step is easy: the phase conventions on the sections used in the pairing implied for example that $s^{ka,ka} = (s^{aa})^{\otimes k}$ and that the integrand was a kth power of another function, the sign above must actually be $(-1)^k$ or ± 1.

To do better requires a frustrating amount of work, which will only be sketched here. Recall that the signs in definition 3 were fixed by writing down explicit polynomial representatives of the trilinear and bilinear invariants. If we remove a suitable branch cut from each copy of the sphere in M, leaving a contractible manifold, we can extend this de nition to allow real values of the variables $a; b; \ldots; f$, and extend $([\ ])$ to a *real-analytic* function of these variables (at least locally). It is obtained by pairing two holomorphic sections of a trivial bundle with a hermitian structure which still satis es the quantization formula (4). There is another way of computing the phase di erence above, based on choosing two paths $\ ! \ {}'$ in M, one on which $= 0$ and one on which $= 0$, and computing the holonomy around the resulting loop. This can be done by computing the symplectic area of a bounded disc. To carry this out appropriately one must be very careful with *which* disc: once the symplectic form no longer has integral periods on $H_2(M)$, this matters. Further, the most obvious paths and disc intersect the branch cuts, so one must account for this too. Ultimately one obtains an analytic expression

$$([\ {}']) = \ e^{\frac{1}{2}ik\sum a \ _a + i\sum a}$$

where the analyticity restricts this sign to a single *overall* ambiguity. (Note that at integral values of the lengths, we can see the sign $(-1)^{\sum a}$ appearing.) One could compute this sign using a single example, but as the reader will judge from this terse paragraph, the author is so bored with xing signs that he no longer cares to! The experimental evidence in [14] con rms that the sign is positive.

Therefore

$$([\ {}'] = (-1)^{\sum a} e^{\frac{1}{2}ik\sum a \ _a} \quad \text{and} \quad ([\ ]) = (-1)^{\sum a} e^{-\frac{1}{2}ik\sum a \ _a}. \qquad (13)$$

5.10 Putting it all together

We combine the original integral de nition (7) with the asymptotic normalisation factors (8), (9), the reduction (10), the stationary phase evaluation (6) incorporating the Hessian (12) and 0–order terms (13).

$$\begin{matrix} ka & kb & kc \\ kd & ke & kf \end{matrix} \qquad \prod \sqrt{ka+1} \left(\frac{k}{2}\right)^{-6} (\mathrm{vol}(\ ^{-1}(0))\,\mathrm{vol}(\ ^{-1}(0)))^{-\frac{1}{2}} k^9 \left(\frac{k}{2}\right)^{\frac{3}{2}}$$

$$\left(\frac{2}{k}\right)^9 \left\{ \frac{e^{-\frac{1}{2}ik\sum a \ _a}}{\sqrt{-\mathrm{Hess}_{[\ ]}(\ )}} + \frac{e^{\frac{1}{2}ik\sum a \ _a}}{\sqrt{-\mathrm{Hess}_{[\ {}']}(\ )}} \right\} :$$

The terms from the Hessian and the normalisation involving

$$\mathrm{vol}(\ ^{-1}(0)) = \mathrm{vol}(G^4_{\ }) = \mathrm{vol}(G^4_{\ '}) \text{ and } \mathrm{vol}(\ ^{-1}(0)) = \mathrm{vol}(G^6_{\ }) = \mathrm{vol}(G^6_{\ '})$$

cancel. The normalisation factor $(\prod(ka+1))^{\frac{1}{2}}$ cancels with the term in the Hessian involving $\prod a$, contributing asymptotically simply k^3. What remains is

$$(2\pi)^{\frac{9}{2}}2^{\frac{11}{2}}k^{-\frac{3}{2}}[ace]^{-\frac{1}{2}}\,\mathrm{vol}\,(G)^{-1}\,\mathrm{vol}\,(G/T)^{-3}\cos\Big(\sum(ka+1)\frac{\theta_a}{2}+\frac{\pi}{4}\Big):$$

Substituting in the volumes $8\pi^2$ and 4π of G and G/T gives

$$\begin{Bmatrix} ka & kb & kc \\ kd & ke & kf \end{Bmatrix} \sim \sqrt{\frac{2}{3\pi k^3 V}}\cos\Big(\sum(ka+1)\frac{\theta_a}{2}+\frac{\pi}{4}\Big)$$

where $V = \frac{1}{6}[ace]$ is the (scaling-independent) volume of τ. This completes the proof of the theorem.

6 Further geometrical remarks

6.1 Comparison with the Ponzano–Regge formula

It is important to note that the formula (2) is *not* the same as the original Ponzano–Regge formula. There are two main differences, apart from the trivial fact that they label their representations by half-integers instead of integers.

Their claim, in our integer-labelling notation, is that for large a, b, c, d, e, f:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \sim \begin{cases} \sqrt{\dfrac{2}{3\pi V'}}\cos\Big(\sum(a+1)\dfrac{\theta'_a}{2}+\dfrac{\pi}{4}\Big) & \text{if } \tau' \text{ is Euclidean,} \\ \text{exponentially decaying} & \text{if } \tau' \text{ is Minkowskian,} \end{cases} \qquad (14)$$

where τ' is a tetrahedron whose edges are $a+1, b+1, \ldots, f+1$ and whose dihedral angles θ'_a and volume V' are therefore slightly different from those of our τ. This difference is worrying, as it is quite possible to find sextuples of integers such that τ' is Euclidean yet τ is Minkowskian, in which case the formulae seem to conflict: is the $6j$–symbol exponentially or polynomially decaying in this case?

The second difference explains this. The Ponzano–Regge formula (14) is only claimed as an *approximation* for large irreducibles, rather than an asymptotic expansion in a strict sense as in theorem 1. Therefore the only meaningful comparison between the two formulae is to examine how their function behaves as we rescale a, b, c, d, e, f by $k \to \infty$ in the precise sense of our theorem.

Although for small k it is possible that τ' might be Euclidean when τ is not, eventually the shift in edge-lengths becomes insignificant and either both are Euclidean or neither is. Therefore there is no inconsistency between cases in the two formulations.

As for comparing the actual formulae, the asymptotic behaviour of the Ponzano–Regge function is

$$\sqrt{\frac{2}{3\pi k^3 V}} \cos\left(\sum (ka+1)\frac{\theta'_a}{2} + \frac{\pi}{4}\right); \tag{15}$$

because V and V' agree to leading order in k. The only problem is the dihedral angles relating to slightly different tetrahedra. Fortunately we may easily show that

$$e^{ik\sum(a+1)\theta'_a} \sim e^{ik\sum(a+1)\theta_a}$$

by applying the *Schläfli identity* (see Milnor [11]), which says that the differential form $\sum a\, d\theta_a$ vanishes identically on the space of Euclidean tetrahedra. Therefore there is no inconsistency.

Remark The case of a flat tetrahedron is not covered by either formula.

6.2 Regge symmetry and scissors congruence

Suppose one picks out a pair of opposite sides of the tetrahedron denoting the $6j$–symbol (as in figure 1), say a, d. Let s be half the sum of the other four labels (twice their average). Define:

$$\begin{aligned} a' &= a & b' &= s - b \\ d' &= d & c' &= s - c \\ & & e' &= s - e \\ & & f' &= s - f \end{aligned} \tag{16}$$

Regge discovered that the $6j$–symbols are invariant under this algebraic operation (the easiest way to see this is to look at the generating function for $6j$–symbols, [15]):

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Bmatrix} a' & b' & c' \\ d' & e' & f' \end{Bmatrix}$$

We can also consider this as a *geometric* operation on a tetrahedron, altering its side lengths according to the above scheme. It is not meant to be obvious that the result of applying this to a Euclidean tetrahedron will return a Euclidean one!

Regge and Ponzano considered the effect of this symmetry on the geometrical quantities occurring in their asymptotic formula, mainly as another check on its plausibility. They discovered that the volume and phase term associated to a Euclidean tetrahedron are indeed exactly invariant. This is amazing, given that it would be consistent with their appearance in an *asymptotic* expansion for them to change, but by lower-order contributions.

Let us reconsider this surprising geometric symmetry. First note that the symmetry is an involution: if one thinks geometrically, it corresponds to reflecting the lengths of the four chosen sides about their common average. These involutions, together with the tetrahedral symmetries, form a group of 144 symmetries of the $6j$–symbol, isomorphic to $S_4 \times S_3$ (see [15]).

V. G. Turaev pointed out to me that the term $\sum l_i \theta_i$ in the phase part of the formula (2) for a tetrahedron Δ is reminiscent of the Dehn invariant $\delta(\Delta)$. Actually it would be fairer to say that it is the "Hadwiger measure" (or "Steiner measure") $\phi_1(\Delta)$. Both invariants are connected with problems of equidissection of three-dimensional polyhedra.

Two polyhedra are *scissors congruent* if one may be dissected into finitely-many subpolyhedra which may be reassembled to form the other. (Hilbert's third problem was to determine whether three-dimensional polyhedra with equal volumes were, as is the case in two dimensions, scissors-congruent. Dehn used his invariant to solve this problem in the negative.)

The modern way of looking at the problem is to define a Grothendieck group of polyhedra $\mathcal{P}$. We take $\mathbb{Z}$–linear combinations of polyhedra in $\mathbb{R}^3$ with the relations:

$$P \cup Q = P + Q - P \cap Q \tag{17}$$

$$P = 0 \quad \text{if } P \text{ is degenerate} \tag{18}$$

$$P = Q \quad \text{if } P, Q \text{ are congruent} \tag{19}$$

Volume is an obvious homomorphism $\mathcal{P} \to \mathbb{R}$. The Dehn invariant is a less obvious one $\mathcal{P} \to \mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z})$, defined for a polyhedron by summing, over its edges, their lengths tensor dihedral angles:

$$\delta(P) = \sum l_i \otimes \theta_i$$

Sydler proved that these two invariants suffice to *classify* polyhedra up to scissors-congruence: two such are scissors-congruent *if and only if* they have the same volume and Dehn invariant. See Cartier [2] for more details.

If we look for homomorphisms $\mathcal{P} \to \mathbb{R}$ which are *continuous* under small perturbations of vertices of a polyhedron, then volume is the only one (up to scaling). However, if we remove the condition (18) on degenerate polyhedra from the axioms defining $\mathcal{P}$, there is a four-dimensional vector space of continuous homomorphisms, spanned by the following *Hadwiger measures* (picked out as eigenvectors under dilation):

$$\mu_3(P) = \mathrm{vol}(P) \tag{20}$$

$$\mu_2(P) = \tfrac{1}{2}\mathrm{vol}(\partial P) \tag{21}$$

$$\mu_1(P) = \sum l_i \theta_i \tag{22}$$

$$\mu_0(P) = \chi(P) \quad \text{(the Euler characteristic)} \tag{23}$$

See Milnor [10] or Klain and Rota [8] for more on these beautiful functions.

The relationship with the Regge symmetry of tetrahedra is as follows:

Theorem 9 *The Regge symmetry* (16) *takes Euclidean tetrahedra to Euclidean tetrahedra, preserving volume, Dehn invariant and Hadwiger measure* μ_1. (*Remark: simple examples show that Regge symmetry does not preserve the surface area measure* μ_2.)

Proof The tour-de-brute-force of trigonometry in appendices B and D of [14] contains all the calculations necessary to prove this. They demonstrate that under the Regge symmetry, which is a rational linear transformation A of the six edge lengths, the dihedral angles also transform according to A. The orthogonality of this matrix and the fact that we may view the Dehn invariant as being in $\mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\pi\mathbb{Z}) \cong \mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z})$ demonstrate its invariance, as well as that of μ_1. The volume is checked by straightforward calculation using the Cayley determinant. □

Corollary 10 *The orbit under the group of 144 symmetries of a generic tetrahedron consists of* twelve *distinct congruence classes of tetrahedra, all of which are scissors-congruent to one another.*

Remark The fact that a tetrahedron is scissors-congruent to its mirror-image was proved by Gerling in 1844 (see Neumann [13]), using perpendicular barycentric subdivision about the circumcentre. One would expect that for the Regge symmetry, which is also a \generic" scissors congruence (as opposed to a \random" coincidence of volume and Dehn invariant for two speci c tetrahedra), a similar general construction might be given. What is it?

6.3 Further questions

1. Ponzano and Regge give an explicit formula for the exponential decay of the $6j${symbol, in the case when no Euclidean tetrahedron exists. Amazingly, it is an analytic continuation of the main formula, incorporating the volume of the *Minkowskian* tetrahedron which exists instead, and with the oscillatory phase term converted into a decaying hyperbolic function. Can this be extracted from a similar procedure?

2. Can similar geometrically-meaningful formulae be obtained for general spin networks, the so-called $3nj${symbols?

3. The calculation in this paper is comparatively crude, since it computes a pairing of 12{linear invariants when one could really do with a pairing of 4{linear invariants (see the remark of section 2.2). The space whose quantization gives quadrilinear invariants is 2{dimensional, in fact a sphere with a non-standard Kähler structure. Can one work directly on this space instead? (Possibly any advantage in dimensional reduction is lost when one needs to do explicit calculations, which end up like the ones here).

4. Can similar formulae be obtained for other groups, and does their associated geometry have any physical meaning? The $6j${symbols are scalars only for multiplicity-free groups such as $SU(2)$. In general they live in the tensor product of four trilinear invariant spaces, in which one would need preferred bases.

5. Can one obtain similar formulae for quantum $6j${symbols, which arise as pairings in the quantization of moduli spaces of flat connections on the 4{ punctured sphere?

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