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Increasing trees and Kontsevich cycles

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Abstract

It is known that the combinatorial classes in the cohomology of the mapping class group of punctures surfaces defined by Witten and Kontsevich are polynomials in the adjusted Miller–Morita–Mumford classes. The leading coefficient was computed in [4]. The next coefficient was computed in [6]. The present paper gives a recursive formula for all of the coefficients. The main combinatorial tool is a generating function for a new statistic on the set of increasing trees on $2n+1$ vertices. As we already explained in [6] this verifies all of the formulas conjectured by Arbarello and Cornalba [1]. Mondello [10] has obtained similar results using different methods.

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Introduction

This is the last of three papers on the relationship between the adjusted Miller–Morita–Mumford (MMM) classes $\widetilde{\kappa}_n$, also known as *tautological classes* (times $(-1)^{n+1}$), in the integral cohomology of the mapping class group and certain combinatorial classes defined by Witten and Kontsevich. In the first paper [4] we showed that these combinatorial classes $[W_\lambda^*]$, are polynomials in the MMM classes and we computed the leading coefficient:

$$[W_\lambda^*] = \prod_{i=1}^{r} \frac{((-2)^{k_i+1}(2k_i+1)!!)^{n_i}}{n_i!} \widetilde{\kappa}_\lambda + \text{ lower terms} \tag{1}$$

if $\lambda = k_1^{n_1} k_2^{n_2} \cdots k_r^{n_r}$ is a partition of $\sum n_i k_i$ into $\sum n_i$ parts. Here we use the notation of our second paper [6]

$$\widetilde{\kappa}_\lambda = \prod_{i=1}^{r} \widetilde{\kappa}_{k_i}^{n_i}.$$

The formula (1) was conjectured by Arbarello and Cornalba [1] and answers questions posed by Witten and Kontsevich [8]. The introduction of [4] gives a more detailed history of the problem.

In the next paper [6] we rephrased the theorem (1) above in terms of graph cohomology using an integral version of Kontsevich's theorem that the cohomology of the mapping class group is rationally isomorphic to the double dual of the graph homology of connected ribbon graphs. We also computed $a_{n,1}^{n+1}$ which is the next case of a coefficient in the polynomial (1) and the dual coefficient $b_{n,1}^{n+1}$. The notation is:

$$[W_\lambda^*] = \sum_\mu a_\lambda^\mu \widetilde{\kappa}_\mu, \qquad \widetilde{\kappa}_\mu^\lambda = \sum_\lambda b_\mu^\lambda [W_\lambda^*] \tag{2}$$

where a_λ^μ and b_μ^λ are rational numbers.

The formula proved in [6] is

$$a_{n,1}^{n+1} = \frac{-12a_n - (2n+5)a_{n+1}}{\mathrm{Sym}(n,1)}, \qquad b_{n,1}^{n+1} = \frac{2n+5}{12a_n} + \frac{1}{a_{n+1}} \tag{3}$$

where $a_n = (-2)^{n+1}(2n+1)!!$ and $\mathrm{Sym}(n,1) = 1 + \delta_{n1}$ is the number of symmetries of $(n,1)$ (equal to 2 if $n=1$ and 1 otherwise).

The purpose of the present paper is to complete this project by giving an algorithm for computing all of the coefficients $a_\lambda^\mu, b_\lambda^\mu$ and, as an example, obtaining the following generalization of (3) conjectured in [6].

$$a^{n+k}_{n,k} = \frac{-(2n+2k+3)a_{n+k} - a_n a_k}{\mathrm{Sym}(n,k)}, \qquad b^{n+k}_{n,k} = \frac{2n+2k+3}{a_n a_k} + \frac{1}{a_{n+k}} \tag{4}$$

In the meantime, Gabriele Mondello has also obtained the same result [10].

The contents of this paper are as follows. The first section summarizes the definitions and results of the previous two papers. In section 2 we study the degenerate case corresponding to degree 0 MMM class $\widetilde{\kappa}_0$ which is equal to the Euler characteristic considered as a function (0–cocycle) on the space of ribbon graphs. This is related in a simple way to the degenerate dual Witten cycle W_0^* which counts the number of trivalent vertices of a ribbon graph. The formula involves Stirling numbers of the first and second kind.

In the third section we show that the determination of the numbers a^μ_λ and b^μ_λ is equivalent to the determination of the cup product structure of the dual Kontsevich cycles. This is more or less obvious. The coefficients in the product are not all integers since the dual Kontsevich cycles are not integral generators.

The coefficients a^μ_λ are determined by the coefficients of the inverse matrix b^μ_λ which, by the sum of products formula, are determined by the special cases b^n_λ. Section 4 gives a formula for these coefficients b^n_λ in terms of the category of ribbon graphs. In the next section this is reduced to a formula involving *tree polynomials*. As an example we show in Corollary 5.9 that

$$[W^*_{111}] = 288\widetilde{\kappa}_1^3 + 4176\widetilde{\kappa}_2\widetilde{\kappa}_1 + 20736\widetilde{\kappa}_3 \tag{5}$$

This formula, together with (1) and (4), verifies all values of the coefficients a^μ_λ conjectured by Arbarello and Cornalba in [1].

In Section 6 we compute the tree polynomial in the case when almost all of the variables are equal to 1. The main application is Section 7, where we prove the formula (4) for $b^{r+k}_{r,k}$. The problem becomes one of finding the closed form for a double sum of a hypergeometric term.

In Section 8 we obtain the following description of the what we call the *reduced tree polynomial*. Suppose that T is an *increasing tree* with vertices $0, 1, \cdots, 2k$ in the sense that, for every $0 \le j \le 2k$ the vertices $0, 1, \cdots, j$ span a connected subgraph of T. Then we associate to T the monomial

$$x^T = x_0^{n_0} x_1^{n_1} \cdots x_{2k}^{n_{2k}}$$

where n_j is the number of components of $T - \{j\}$ with an even number of vertices. The reduced tree polynomial is defined to be

$$\widetilde{T}_k(x_0, \cdots, x_{2k}) = \sum_T x^T \tag{6}$$

where the sum is over all increasing trees with vertices $0, \cdots, 2k$. We also show that the reduced tree polynomial $\widetilde{T}_k$ is related to the tree polynomial T_k of the previous section by the formula

$$T_k = x_0 \widetilde{T}_k.$$

This tells us several things that were not obvious before. For example, T_k is a homogeneous polynomial of degree $2k+1$ with nonnegative integer coefficients adding up to $(2k)!$. In Section 9 we give a recursive formula for the reduced tree polynomial. By Theorem 5.5 this gives a recursive formula for b_λ^n. By the sum of products rule (Lemma 1.4) this gives a formula for b_λ^μ and thus for the a_λ^μ. Examples are given in the last section.

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The section titles are:

1 Preliminaries
2 Sterling numbers and the degenerate case
3 Cup product structure of Kontsevich cycles
4 Formula for b_λ^n
5 Reduction to the tree polynomial
6 First formula for T_k
7 A double sum
8 Reduced tree polynomial
9 Recursion for $\widetilde{T}_k$
10 Examples of $\widetilde{T}_k$

1 Preliminaries

We work in the category of *ribbon graphs*. These are defined to be graphs with a designated cyclic ordering of the half edges incident to each vertex. We consider only finite connected ribbon graphs. We use the Conant–Vogtmann defnition [3] for the Kontsevich orientation of a connected graph. This is an ordering up to even permutation of the set consisting of the vertices and half-edges of the graph.

Suppose that Γ is an oriented ribbon graph and e is an edge of Γ which is not a loop (ie, the half-edges e_1, e_2 of e are incident to distinct vertices v_1, v_2. Then the graph Γ/e obtained from Γ by collapsing e to a point v_* has the structure of a ribbon graph and also has an *induced orientation* which is given by $v_*(etc.)$ if the orientation of Γ is written as $v_1v_2e_1e_2(etc.)$. If Γ is obtained from a trivalent graph by collapsing n edges we say that Γ has *codimension* n.

The category of connected ribbon graphs is denoted $\mathcal{F}at$. The morphisms of this category are compositions of collapsing maps $\Gamma \to \Gamma/e$ and isomorphisms. The main property of this category is that its geometric realization is integrally homotopy equivalent to the disjoint union of all mapping class groups M_g^s of punctured surfaces (with $s \geq 1$ punctures and genus g) except for the once and twice punctured sphere:

$$|\mathcal{F}at| \simeq \coprod_{s\geq 1,\ (s\geq 3 \text{ if } g=0)} BM_g^s$$

This theorem is usually attributed to Strebel [14]. A topological proof using *Outer Space* (from [2]) can be found in [5].

By a theorem of Kontsevich proved in [3] and refined in [6], the cohomology of $\mathcal{F}at$ (or equivalently, M_g^s) is rationally isomorphic to the cohomology of the associative graph cohomology complex. We work in the *integer subcomplex* of the rational associative graph cohomology complex generated by the cochains

$$\langle\Gamma\rangle := |\mathrm{Aut}(\Gamma)|[\Gamma]^*$$

This is a $\mathbb{Z}$–augmented complex of free abelian groups which can be described as follows.

Definition 1.1 For all $n \geq 0$ let $G_n^{\mathbb{Z}}$ be the free abelian group generated by all isomorphism classes $\langle\Gamma\rangle$ of oriented connected ribbon graphs Γ of codimension n without orientation reversing automorphisms modulo the relation $\langle -\Gamma\rangle = -\langle\Gamma\rangle$. For $n \geq 1$ let $d\colon G_n^{\mathbb{Z}} \to G_{n-1}^{\mathbb{Z}}$ be given by

$$d\langle\Gamma\rangle = \sum \langle\Gamma_i\rangle$$

where the sum is over all isomorphism classes of oriented ribbon graphs Γ_i over Γ with one extra edge e_i so that $\Gamma \cong \Gamma/e_i$ with the induced orientation.

Theorem 1.2 (Kontsevich [3]) $H^*(\coprod BM_g^s; \mathbb{Q}) \cong H^*(G_*^{\mathbb{Z}}; \mathbb{Q})$.

The refinement of this theorem proved in [6] is:

Theorem 1.3 *This rational equivalence is induced by an augmented integral chain map*

$$\phi\colon\ C_*(\mathcal{F}at) \to G_*^{\mathbb{Z}}$$

where $C_*(\mathcal{F}at)$ is the cellular chain complex of the nerve of $\mathcal{F}at$.

If $\lambda = 1^{r_1}2^{r_2}\cdots$ is a partition of $n = \sum ir_i$, the *dual Kontsevich cycles* W_λ^* is the integral $2n$ cocycle on the integral cohomology complex $G_*^{\mathbb{Z}}$ given as follows:

$$W_\lambda^*(\langle\Gamma\rangle) = o(\Gamma) = \pm 1$$

if Γ is an oriented ribbon graph of codimension $2n$ having exactly r_i vertices of valence $2i+3$ and no even valence vertices. The sign is $+$ if Γ has the *natural orientation* (given by taking each vertex followed by the incident half edges in cyclic order) and $-$ is not. This set of ribbon graphs is denoted W_λ and called the *Kontsevich cycle*. If Γ is not in W_λ then $W_\lambda^*(\langle\Gamma\rangle) = 0$.

Recall that the *Miller–Morita–Mumford class* $\kappa_n \in H^{2n}(BM_g, \mathbb{Z})$ is defined topologically ([9], [11]) as the image under the transfer

$$p_*\colon\ H^{2n+2}(E) \to H^{2n}(BM_g)$$

of the $n-1^{\text{st}}$ power e^n of the Euler class $e \in H^2(E)$ of the vertical tangent bundle of the universal surface bundle over BM_g with fiber an oriented surface Σ_g of genus g. If we pull this surface bundle back to the space $B = BM_g^s$ which maps to BM_g, we get s points in each fiber forming an s–fold covering space $\widetilde{B}$ over B. The *adjusted* or *punctured* Miller–Morita–Mumford class is given by

$$\widetilde{\kappa}_n = \kappa_n - p_*(c^n)$$

where $c \in H^2(\widetilde{B})$ is the Euler class of the vertical tangent bundle of E pulled back to $\widetilde{B}$. (See [7] for more details about this construction and its relationship to higher Franz–Reidemeister torsion.) Arbarello and Cornalba [1] showed that these are the correct versions of the MMM classes which should be compared to the combinatorial classes of Witten and Kontsevich.

In [4] it was shown that the adjusted MMM classes are represented by the *cyclic set cocycle* $c^n_{\mathcal{F}at}$ adjusted by a factor of -2:

$$\widetilde{\kappa}_n = -\frac{1}{2}[c^n_{\mathcal{F}at}].$$

Therefore, $\widetilde{\kappa}_n$ is represented by the *adjusted cyclic set cocycle*

$$\widetilde{c}_n = -\frac{1}{2}c^n_{\mathcal{F}at}.$$

This cocycle can be defined as follows. Take any $2n$–simplex

$$\Gamma_*\colon\ \Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_{2n}$$

in the category of ribbon graphs. Then

$$\widetilde{c}_n(\Gamma_*) = -\frac{1}{2}\sum_v m(v) \sum \frac{\operatorname{sgn}(a_0, a_1, \cdots, a_{2n})}{|C_0|\cdot|C_1|\cdots|C_{2n}|}$$

where the first sum is over all vertices v of Γ_0, $m(v)$ is the valence of v minus 2, and the second sum is over all choices of angles a_i of the vertex v_i which is the image of v in Γ_i. The denominator has the sizes $|C_i|$ are the sets C_i of angles about v_i (so $a_i \in C_i$ for each i). The sign is the sign of the permutation of the images of a_i in the final set C_{2n}. When these angles are not distinct, the sign is zero and, more generally, the sign sum is equal to the partial sum given by choosing each a_i in the complement of the image of C_{i-1} in C_i. For more details, see [4].

The relationship between the adjusted MMM classes $\widetilde{\kappa}_n$ and the *dual Witten cycles* $[W_n^*]$ is given ([4]) by

$$[W_n^*] = a_n \widetilde{\kappa}_n, \qquad \widetilde{\kappa}_n = b_n [W_n^*]$$

where

$$a_n = \frac{1}{b_n} = (-2)^{n+1}(2n+1)!!$$

To compute the other coefficients in (2) we need the following formula proved in [6], Lemma 3.15.

Lemma 1.4 (Sum of products rule) *If $\lambda = (\ell_i, \cdots, \ell_r)$ is a partition of n into r parts and $\mu = (m_1, \cdots, m_s)$ is a partition of the same number n into s parts then the coefficient b_λ^μ in equation (2) is equal to the sum*

$$b_\lambda^\mu = \sum_f \prod_{j=1}^s b_{\lambda_{\pi(j)}}^{m_j}$$

over all epimorphisms

$$f\colon\ \{1, \cdots, r\} \twoheadrightarrow \{1, \cdots, s\}$$

having the property that the sum of the numbers ℓ_i over all $i \in \pi(j) = f^{-1}(j)$ is equal to m_j of the product over all $1 \le j \le s$ of the coefficient $b_{\lambda_{\pi(j)}}^{m_j}$ where $\lambda_{\pi(j)}$ is the partition of m_j given by the numbers ℓ_i for $i \in \pi(j)$.

By this formula it suffices to compute the numbers b_λ^m.

2 Sterling numbers and the degenerate case

We start with an examination of the degenerate case $W^*_{0^n}$. These are polynomials in the 0^{th} adjusted cyclic set cocycle $\widetilde{c}_0$, equal to the 0^{th} (topological) Miller–Morita–Mumford class $\widetilde{\kappa}_0$, which is the Euler characteristic. If Γ is trivalent with the natural orientation, then

$$\widetilde{c}\,\langle\Gamma\rangle = \chi(\Gamma) = \frac{v}{-2}$$

where v is the number of vertices of Γ. (In general we need to count the number of vertices with *multiplicity*, ie, valence minus 2.)

We interpret the 0's in $W^*_{0^n}$ as counting the number of vertices with multiplicity:

$$W^*_{0^n}\,\langle\Gamma\rangle = \binom{v}{n} = \binom{-2\widetilde{c}_0}{n} = \frac{1}{n!}\sum_{i=0}^{n} S_1(n,i)(-2\widetilde{c}_0)^i \tag{7}$$

where $S_1(n,i)$ is the Stirling number of the first kind. This can be solved for the $\widetilde{c}_0^{\,i}$ to give:

$$\widetilde{c}_0^{\,m} = \frac{1}{(-2)^m}\sum_{n=0}^{m} n!S_2(m,n)W^*_{0^n} \tag{8}$$

where $S_2(m,n)$ are the Stirling numbers of the second kind.

In the notation of [6], this is

$$\widetilde{c}_0^{\,m} = \sum_{n=0}^{m} b^{0^n}_{0^m} W^*_{0^n}$$

where

$$b^{0^n}_{0^m} = \frac{n!S_2(m,n)}{(-2)^m}. \tag{9}$$

This is consistent with the formula

$$b^{0^n}_{0^m} = \sum_f \prod_{j=1}^{n} b^{0}_{0^{m_j}} = \sum_f \prod_{j=1}^{n} \frac{1}{(-2)^{m_j}}$$

where the sum is taken over all surjective mappings

$$f\colon\ \{1,2,\cdots,m\} \twoheadrightarrow \{1,2,\cdots,n\}$$

with m_j being the number of elements in $\pi(j) = f^{-1}(j)$. Since there are $n!S_2(m,n)$ such mappings f, this agrees with (9).

Assume for a moment that the sum of products formula (Lemma 1.4) holds more generally for all partitions with 0's. Thus, if $\mu = (\mu_1,\mu_2,\cdots,\mu_r)$ and

$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s)$ are partitions of the same number n then we have the following which we take as a definition. (It agrees with the previously defined terms b_λ^μ when $p = q = 0$.)

$$b_{\lambda 0^p}^{\mu 0^q} := \sum_f b_{\lambda_{\pi(1)}}^{\mu_1} \cdots b_{\lambda_{\pi(r)}}^{\mu_r} b_{\lambda_{\pi(r+1)}}^0 \cdots b_{\lambda_{\pi(r+q)}}^0 \tag{10}$$

where the sum is over all surjective mappings

$$f\colon \{1, 2, \cdots, s+p\} \twoheadrightarrow \{1, 2, \cdots, r+q\}$$

having the property that the sum of the parts λ_j of λ for $j \in \pi(i) = f^{-1}(i)$ is equal to μ_i:

$$\mu_i = \sum_{j \in \pi(i)} \lambda_j$$

where $\lambda_j = 0$ for $i > s$ and $\mu_i = 0$ for $i > r$. When the superscript of b is 0 the subscript must be 0^m for some $m \geq 1$ and we have

$$b_{0^m}^0 = \frac{1}{(-2)^m}.$$

If the superscript is $\mu_i \neq 0$ then the subscript is a partition of μ_i, say ν, plus any number of 0's. We define

$$b_{\nu 0^m}^{\mu_i} := \frac{(2\mu_i + 1)^m}{(-2)^m} b_\nu^{\mu_i}.$$

This makes sense since it is supposed to be the contribution of a vertex of valence $2\mu_i + 3$ to the cup product

$$\widetilde{\kappa}_{\nu 0^m} = \widetilde{\kappa}_\nu \widetilde{\kappa}_0^m$$

But each $\widetilde{\kappa}_0$ is given by

$$\frac{v}{-2} = \frac{2\mu_i + 1}{-2}.$$

Putting these together in (10) we get the following.

Proposition 2.1

$$b_{\lambda 0^p}^{\mu 0^q} = \sum_{m=0}^{p-q} \binom{p}{m} q! S_2(p-m, q) \frac{(2n+r)^m}{(-2)^p} b_\lambda^\mu$$

We claim that these are the coefficients which convert monomials in the adjusted Miller–Morita–Mumford classes into linear combinations of dual Kontsevich cycles with 0's.

Definition 2.2 Let $\lambda = 1^{n_1}2^{n_2}\cdots$ be a partition of $n = \sum in_i$ into $r = \sum n_i$ parts. We define the *degenerate Kontsevich cycles* $W^*_{\lambda 0^m}$ to be the integer cocycle of degree $2n$ on the integer subcomplex of associative graph cohomology given by

$$W^*_{\lambda 0^m}\langle\Gamma\rangle = o(\Gamma)\binom{n_0}{m}$$

provided that Γ is a connected oriented ribbon graph having exactly n_i vertices of valence $2i+1$ for all $i \geq 0$ and no vertices of even valence. The orientation $o(\Gamma)$ is ± 1 depending on whether or not the orientation of Γ is the natural one.

It is easy to express $W^*_{\lambda 0^m}$ in terms of the Euler characteristic

$$\chi = \widetilde{c}_0 = \frac{n_0 + 2n + r}{-2}$$

and the nondegenerate Kontsevich cycle W^*_λ:

$$W^*_{\lambda 0^m} = \frac{1}{m!}\sum_{j=0}^{m} S_1(m,j)(-2\widetilde{c}_0 - 2n - r)^j W^*_\lambda$$

$$= \frac{1}{m!}\sum_{0\leq i\leq j\leq m} S_1(m,j)\binom{j}{i}(-2n-r)^{j-i}(-2\widetilde{c}_0)^i W^*_\lambda$$

Passing to cohomology classes, this can be written as follows.

Theorem 2.3 *The degenerate Kontsevich cycles are related to the adjusted Miller–Morita–Mumford classes by*

$$[W^*_{\lambda 0^m}] = \sum_{\mu,i} a^{\mu 0^i}_{\lambda 0^m}\widetilde{\kappa}_\mu \widetilde{\kappa}_0^i$$

and

$$\widetilde{\kappa}_\lambda \widetilde{\kappa}_0^p = \sum_{\mu,q} b^{\mu 0^q}_{\lambda 0^p}[W^*_{\mu 0^q}]$$

where

$$a^{\mu 0^i}_{\lambda 0^m} = \frac{1}{m!}\sum_{j=i}^{m} S_1(m,j)\binom{j}{i}(-2n-r)^{j-i}(-2)^i a^\mu_\lambda$$

and $b^{\mu 0^q}_{\lambda 0^p}$*, defined by (10), is given by Proposition 2.1.*

Proof Using the duality between the first and second Stirling numbers it is easy to see that the matrices with coefficients $b^{\mu 0^q}_{\lambda 0^p}$, $a^{\lambda 0^p}_{\mu 0^q}$ are inverse to each other. □

3 Cup product structure of Kontsevich cycles

Using Kontsevich's theorem (1.2) the rational cohomology of $G_*^{\mathbb{Z}}$ inherits a ring structure.

Theorem 3.1 *The determination of the conversion coefficients a_λ^μ and b_μ^λ is equivalent to finding the coefficients $m_{\lambda\mu}^\nu$ giving the cup product of the Kontsevich cocycles:*

$$[W_\lambda^*] \cup [W_\mu^*] = \sum_\nu m_{\lambda\mu}^\nu [W_\nu^*] \in H^*(G_*;\mathbb{Q})$$

Remark 3.2 Note that rational numbers $m_{\lambda\mu}^\nu$ are well-defined since $[W_\lambda^*]$ are linearly independent over $\mathbb{Q}$ and span the same vector subspace as the monomials in the adjusted Miller–Morita–Mumford classes $\widetilde{\kappa}_\lambda$. We also note that these numbers are not all integers. The simplest example is

$$[W_1^*] \cup [W_1^*] = 2[W_{1,1}] + \frac{29}{5}[W_2^*]$$

which follows from the equations:

$$\begin{aligned} [W_1^*] &= a_1\widetilde{\kappa}_1 = 12\widetilde{\kappa}_1 \\ \widetilde{\kappa}_1^2 &= 2(b_1)^2[W_{1,1}^*] + b_{1,1}^2[W_2^*] \\ &= \frac{2}{144}[W_{1,1}^*] + \left(\frac{7}{144} - \frac{1}{120}\right)[W_2^*] \end{aligned}$$

Proof In one direction this is clear. If we know the numbers a_λ^μ and b_μ^λ then we can convert $[W_\lambda^*] = \sum a_\lambda^\alpha \widetilde{\kappa}_\alpha$ and $[W_\mu^*] = \sum a_\mu^\beta \widetilde{\kappa}_\beta$, multiply and convert back. Thus,

$$m_{\lambda\mu}^\nu = \sum_{\alpha,\beta} a_\lambda^\alpha a_\mu^\beta b_{\alpha\beta}^\nu. \tag{11}$$

The other direction is also easy. Suppose we know the numbers $m_{\lambda\mu}^\nu$ and we want to find a_λ^μ, b_λ^μ. We proceed by induction on the number of parts of λ. When $\lambda = n$ is a partition of n with one part, then μ must also be equal to n since μ cannot have more parts than λ. But we know these numbers:

$$a_n^n = \frac{1}{b_n^n} = (-2)^{n+1}(2n+1)!!$$

Suppose by induction that we know a_λ^μ, b_λ^μ for all partitions λ with r or fewer parts. Then setting $\mu = n$ in (11) there will be only one term on the right hand side (when $\alpha = \lambda$ and $\beta = \mu = n$) which is unknown. This gives $b_{\lambda n}^\nu$. Taking the inverse matrix we also get all $a_{\lambda n}^\nu$. □

4 Formula for b^n_λ

Using the sum of products rule (Lemma 1.4), the calculation of the numbers b^μ_λ is reduced to the case when $\mu = n$ is a partition of n with one part. If $\lambda = (\lambda_1, \cdots, \lambda_r)$ is a partition of n into r parts then the number b^n_λ is given by

$$b^n_\lambda = (-1)^n \widetilde{c}_\lambda D(\Gamma) = (-1)^n (\widetilde{c}_{\lambda_1} \cup \cdots \cup \widetilde{c}_{\lambda_r}) D(\Gamma)$$

where Γ is a ribbon graph with natural orientation having one vertex of valence $2n+3$ and all other vertices trivalent and $D(\Gamma)$ is any dual cell of Γ.

$$D(\Gamma) = \sum_{\Gamma_*} o(\Gamma_*)(\Gamma_0 \to \cdots \to \Gamma_{2n} = \Gamma)$$

where the sum is over all sequences of morphisms over Γ between representatives Γ_i of the isomorphism classes of ribbon graphs over Γ and $o(\Gamma_*) = \pm 1$ is positive iff the natural orientations of $\Gamma = \Gamma_{2n}$ agrees with the orientation induced from the natural orientation of the trivalent graph Γ_0 by the collapsing morphisms in the sequence $\Gamma_* = (\Gamma_0 \to \cdots \to \Gamma_{2n} = \Gamma)$ which we abbreviate as $(\Gamma_0, \cdots, \Gamma_{2n})$.

Combining these we get

$$b^n_\lambda = (-1)^n \sum_{\Gamma_*} o(\Gamma_*) \widetilde{c}_{\lambda_1}(\Gamma_0, \cdots, \Gamma_{2\lambda_1}) \cdots \widetilde{c}_{\lambda_r}(\Gamma_{2n-2\lambda_r}, \cdots, \Gamma_{2n})$$

We use the notation

$$\lambda[i] = \lambda_1 + \lambda_2 + \cdots + \lambda_i$$

(with $\lambda[0] = 0$ and $\lambda[r] = n$). Then the i^{th} factor in the expression for b^n_λ is

$$\widetilde{c}_{\lambda_i}(\Gamma_{2\lambda[i-1]}, \Gamma_{2\lambda[i-1]+1}, \cdots, \Gamma_{2\lambda[i]}) \tag{12}$$

We will factor the sign terms $(-1)^n$ and $o(\Gamma_*)$ into r factors and associate each factor to one of the factors (12).

First, we note that the graphs $\Gamma_{\lambda[i]}$ must all be odd valent in the sense that they have no even valent vertices. If not then one of the $\widetilde{c}_{\lambda_i}$ factors (12) would be zero. Consequently, the orientation term $o(\Gamma_*)$ can be factored as:

$$o(\Gamma_*) = \prod_{i-1}^{r} o(\Gamma_{2\lambda[i-1]}, \cdots, \Gamma_{2\lambda[i]}).$$

The sign $(-1)^n$ also factors:

$$(-1)^n = \prod (-1)^{\lambda_i}.$$

So, b_λ^n is a sum of products. Each product has r factors where the i^{th} factor has the form

$$(-1)^{\lambda_i} o(\Gamma_{2\lambda[i-1]}, \cdots, \Gamma_{2\lambda[i]})\widetilde{c}_{\lambda_i}(\Gamma_{2\lambda[i-1]}, \cdots, \Gamma_{2\lambda[i]}) \tag{13}$$

which we abbreviate as $(-1)^{\lambda_i} o(\Gamma_*^i)\widetilde{c}_{\lambda_i}(\Gamma_*^i)$. But, the graphs $\Gamma_{2\lambda[i-1]+j}$ for $1 \le j < \lambda_i$ occur only in the i^{th} factor (13). Thus, we have the following.

Lemma 4.1 *b_λ^n can be expressed as a sum of products of sums:*

$$b_\lambda^n = \sum_{(\Gamma_0, \Gamma_{2\lambda[1]}, \cdots, \Gamma_{2\lambda[r]})} \prod_{i=1}^{n} \sum_{\Gamma_*^i} (-1)^{\lambda_i} o(\Gamma_*^i)\widetilde{c}_{\lambda_i}(\Gamma_*^i) \tag{14}$$

The first summation is over all sequences $\Gamma_{2\lambda[i]}$, $i = 0, \cdots, r$ of (representatives of) isomorphism classes of odd valent graphs over $\Gamma = \Gamma_{2n} = \Gamma_{2\lambda[r]}$ and the second sum is over all sequences of morphisms

$$\Gamma_*^i = (\Gamma_{2\lambda[i-1]} \to \Gamma_{2\lambda[i-1]+1} \to \cdots \to \Gamma_{2\lambda[i]})$$

where each Γ_j is from a fixed set of representatives from the set of isomorphism classes of oriented ribbon graphs over Γ.

Now we examine the possibilities for the graphs $\Gamma_{2\lambda[i]}$. Since Γ_0 has codimension 0 it must be trivalent. In order for the first factor in (14) to be nonzero, we must have that $\Gamma_{2\lambda[1]}$ is trivalent except at one vertex of valence $2\lambda_1 + 3$. More generally, we have the following.

Lemma 4.2 *Suppose that the nontrivalent vertices of $\Gamma_{2\lambda[i]}$ have valences $2n_{i1} + 3, \cdots, 2n_{ik_i} + 3$. Then, in order for the corresponding terms in (14) to be nonzero we must have the following.*

(1) $\lambda(i) = n_{i1} + \cdots + n_{ik_i}$

(2) *For each $i \ge 1$ and each $j < k_i$ there is an index $\phi(j)$ so that $n_{i-1,\phi(j)} = -n_{ij}$ and ϕ is an injective function.*

Proof In order for the term $\widetilde{c}_{\lambda_i}(\Gamma_{2\lambda[i-1]} \to \cdots \to \Gamma_{2\lambda[i]})$ to be nonzero, the inverse images in $\Gamma_{2\lambda[i-1]}$ of the vertices of $\Gamma_{2\lambda[i]}$ must all be vertices (necessarily with the same valence) with only one exception. The exceptional vertex must have valence at least $2\lambda_i + 3$ and its inverse image must be a tree in $\Gamma_{2\lambda[i-1]}$ with that many leaves. □

Next, we look at the factors in (14) for $i = 1, \cdots, r$. The first factor is easy to compute:

$$\sum_{\Gamma^1_*} (-1)^{\lambda_1} o(\Gamma^1_*)\widetilde{c}_{\lambda_i}(\Gamma^1_*) = b_{\lambda_1} = \frac{1}{(-2)^{\lambda_1+1}(2\lambda_1+1)!!}$$

The last factor ($i = r$) is more difficult. It is also *universal* in the sense that, if we can compute the last factor, we can compute all the factors. We make this statement more precise using the tree polynomial.

5 Reduction to the tree polynomial

Suppose that $n_0, n_1, \cdots, n_{2k}$ are positive odd integers. Then we will define an integer $T_k(n_0, \cdots, n_{2k})$. We will then show that this integer is given by a homogeneous polynomial in the variables $n_0, \cdots, n_{2k}$ with nonnegative integer coefficients. We call this the *tree polynomial*. We will also give a formula for the numbers b^n_λ in terms of these polynomials.

Definition 5.1 Let $Sh_k(n_0, \cdots, n_{2k})$ be the set of permutations σ of the numbers $1, 2, \cdots, n$, where $n = \sum n_i$, so that

(1) $\sigma(1) = 1$,

(2) $\sigma(n_i + 1) < \sigma(n_i + 2) < \cdots < \sigma(n_{i+1})$ for $i = -1, \cdots, 2k - 1$ where $n_{-1} = 0$ and

(3) $\sigma(n_i + 1) < \sigma(j) < \sigma(n_{i+1})$ only when $j > n_i$.

We call these permutations *cyclic shuffles*.

Cyclic shuffles can be described as follows. Take the letters $a_1, a_2, \cdots, a_{n_0}$ in that order. Then insert the letters $b_1, b_2, \cdots, b_{n_1}$ in one block between two of the a's or after the last a. There are n_0 ways to do this. Next, insert the letters $c_1, c_2, \cdots, c_{n_3}$ in one block between two letters in the sequence so far or after the last letter. There are $n_0 + n_1$ ways to do this. Thus the number of elements in this set is

$$|Sh_k(n_0, \cdots, n_{2k})| = n_0(n_0 + n_1)(n_0 + n_1 + n_2) \cdots (n_0 + \cdots + n_{2k-1}).$$

Cyclic shuffles have several signs associated to them. The ordinary sign of σ will be called its *orientation*. We also have the *selected sign* denoted $\mathrm{sgn}_\sigma(a_i, b_j, \cdots)$ which are the sign of σ restricted to a subset given by selecting one letter of each kind. For example, take the cyclic shuffle

$$\sigma = a_1 a_2 b_1 c_1 c_2 c_3 c_4 b_2 a_3.$$

The orientation is $\operatorname{sgn}(\sigma) = -1$ and there are $3 \cdot 2 \cdot 4$ selected signs

$$\operatorname{sgn}_\sigma(a_i, b_j, c_k) = (-1)^{(j=2)},$$

ie, the selected sign is negative iff b_2 is selected.

The sum of all selected signs will be called the *sign sum* of σ. By the *oriented sign sum* we mean the product of the sign sum with the orientation of σ:

$$\operatorname{sgn}(\sigma)\sum \operatorname{sgn}_\sigma(a_i, b_j, \cdots) = \operatorname{sgn}(\sigma)\sum \operatorname{sgn}_\sigma(\sigma(i), \sigma(n_0 + j), \cdots). \qquad (15)$$

This has $\prod n_i$ terms. (The sum is for $i = 1, \cdots, n_0, j = 1, \cdots, n_1$, etc.) It is easy to see that the oriented sign sum is divisible by n_{2k} since the selected sign $\operatorname{sgn}_\sigma(a_i, b_j, \cdots, y_p)$ is independent of the last index p. (Note that the English language has an even number of letters so the $2k + 1^{\text{st}}$ letter cannot be z.)

Definition 5.2 Let $T_k(n_0, \cdots, n_{2k})$ be the sum over all cyclic shuffles σ of the oriented sign sum of σ:

$$T_k(n_0, \cdots, n_{2k}) = \sum_{\sigma \in Sh_k(n_0, \cdots, n_{2k})} \operatorname{sgn}(\sigma)\sum \operatorname{sgn}_\sigma(a_i, b_j, \cdots)$$

Let

$$Q_k(n_0, \cdots, n_{2k}) = \frac{T_k(n_0, \cdots, n_{2k})}{|Sh_k(n_0, \cdots, n_{2k})|}$$

be the average (expected value) of the oriented sign sum over all cyclic shuffles σ.

We call $T_k(n_0, \cdots, n_{2k})$ the *tree polynomial* since it is a homogeneous polynomial in $n_0, \cdots, n_{2k}$ with nonnegative integer coefficients. (Theorem 8.6 below). In Section 8 we will show that this polynomial is in fact the generating function for a statistic on the set of increasing trees with labels $0, \ldots, 2k$. First we record some obvious properties of the tree polynomial.

Proposition 5.3 *For all positive odd integers $n_0, \cdots, n_{2k}$ we have:*

1. $T_k(n_0, \cdots, n_{2k})$ *is an integer.*
2. $T_k(n_0, \cdots, n_{2k}) = n_{2k} T_k(n_0, \cdots, n_{2k-1}, 1)$.
3. $T_k(n_0, \cdots, n_{2k})$ *is divisible by n_0 and the quotient $T_k(n_0, \cdots, n_{2k})/n_0$ is the sum of $\operatorname{sgn}(\sigma)\sum \operatorname{sgn}_\sigma(a_i, b_j, \cdots)$ over all cyclic shuffles σ which insert the b's after the a's.*

This allows us to compute the first nontrivial tree polynomial. (The trivial case is $T_0(n_0) = Q_0(n_0) = n_0$.)

Corollary 5.4 $T_1(n_0, n_1, n_2) = n_0(n_0 + n_1)n_2$, *so* $Q_1(n_0, n_1, n_2) = n_2$.

Proof Since $T_1(n_0, n_1, n_2) = n_2 T_1(n_0, n_1, 1)$ it suffices to show that

$$\frac{T_1(n_0, n_1, 1)}{n_0} = n_0 + n_1.$$

By Proposition 5.3(3), this is given by

$$\frac{T_1(n_0, n_1, 1)}{n_0} = \sum_{i=1}^{n_0} (-1)^i (n_0 - 2i)n_1 + \sum_{j=1}^{n_1} (-1)^{j+1} n_0(2j - n_1) = n_1 + n_0. \qquad \square$$

The following theorem tells us that the numbers b_λ^n (and thus all b_λ^μ and a_μ^λ) are determined by the tree polynomials.

Theorem 5.5 $b_{\lambda,k}^n$ *is equal to the sum*

$$b_{\lambda,k}^n = \sum_{(m_0, \cdots, m_{2k})} b_\lambda^\mu \frac{(2m_0 + 1)Q_k(2m_0 + 3, 2m_1 + 1, \cdots, 2m_{2k} + 1)}{(2m_0 + 3)(-2)^{k+1}(2k - 1)!!}$$

where the sum is over all $2k + 1$ *tuples of nonnegative integers* $(m_0, \cdots, m_{2k})$ *which add up to* $n - k$ *and* μ *is the partition of* $n - k$ *given by the nonzero* m_i.

Example 5.6 When $k = 1$ this formula becomes

$$b_{\lambda,1}^n = \sum_{\substack{a+b+c=n-1 \\ a,b,c \geq 0}} b_\lambda^{[a,b,c]} \frac{(2a + 1)(2c + 1)}{(2a + 3)4} \tag{16}$$

where $[a, b, c]$ denotes the multiset $\{a, b, c\}$ with the zero's deleted. For example, if $\lambda = 1$ there are three terms with $[a, b, c] = [1, 0, 0] = \{1\}$ and (16) is

$$b_{1,1}^2 = \frac{b_1^1}{4}\left(\frac{3}{5} + \frac{1}{3} + \frac{3}{3}\right) = \frac{29}{60} b_1 = \frac{29}{720}.$$

Proof The number $b_{\lambda,k}^n$ is given by evaluating the cup product $\widetilde{\kappa}_\lambda \cup \widetilde{\kappa}_k$ on a dual cell of any graph Γ_{2n} (with natural orientation) in the Kontsevich cycle W_{2n}. This is given by

$$\sum o_1 o_2 \widetilde{c}_\lambda(\Gamma_0, \cdots, \Gamma')\widetilde{c}_k(\Gamma', \cdots, \Gamma_{2n})$$

where $o_1 = o(\Gamma_0, \cdots, \Gamma')$, $o_2 = o(\Gamma', \cdots, \Gamma_{2n})$ are the orientations of the front and back face of the $2n$–simplex $(\Gamma_0, \cdots, \Gamma_{2n})$. The sum over all sequences $(\Gamma_0, \cdots, \Gamma')$ times o_1 is the dual cell of Γ':

$$\sum o_1(\Gamma_0, \cdots, \Gamma') = D(\Gamma').$$

Consequently,

$$\sum o_1 \widetilde{c}_\lambda(\Gamma_0, \cdots, \Gamma') = b^\mu_\lambda \tag{17}$$

if Γ' lies in the Kontsevich cycle W_μ.

For the other factor, we note that the adjusted cyclic set cocycle $\widetilde{c}_k$ is a sum of two terms, one for each of the two vertices of $\Gamma' = \Gamma_{2n-2k}$ which collapse to a point in the next graph Γ_{2n-2k}. Each of these vertices gives a pointed $2k$–simplex. For each such pointed $2k$–simplex, let v_0, v_1 be the two vertices which collapse at the first step and let $v_2, \cdots, v_{2k}$ be the other vertices of Γ', indexed according the order in which they merge with v_0.

Since Γ' must lie in a Kontsevich cycle W_μ, its vertices v_i must have codimensions $2m_i$ with $m_i \geq 0$ so that the nonzero m_i make up the parts of the partition μ. For each such sequence $(m_0, \cdots, m_{2k})$ we get a subtotal

$$\sum o_1 \widetilde{c}_k(\Gamma', \cdots, \Gamma_{2n}) =$$
$$\left(\frac{2n+3}{2m_0+3}\right)\left(\frac{(2m_0+1)T_k(2m_0+3, 2m_1+1, \cdots, 2m_{2k}+1)}{(-2)^{k+1}(2k-1)!!(2m_0+3)(2m_0+2m_1+4)\cdots(2n+3)}\right)$$
$$= \left(\frac{2m_0+1}{2m_0+3}\right)\left(\frac{Q_k(2m_0+3, 2m_1+1, \cdots, 2m_{2k}+1)}{(-2)^{k+1}(2k-1)!!}\right)$$

since there is a $(2n+3)$-to-$(2m_0+3)$ correspondence between pointed $2k$–simplices and cyclic shuffles. Combine this with (17) and sum over all sequences $(m_0, \cdots, m_{2k})$ to get the result. □

Example 5.6 allows us to obtain a recursive formula for $b^n_{1^n}$.

Corollary 5.7 *For all positive n we have*

$$b^n_{1^n} = 4^{-n} n! h(n)$$

where $h(n)$ is given recursively by $h(0) = 1$ and

$$h(n+1) = \sum_{\substack{a+b+c=n \\ a,b,c \geq 0}} h(a)h(b)h(c)\frac{(2a+1)(2c+1)}{(2a+3)(n+1)}.$$

Proof In the recursion (16) we note that, by the sum of products formula for b^μ_λ, we have

$$b^{[a,b,c]}_{1^n} = \frac{n!}{a!\, b!\, c!} f(a) f(b) f(c)$$

where $f(n) = b^n_{1^n}$ for $n \geq 1$ and $f(0) = 1$. Then (16) becomes

$$f(n+1) = \sum_{\substack{a+b+c=n \\ a,b,c\geq 0}} \frac{n!}{a!\,b!\,c!} f(a)f(b)f(c)\frac{(2a+1)(2c+1)}{(2a+3)4}.$$

Substitute $f(n) = 4^{-n}n!h(n)$ to get the recursion for $h(n)$. □

Example 5.8

$$h(1) = \frac{1}{3}, \qquad b^1_1 = \frac{1}{12}$$
$$h(2) = \frac{29}{90}, \qquad b^2_{11} = \frac{29}{720}$$
$$h(3) = \frac{263}{630}, \qquad b^3_{111} = \frac{263}{6720}$$
$$h(4) = \frac{23479}{37800}, \qquad b^4_{1111} = \frac{23479}{403200}$$

The value of b^3_{111} allows us to compute the expansion of $[W^3_{111}]$ as conjectured by Arbarello and Cornalba [1] and promised in [6].

Corollary 5.9 $[W^*_{111}] = 288\widetilde{\kappa}^3_1 + 4176\widetilde{\kappa}_2\widetilde{\kappa}_1 + 20736\widetilde{\kappa}_3$

Proof By the sum of products formula we have

$$b^{21}_{111} = 3b^2_{11}b^1_1 = 3 \cdot \frac{29}{720} \cdot \frac{1}{12} = \frac{29}{2880}$$
$$b^{21}_{21} = b_2 b_1 = \frac{1}{-120 \cdot 12} = -\frac{1}{1440}.$$

By Equation (3) in the introduction which was proved in [6] but which also follows from Example 5.6 above, we have

$$b^3_{21} = -\frac{19}{3360}.$$

Therefore, the coefficients of the expansion

$$[W^*_{111}] = a^{111}_{111}\widetilde{\kappa}^3_1 + a^{21}_{111}\widetilde{\kappa}_2\widetilde{\kappa}_1 + a^3_{111}\widetilde{\kappa}_3$$

are given by

$$a^{111}_{111} = \frac{12^3}{3!} = 288$$
$$a^{21}_{111} = -\frac{a^{111}_{111}b^{21}_{111}}{b^{21}_{21}} = 4176$$
$$a^3_{111} = -\frac{a^{21}_{111}b^3_{21} + a^{111}_{111}b^3_{111}}{b_3} = 20736.$$

□

6 First formula for T_k

We will compute the tree polynomial in the case when most of the entries are equal to 1.

Theorem 6.1

$$T_k(n, 1, ..., 1, m) = (2k-1)!!mn(n+1)(n+3)...(n+2k-1)$$

Proof Dividing by $n(n+1)(n+2)\cdots(n+2k-1)$ and restricting to the case $m=1$, it suffices to show that

$$Q_k(n, 1, ..., 1, 1) = \frac{(2k-1)!!n!!}{(n+2k-2)!!} \tag{18}$$

But $Q_k(n, 1, \cdots, 1)$ is the expected value of the oriented sign sum

$$\operatorname{sgn}(\sigma)\sum \operatorname{sgn}_\sigma(a_i, b_1, b_2, \cdots, b_{2k})$$

for a random cyclic shuffle σ. Since any change in the order of the b_i leaves this sum invariant, we may assume that the b's are in correct cyclic order. By Proposition 5.3(3), we may also assume that b_{2k} comes after all the a's. Cyclic shuffles of this kind are in 1–1 correspondence with ordinary shuffles of $a_1, \cdots, a_n$ with $b_1, \cdots, b_{2k-1}$ whose oriented sign sums have expectation values tabulated in the lemma below:

$$Q_k(2j-1, 1, \cdots, 1) = E_0(2j-1, 2k-1) = \frac{(2j-1)!!(2k-1)!!}{(2j+2k-3)!!}$$

This gives (18) proving the theorem. □

Lemma 6.2 *Consider all shuffles σ of $a_1, \cdots, a_n$ with $b_1, \cdots, b_m$ where n, m are nonnegative integers. Then the sum of the oriented sign sum*

$$X_0(n, m) = \sum_\sigma \operatorname{sgn}(\sigma) \sum_{i=1}^{n} \operatorname{sgn}_\sigma(a_i, b_1, \cdots, b_m)$$

and its expected value

$$E_0(n, m) = \frac{X_0(n, m)}{\binom{n+m}{n}}$$

depend on the parity of n, m and are given in the following table.

n	m	$X_0(n,m)$	$E_0(n,m)$
$2j$	$2k$	$2j\binom{j+k}{j}$	$\frac{(2j)(2j-1)!!(2k-1)!!}{(2j+2k-1)!!}$
$2j$	$2k-1$	0	0
$2j-1$	$2k$	$(2j+2k-1)\binom{j+k-1}{k}$	$\frac{(2j-1)!!(2k-1)!!}{(2j+2k-3)!!}$
$2j-1$	$2k-1$	$2k\binom{j+k-1}{k}$	$\frac{(2j-1)!!(2k-1)!!}{(2j+2k-3)!!}$

Proof Shuffles σ are in 1–1 correspondence with the ways of writing m as the sum of an $n+1$–tuple of nonnegative integers:

$$m = m_0 + m_1 + \cdots + m_n.$$

(The corresponding shuffle is $b^{m_0}a_1b^{m_1}a_2\cdots a_nb^{m_n}$.) The terms in the oriented sign sum are the product of

$$\operatorname{sgn}(\sigma) = (-1)^{m_{n-1}+m_{n-3}+m_{n-5}+\cdots}$$

$$\operatorname{sgn}_\sigma(a_i, b_1, b_2, \cdots, b_m) = (-1)^{m_0+m_1+\cdots+m_{i-1}}.$$

Note that there are $j = \lceil \frac{n}{2} \rceil$ terms m_i in the exponent for $\operatorname{sgn}(\sigma)$. And, when i is even, $\operatorname{sgn}(\sigma)\operatorname{sgn}_\sigma(a_i, b_1, \cdots, b_m)$ has the same form. Thus

$$E(\operatorname{sgn}(\sigma)) = E(\operatorname{sgn}(\sigma)\operatorname{sgn}_\sigma(a_{2i}, b_1, \cdots, b_m)).$$

Similarly, $\operatorname{sgn}(\sigma)\operatorname{sgn}_\sigma(a_{odd}, b_1, \cdots, b_m)$ is equal to -1 to the power a sum of $j + (-1)^n$ terms m_i so its expected value is independent of the subscript of a which can have $\lceil \frac{n}{2} \rceil$ different values. The expected value of the oriented sign sum is thus given by:

$$E_0(n,m) = \left\lfloor \frac{n}{2} \right\rfloor E(\operatorname{sgn}(\sigma)) + \left\lceil \frac{n}{2} \right\rceil E(\operatorname{sgn}(\sigma)\operatorname{sgn}_\sigma(a_1, b_1, \cdots, b_m))$$

$$= \frac{1}{\binom{n+m}{n}} \left(\left\lfloor \frac{n}{2} \right\rfloor \sum_\sigma \operatorname{sgn}(\sigma) + \left\lceil \frac{n}{2} \right\rceil \sum_\sigma \operatorname{sgn}(\sigma)\operatorname{sgn}_\sigma(a_1, b_1, \cdots, b_m) \right).$$

The lemma now follows from the following eight computations.

$$
\begin{array}{cccc}
n & m & \sum \mathrm{sgn}(\sigma) & \sum \mathrm{sgn}(\sigma)\,\mathrm{sgn}_\sigma(a_1,b_1,\cdots,b_m) \\
\hline
2j & 2k & \binom{j+k}{j} & \binom{j+k}{j} \\
2j & 2k-1 & \binom{j+k-1}{j} & -\binom{j+k-1}{j} \\
2j-1 & 2k & \binom{j+k-1}{k} & 2\binom{j+k}{k}-\binom{j+k-1}{k} \\
2j-1 & 2k-1 & 0 & 2\binom{j+k-1}{j}
\end{array}
\tag{19}
$$

We verify the entries in this table starting at the bottom left. When both n, m are odd there is a fixed point free involution on the set of shuffles given by switching $m_{2i} \leftrightarrow m_{2i-1}$. This always changes the sign of σ so $\sum \mathrm{sgn}(\sigma) = 0$.

When $n = 2j-1, m = 2k$ we have

$$\mathrm{sgn}(\sigma) = (-1)^{m_0+m_2+\cdots+m_{2j-2}}.$$

Take the involution on the set of shuffles given as follows. Take the largest i so that $m_{2i} + m_{2i+1}$ is odd and switch $m_{2i} \leftrightarrow m_{2i+1}$. If these sums are all even then take the largest i so that m_{2i} is nonzero. If it is even, subtract 1 from m_{2i} and add 1 to m_{2i+1}. If m_{2i} is odd, add 1 to it and subtract 1 from m_{2i+1}. This sign reversing involution does not contain all shuffles. The remaining ones have $m_{even} = 0$ and m_{odd} are all even. These shuffles have positive sign and there are $\binom{j+k-1}{k}$ of them.

The other term on the third line is the sum of

$$\mathrm{sgn}(\sigma)\,\mathrm{sgn}_\sigma(a_1, b_1, \cdots, b_m) = (-1)^{m_2+m_4+\cdots+m_{2j-2}}$$

Here we apply the involution above to $m_2, \cdots, m_{2j-1}$. The remaining terms are all positive and have $m_2 = m_4 = \cdots = 0$ and $m_3, m_5, \cdots$ all even and m_0, m_1 are arbitrary (with even sum). There are $\binom{j+k}{k}$ such terms where both m_0, m_1 are even. If they are unequal we can subtract 1 from the larger and add 1 to the smaller, making them both odd. This however overcounts the terms where m_0, m_1 are odd and equal. Thus there are

$$\binom{j+k}{k} - \binom{j+k-1}{k}$$

terms where both m_0, m_1 are odd. The term $\binom{j+k-1}{k}$ counts shuffles where $m_0 = m_1$ are odd or even, both kinds being overcounted once.

By symmetry (switch $n \leftrightarrow m$ and $j \leftrightarrow k$) we get $\sum \operatorname{sgn}(\sigma)$ for $n = 2j, m = 2k-1$. Since $m = 2k-1$ is odd,

$$\operatorname{sgn}(\sigma) = (-1)^{m_1+m_3+\cdots+m_{2j-1}} = -(-1)^{m_0+m_2+\cdots+m_{2j}}.$$

This accounts for the $-\binom{j+k-1}{j}$ in the chart. The remaining three terms are similar. □

Lemma 6.3 *Consider all shuffles σ of $a_0, \cdots, a_n$ with $b_1, \cdots, b_m$ so that a_0 stays on the left (ie, $m_0 = 0$). Then the sum $X_1(n,m)$ and average $E_1(n,m)$ of the oriented sign sum*

$$\operatorname{sgn}(\sigma)\sum_{i=0}^{n} \operatorname{sgn}_\sigma(a_i, b_1, \cdots, b_m)$$

are given by

n	m	$X_1(n,m)$	$E_1(n,m)$
$2j$	$2k$	$(2j+1)\binom{j+k}{j}$	$\frac{(2j+1)!!(2k-1)!!}{(2j+2k-1)!!}$
$2j$	$2k-1$	$\binom{j+k-1}{j}$	$\frac{(2j-1)!!(2k-1)!!}{(2j+2k-1)!!}$
$2j-1$	$2k$	$(2j+2k)\binom{j+k-1}{k}$	$\frac{(2j+2k)(2j-1)!!(2k-1)!!}{(2j+2k-1)!!}$
$2j-1$	$2k-1$	$2k\binom{j+k-1}{k}$	$\frac{(2j-1)!!(2k-1)!!}{(2j+2k-3)!!}$

Remark 6.4 Note that Proposition 5.3 (3) can be rephrased as:

$$Q_k(2j+1, 1, \cdots, 1) = E_1(2j, 2k) = E_0(2j+1, 2k-1).$$

Proof The shuffles in this lemma are the same as those in Lemma 6.2. The only difference is that the oriented sign sum has one more term. The extra term is

$$\operatorname{sgn}(\sigma)\operatorname{sgn}_\sigma(a_0, b_1, \cdots, b_m) = \operatorname{sgn}(\sigma).$$

Therefore

$$E_1(n,m) = E(n,m) + E(\operatorname{sgn}(\sigma)).$$

The first term is given by Lemma 6.2. The second term is given by the first column of (19) divided by $\binom{n+m}{n}$. □

Lemma 6.5 *Consider all shuffles σ of $a_0, \cdots, a_{n+1}$ with $b_1, \cdots, b_m$ so that a_0 is the first letter and a_{n+1} is the last. Let $X_2(n,m)$ and $E_2(n,m)$ be the sum and average value of the oriented sign sum*

$$\operatorname{sgn}(\sigma) \sum_{i=0}^{n+1} \operatorname{sgn}_\sigma(a_i, b_1, \cdots, b_m).$$

Then

n	m	$X_2(n,m)$	$E_2(n,m)$
$2j$	$2k$	$(2j+2)\binom{j+k}{j}$	$\frac{(2j+2)(2j-1)!!(2k-1)!!}{(2j+2k-1)!!}$
$2j$	$2k-1$	0	0
$2j-1$	$2k$	$(2j+2k+1)\binom{j+k-1}{k}$	$\frac{(2j+2k+1)(2j-1)!!(2k-1)!!}{(2j+2k-1)!!}$
$2j-1$	$2k-1$	$-2k\binom{j+k-1}{k}$	$-\frac{(2j-1)!!(2k-1)!!}{(2j+2k-3)!!}$

Proof The shuffles in this lemma are the same as those in Lemma 6.2 with a_0 added on the left and a_{n+1} added on the right. The a_{n+1} on the right changes the sign by $(-1)^m$ and there are two extra terms in the oriented sign sum given by

$$\operatorname{sgn}(\sigma)\left(\operatorname{sgn}_\sigma(a_0, b_1, \cdots, b_m) + \operatorname{sgn}_\sigma(a_{n+1}, b_1, \cdots, b_m)\right) = (1+(-1)^m)\operatorname{sgn}(\sigma).$$

Therefore

$$E_0(n,m) = (-1)^m E(n,m) + (1+(-1)^m)E(\operatorname{sgn}(\sigma)).$$

The first term is given by Lemma 6.2. The second term is given by the first column of (19) times $1+(-1)^m$ divided by $\binom{n+m}{n}$. □

Theorem 6.6 *If $p+q = 2k-1$ and $n = 2r+1$ we get:*

$$T_k(3, 1^p, n, 1^q) = \sum_{s=0}^{\lceil q/2 \rceil} \frac{q!}{(q-2s+1)!} \binom{r-1+s}{s} (2k-2s)!\, 3\,(k-s+1) \cdot \big[(q-2s+1)(2r+2s+1) - 2s(2k-2s+3)\big]$$

where we use the notation

$$T_k(3, 1^p, n, 1^q) = T_k(3, \overbrace{1, \ldots, 1}^{p}, n, \overbrace{1, \ldots, 1}^{q}).$$

Proof Take cyclic shuffles σ of

$$a_1a_2a_3b^1b^2\cdots b^pc_1\cdots c_nd^1\cdots d^q$$

Then

$$T_k(3,1^p,n,1^q)=\sum_\sigma \mathrm{sgn}(\sigma)\sum_{i=1}^3\sum_{j=1}^n \mathrm{sgn}_\sigma(a_ib^1\cdots b^pc_jd^1\cdots d^q)$$

The cyclic shuffle σ permutes the b's and shuffles them with the a's, inserts $c_1\cdots c_n$ as one block, then permutes the d's and shuffles them in.

However, any permutation of the b's will not change the oriented sign sum since it changes both the orientation and the sign sum by the same sign. Therefore, it suffices to consider those σ which do not permute the $b's$ and multiply the result by $p!$.

Similarly, we may assume that the q are in a fixed order, so that $q^1,\cdots,q^\ell$ are shuffled between the c's and $d^{\ell+1},\cdots,d^q$ are shuffled into the a's and b's. The resulting sum should be multiplied by $q!$. The oriented sign sum for the shuffles of the d's between the c's is $X_2(n-2,\ell)$. The value of this terms and the remaining factrors depends on the parity of ℓ.

Case 1 $\ell=2s$ is even In this case there are an odd number of letters and an odd number of kinds of letters in the set

$$S=\{c_1,\cdots,c_n,d^1,\cdots,d^\ell\}.$$

So, S behaves like a single letter and we have $p+1+q-\ell=2k-2s$ letters shuffled together in $(2k-2s)!/(p!(q-\ell)!)$ ways and then shuffled with a_1,a_2,a_3 keeping a_1 first. The contribution to the tree polynomial given by these shuffles is then

$$p!q!\frac{(2k-2s)!}{p!(q-\ell)!}X_1(2,2k-2s)X_2(2r-1,2s). \tag{20}$$

By Lemmas 6.2 and 6.5 this is equal to

$$\frac{q!(2k-2s)!}{(q-2s)!}3(k-s+1)(2r+2s+1)\binom{r+s-1}{s} \tag{21}$$

Case 2 $\ell=2s-1$ is odd In this case the set S has an even number of letters and an even number of kinds of letters. Therefore, S can be placed anywhere with the same effect. Since there are $3+p+q-\ell=2k-2s+3$ remaining letters we multiply by this factor. There are p b's and $q-\ell$ d's shuffled together in

$$\binom{p+q-\ell}{p}=\frac{(2k-2s)!}{p!(q-\ell)!}$$

ways. So the contribution to the tree polynomial of these shuffles is

$$p!q!(2k-2s+3)\frac{(2k-2s)!}{p!(q-\ell)!}X_1(2,2k-2s)X_2(2r-1,2s-1). \tag{22}$$

By Lemmas 6.2 and 6.5 this is equal to

$$-(2k-2s+3)\frac{q!(2k-2s)!}{(q-2s+1)!}3(k-s+1)2s\binom{r+s-1}{s} \tag{23}$$

Therefore, $T_k(3,1^p,2r+1,1^q)$ is equal to the sum of (21) for $s=0\cdots\left\lfloor\frac{q}{2}\right\rfloor$ and (23) for $s=1\cdots\left\lceil\frac{q}{2}\right\rceil$. Since (21) and (23) are so similar we can simplify the sum by adding them together to get

$$\sum_{s=0}^{\lceil q/2\rceil}\frac{q!}{(q-2s+1)!}\binom{r-1+s}{s}(2k-2s)!\,3\,(k-s+1)\cdot \tag{24}$$
$$\big[(q-2s+1)(2r+2s+1)-2s(2k-2s+3)\big]\,.$$

The polynomial on the second line consists of the places where the $\ell=2s$ and $\ell=2s-1$ terms differ. The sum now runs from $s=0$ to $s=\lceil q/2\rceil$, which means we have introduced the extra terms corresponding to $\ell=-1$ and, when q is odd, $\ell=q+1$, but both of these are zero. □

7 A double sum

Using the formula (24) for the tree polynomial $T_k(3,1^p,2r+1,1^q)$ we are now in a position to compute the coefficient $b_{r,k}^{r+k}$ for any $r,k\geq 0$. This section benefitted greatly from the advice of Christian Krattenthaler, who pointed out that the techniques of summation in an earlier version were unnecessarily complicated.

By Theorem 5.5 we have:

$$\begin{aligned} b_{r,k}^{r+k} &= \sum b_r\frac{(2m_0+1)Q_k(2m_0+3,2m_1+1,\cdots,2m_{2k}+1)}{(2m_0+3)(-2)^{k+1}(2k-1)!!} \\ &= \frac{(2r+1)Q_k(2r+3,1,\cdots,1)}{(2r+3)a_r(-2)^{k+1}(2k-1)!!}+\sum_{p+q=2k-1}\frac{Q_k(3,1^p,2r+1,1^q)}{3a_r(-2)^{k+1}(2k-1)!!} \end{aligned} \tag{25}$$

where $a_r=1/b_r=(-2)^{r+1}(2r+1)!!$. By Theorem 6.1 we know that

$$Q_k(2r+1,1,\cdots,1)=\frac{(2k-1)!!(2r+3)!!}{(2r+2k+1)!!}$$

so the first term in (25) above is equal to

$$\frac{2r+1}{a_r(-2)^{k+1}(2k-1)!!}\frac{(2k-1)!!(2r+1)!!}{(2r+2k+1)!!} = \frac{2r+1}{(-2)^{r+k+2}(2r+2k+1)!!}.$$

Lemma 7.1

$$\sum_{p+q=2k-1} Q_k(3,1^p,2r+1,1^q) = 3\frac{(2k+2r+3)}{2k+1} - 3\frac{(2r+3)!!\,(2k-1)!!}{(2k+2r+1)!!}$$

Suppose for a moment that this is true. Then the second term of (25) is equal to

$$\sum_{p+q=2k-1} \frac{Q_k(3,1^p,2r+1,1^q)}{3a_r(-2)^{k+1}(2k-1)!!} = \frac{2r+2k+3}{a_r a_k} - \frac{2r+3}{(-2)^{r+k+2}(2r+2k+1)!!}.$$

Putting these together we get

$$b_{r,k}^{r+k} = \frac{2r+2k+3}{a_r a_k} + \frac{1}{a_{r+k}}$$

which can be simplified to

$$b_{r,k}^{r+k} = b_r b_k(2r+2k+3) + b_{r+k}.$$

In terms of the adjusted Miller–Morita–Mumford classes this says

$$\widetilde{\kappa}_r\widetilde{\kappa}_k = (b_r b_k(2r+2k+3) + b_{r+k})[W_{r+k}^*] + \operatorname{Sym}(r,k) b_r b_k [W_{r,k}^*] \tag{26}$$

where $\operatorname{Sym}(r,k)$ is 2 for $r=k$ and 1 for $r \neq k$. Solving the equation

$$a_{r,k}^{r+k} b_{r+k} + a_{r,k}^{r,k} b_{r,k}^{r+k} = 0$$

in which $a_{r,k}^{r,k} = a_r a_k / \operatorname{Sym}(r,k)$ we see that the inverse coefficient $a_{r,k}^{r+k}$ is given by:

$$\begin{aligned} a_{r,k}^{r+k} = -\frac{a_r a_k(b_r b_k(2r+2k+3)+b_{r+k})}{\operatorname{Sym}(r,k) b_{r+k}} &= -\frac{a_r a_k + (2r+2k+3)a_{r+k}}{\operatorname{Sym}(r,k)} \\ &= \frac{(-2)^{r+k+1}}{\operatorname{Sym}(r,k)}(2(2r+1)!!(2k+1)!! - (2r+2k+3)!!) \end{aligned}$$

The Kontsevich cycle $W_{r,k}^*$ is related to the adjusted MMM classes by the formula

$$[W_{r,k}^*] = a_{r,k}^{r+k}\widetilde{\kappa}_{r+k} + a_{r,k}^{r,k}\widetilde{\kappa}_r\widetilde{\kappa}_k.$$

This gives the following equation as conjectured in [6].

Theorem 7.2

$$[W^*_{r,k}] = \frac{(-2)^{r+k+1}}{\mathrm{Sym}(r,k)}(2(2r+1)!!(2k+1)!!(\widetilde{\kappa}_{r+k} - \widetilde{\kappa}_r\widetilde{\kappa}_k) - (2r+2k+3)!!\widetilde{\kappa}_{r+k})$$

Proof of Lemma 7.1 It remains to calculate the sum

$$\sum_{p+q=2k-1} Q_k(3, 1^p, 2r+1, 1^q),$$

where in this case

$$Q_k(3, 1^p, 2r+1, 1^q) = \frac{2(p+2r+3)!}{(p+3)!(p+q+2r+3)!} T_k(3, 1^p, 2r+1, 1^q). \tag{27}$$

By Theorem 6.6 the tree polynomial T_k is given by

$$T_k(3,1^p, 2r+1, 1^q) = \sum_{s=0}^{\lceil q/2 \rceil} \frac{q!}{(q-2s+1)!}\binom{r-1+s}{s}(2k-2s)!\,3\,(k-s+1)\cdot \left[(q-2s+1)(2r+2s+1) - 2s(2k-2s+3)\right]. \tag{28}$$

We combine equations (27) and (28), eliminating the variable $p = 2k-1-q$ and expressing everything in terms of factorials. We seek the double sum:

$$\sum_{q=0}^{2k-1} \sum_{s=0}^{\lceil q/2 \rceil} F(k,r,s,q), \text{ where}$$

$$F(k,r,s,q) = \frac{6(2k+2r-q+2)!\,q!\,(r+s-1)!\,(2k-2s)!\,(k-s+1)}{(2k+2r+2)!\,(2k-q+2)!\,(q-2s+1)!\,s!\,(r-1)!}\cdot \left[(q-2s+1)(2r+2s+1) - (2s)(2k-2s+3)\right]. \tag{29}$$

The summand $F(k,r,s,q)$ is a hypergeometric term in each of its variables, so sophisticated summation techniques are available; see [12] for an introduction. We are grateful to Christian Krattenthaler for suggesting the following path.

The summand $F(k,r,s,q)$ is more manageable as a sum over q, so we will switch the order of the double summation. The result is *almost*

$$\sum_{s=0}^{k} \sum_{q=2s-1}^{2k-1} F(k,r,s,q) \tag{30}$$

except that this introduces one new term where $s = 0$ and $q = -1$. Here F would need delicate handling owing to the $q!$ in its numerator. We proceed by first calculating the sum in (30) formally, and then dealing with the error term that arises when $s = 0$.

The inner summation is now over q with s fixed, and the summand is a q–free part times an expression of the form

$$G(q) = \frac{(A+B-q)!\,q!}{(A-q)!\,(q-C)!}\left[(q-C)(B+C+2)-(C+1)(A-C)\right],$$

where $A = 2k+2$, $B = 2r$, $C = 2s-1$. Gosper's summation algorithm quickly points out that G has a discrete antiderivative: $G(q) = H(q+1)-H(q)$, where

$$H(q) = \frac{-(A+B+1-q)!\,q!}{(A-q)!\,(q-C-1)},$$

a relation easily verified by hand. Any definite sum is now easily computed, and in particular we want $\sum_{q=C}^{A-3} G(q) = H(A-2) - H(C)$. Now note that $H(C) = 0$ owing to the $(-1)!$ in the denominator, and we find that the sum is just

$$H(A-2) = \frac{-(A-2)!\,(B+3)!}{2(A-C-3)!}.$$

Returning to the original variables and replacing the q–free coefficient, we have found that:

$$\sum_{q=2s-1}^{2k-1} F(k,r,s,q) = \frac{-3\,(2k)!\,(2r+3)!}{(r-1)!\,(2k+2r+2)!}\cdot\frac{(k+1-s)\,(r-1+s)!}{s!} \tag{31}$$

Gosper's algorithm reveals that this too has a discrete antiderivative with respect to s:

$$\begin{aligned} H(s) &= \frac{-3\,(2k)!\,(2r+3)!}{(r-1)!\,(2k+2r+2)!}\cdot\frac{(kr-rs+2r+k+1)\,(r+s-1)!}{r(r+1)\,(s-1)!} \\ &= \frac{-3\,(2k)!\,(2r+3)!\,(kr-rs+2r+k+1)\,(r+s-1)!}{(r+1)!\,(2k+2r+2)!\,(s-1)!} \end{aligned}$$

The sum from $s = 0$ to $s = k$ is then $H(k+1) - H(0)$, and again we find that $H(0) = 0$. The full sum is therefore $H(k+1)$, and pulling factorials together, we get:

$$\begin{aligned} \sum_{s=0}^{k}\sum_{q=2s-1}^{2k-1} F(k,r,s,q) &= \frac{-6\,(2k-1)!\,(2r+3)!\,(k+r+1)!}{(k-1)!\,(r+1)!\,(2k+2r+2)!} \\ &= \frac{-3\,(2k-1)!!\,(2r+3)!!}{(2k+2r+1)!!} \end{aligned} \tag{32}$$

Now we compute the error term. At $s = 0$, the left-hand side of (31) is problematic, but the right-hand side which we used for further computations is

$$\frac{-3\,(2k)!\,(2r+3)!\,(k+1)}{(2k+2r+2)!}.$$

The actual desired value can be easily computed since $F(k,r,0,q)$ is a $q-$independent factor times the binomial coefficient $\binom{2k+2r+2-q}{2r}$.

$$\sum_{q=0}^{2k-1} F(k,r,0,q) = \frac{6\,(k+1)\,(2k)!}{(2k+2r+2)!}\left(\frac{(2k+2r+3)!}{(2k+2)!} - \frac{(2r+3)!}{2!}\right)$$

We subtract to find that the error introduced by using the formal answer at $s=0$ was

$$-\frac{3\,(3+2k+2r)}{(2k+1)}. \tag{33}$$

The final answer is the formal sum (32) minus the error term (33), which is precisely the conjectured value:

$$3\frac{(2k+2r+3)}{2k+1} - 3\frac{(2r+3)!!\,(2k-1)!!}{(2k+2r+1)!!}$$ □

8 Reduced tree polynomial

The second formula for the tree polynomial is based on the following lemma.

Lemma 8.1 *The sum over all sequences of positive integers*

$$1 \le z_1, z_2, \cdots, z_s \le n$$

of the quantity

$$(-1)^{z_1+\cdots+z_s}(B(z) - A(z))$$

where $A(z)$ is the number of positive integers j which are $\le z_i$ for an odd number of indices i and $B(z)$ is the number of positive integers $j \le n$ so that $j \le z_i$ for an even number of i is equal to

(1) 1 *if s, n are both odd,*

(2) n *if $s \ge 0$ is even and n is odd, and*

(3) $\frac{n}{2}(-2)^s$ *if $n \ge 0$ is even.*

Proof First note that this sum can be written as

$$\sum_z (-1)^{\sum z_i}(B(z) - A(z)) = \sum_z (-1)^{\sum z_i} \sum_{p=0}^{n} (-1)^p |L_p(z)| \tag{34}$$

where $L_p(z)$ is the set of all $j \in \{1, 2, \cdots, n\}$ so that $j \leq z_i$ for exactly p values of i, ie, so that $z^{-1}[j,n]$ has p elements where $[j,n]$ denotes the set of integers from j through n. This can also be written as

$$\sum_{j=1}^{n} \sum_{z} (-1)^{\sum z_i} (-1)^{|z^{-1}[j,n]|}$$
$$= \sum_{j=1}^{n} \sum_{p=0}^{s} \binom{s}{p} (-1)^{pj} N(j)^{s-p} N(n-j)^p$$

where

$$N(j) = \sum_{i=1}^{j-1} (-1)^i = \sum_{i=1}^{j+1} (-1)^i = \begin{cases} -1 & j \text{ even} \\ 0 & j \text{ odd.} \end{cases}$$

Case 1 n is odd Then either j or $n-j$ is odd for each j. So the summand is nonzero only for $p = 0$ or s:

$$\sum_{p=0}^{s} \binom{s}{p} (-1)^{pj} N(j)^{s-p} N(n-j)^p = N(j)^s + (-1)^{kj} N(n-j)^s = (-1)^{s(j+1)}$$

So the sum (34) is equal to

$$\sum_{j=1}^{n} (-1)^{s(j+1)} = \begin{cases} 1 & \text{if } s \text{ is odd} \\ n & \text{if } s \text{ is even.} \end{cases}$$

Case 2 n is even In this case it is possible for both j and $n-j$ to be even. But then we get

$$\sum_{p=0}^{s} \binom{s}{p} (-1)^{pj} (-1)^s = (-1)^s (1 + (-1)^j)^s = (-2)^s.$$

There are $\frac{n}{2}$ such terms and the other terms, where $j, n-j$ are both odd, are all zero. □

Using this lemma we get another formula for the tree polynomial showing that the monomials correspond to increasing trees. Recall that an *increasing tree* T with vertices $0, 1, 2, \cdots, 2k$ is a tree constructed by attaching the vertices in order. In other words, 0 is the root and children are always larger than their parents. (See [13] for more details about increasing trees.) For each such T take the monomial in the variables $x_0, x_1, \cdots, x_{2k}$ given as follows.

For each vertex $i = 0, 1, \cdots, 2k$ of T let n_i be the number of trees in the forest $T - \{i\}$ with an even number of vertices. Associate to T the *tree monomial*

$$x^T = x_0^{n_0} \cdots x_{2k}^{n_{2k}}.$$

Example 8.2 In the simplest case $k = 1$ there are are two increasing trees: $0-1-2$ and $1-0-2$. The corresponding tree monomials are x_0x_2 and x_1x_2.

Lemma 8.3 *Each tree monomial x^T has degree $2k$.*

Proof Since T has an odd number of vertices we can orient each edge so that it points in the direction in which there are an even number of vertices. Then n_i is the number of outward pointing edges at vertex i. The sum of the n_i must the number of edges which is $2k$. □

Theorem 8.4 *The sum of the tree monomials x^T is related to the tree polynomial by:*

$$x_0 \sum_T x^T = T_k(x_0, x_1, \cdots, x_{2k}).$$

Suppose for a moment that this is true.

Definition 8.5 We will call

$$\widetilde{T}_k(x_0, \cdots, x_{2k}) := \sum_T x^T = \frac{1}{x_0} T_k(x_0, \cdots, x_{2k})$$

the *reduced tree polynomial.*

Since increasing trees are in 1–1 correspondence with permutations of $1, \cdots, 2k$ we get the following.

Corollary 8.6 *The tree polynomial $T_k(x_0, \cdots, x_{2k})$ is a homogeneous polynomial of degree $2k+1$ with nonnegative integer coefficients which add up to $(2k)!$, ie, $T_k(1, 1, \cdots, 1) = (2k)!$.*

Calculations of the reduced tree polynomial tell us something about permutation. For example, we have the following.

Corollary 8.7 *In the special case $x_1 = x_2 = \cdots = x_{2k} = 1$ the reduced tree polynomial is the generating function*

$$\widetilde{T}_k(x, 1, 1, ..., 1) = (2k-1)!!(x+1)(x+3)...(x+2k-1) = \sum_{i=0}^{k} p_i x^i$$

where p_i is the number of permutations of $2k$ with i even cycles.

Proof For every increasing tree T the coefficient of x_0 in the monomial x^T is equal to the number of even cycles in the permutation of $2k$ corresponding to T. □

By the following proposition the first variable x_0 in the reduced tree polynomial $\widetilde{T}_k$ is superfluous.

Proposition 8.8 *The reduced tree polynomial $\widetilde{T}_k(x_0, \cdots, x_{2k})$ is a polynomial in the variables $x_0 + x_1, x_2, x_3, \cdots, x_{2k}$. In other words,*

$$\widetilde{T}_k(x_0, x_1, \cdots, x_{2k}) = \widetilde{T}_k(0, x_0 + x_1, x_2, \cdots, x_{2k}).$$

Remark 8.9 This means that it suffices to compute $\widetilde{T}_k$ in the case when $x_0 = 0$ since we can recover the general polynomial by substituting $x_0 + x_1$ for x_1.

Proof Any increasing tree T contains vertices $0, 1$ connected by an edge together with a certain number of trees $a_1, \cdots, a_r$ with an odd number of vertices and other trees $b_1, \cdots, b_s$ with an even number of vertices. These trees can be attached to either 0 or 1 giving 2^{r+s} different increasing trees. Let S be this set of increasing trees.

Each b_i gives a factor of either x_0 or x_1 for x^T depending on whether it is attached to 0 or 1. Therefore the b_i's altogether give a factor of

$$(x_0 + x_1)^s$$

to the sum of x^T for all increasing trees in S.

The number r must be odd in order for the total number of vertices to be equal to $2k + 1$. Exactly half of the time an odd number of a_i will be attached to 0 and the other half of the time an odd number of a_i will be attached to 1. Consequently, the a_j give a factor of

$$(x_0 + x_1)^{r-1}$$

to the sum of x^T for all increasing trees in S. Thus, the sum of x^T for all $T \in S$ is equal to $(x_0 + x_1)^{r+s-1}$ times a polynomial in the other variables $x_2, \cdots, x_{2k}$. □

Proof of Theorem 8.4 Suppose that σ is a cyclic shuffle of the letters

$$a_1^0, \cdots, a_{x_0}^0, a_1^1, \cdots, a_{x_1}^1, \cdots, a_1^{2k}, \cdots, a_{x_{2k}}^{2k}.$$

Then we associate to σ an increasing tree $T(\sigma)$ as follows.

To each letter a^i we associate the vertex i. We start with $T_0(\sigma)$ being just the root 0 which is associated to $a^0_1 \cdots a^0_{x_0}$. We attach to $T_0(\sigma)$ the vertex 1 corresponding to a^1. This gives $T_1(\sigma)$. There are two possibilities for $T_2(\sigma)$ as in Example 8.2. We get $0-1-2$ if a^2 is inserted after (on the right of) an a^1_i. We get $1-0-2$ is a^2 is inserted after an a^0. Proceeding by induction suppose that we have constructed the increasing tree $T_n(\sigma)$ with vertices $0, 1, \cdots, n$. Then $T_{n+1}(\sigma)$ is obtained from $T_n(\sigma)$ by attaching the new vertex $n+1$ to vertex j if σ inserts a^{n+1} after some a^j_i.

Since there are x_j letters a^j_i, the number of cyclic shuffles σ giving the same increasing tree T is equal to

$$x_0^{m_0} x_1^{m_1} \cdots x_{2k-1}^{m_{2k-1}}$$

where m_j is the number of children that vertex j has.

Claim The sum

$$\sum_{T(\sigma)=T} \operatorname{sgn}(\sigma) \sum_i \operatorname{sgn}_\sigma(a^0_{i_0} a^1_{i_1} \cdots a^{2k}_{i_{2k}}) \tag{35}$$

of the oriented sign sum of σ for all σ with $T(\sigma) = T$ is equal to the tree monomial x^T times x_0.

Since the tree polynomial is the sum of (35) over all increasing trees T, this claim will prove the theorem.

To prove the claim we first consider the unique shuffle σ_0 with $T(\sigma_0) = T$ having the property that each letter is inserted in the last allowed slot (after the last letter corresponding to its parent in the increasing tree T). The oriented sign sum of σ_0 is equal to the product

$$\operatorname{sgn}(\sigma_0) \sum_i \operatorname{sgn}_\sigma(a^0_{i_0} a^1_{i_1} \cdots a^{2k}_{i_{2k}}) = x_0 x_1 \cdots x_{2k} \tag{36}$$

since every summand is equal to 1.

The statement (36) is the base case ($j = 0$) of the following induction hypothesis:

$$\sum_\sigma \operatorname{sgn}(\sigma) \sum_i \operatorname{sgn}_\sigma(a^0_{i_0} a^1_{i_1} \cdots a^{2k}_{i_{2k}}) = x_0^{n_0} \cdots x_{j-1}^{n_{j-1}} x_j \cdots x_{2k} \tag{37}$$

if the sum is taken over all σ so that

(1) $T(\sigma) = \sigma$

(2) for all $i \geq j$ the children of i are inserted in the last allowed slot (after the last a^i).

We recall that n_i is the number of components of $T(\sigma) - i$ having an even number over vertices.

Suppose by induction that (37) holds for j. To extend it to $j+1$ we need to allow the children of vertex j to be inserted at any of the x_j allowed points.

Let $b^1, b^2, \cdots, b^r$ be the letters corresponding to the children of j with an odd number of descendants. Then each b^i has the property that it, together with all its descendants, can be moved to any other slot without changing the oriented sign sum. This is because both the shuffle and the permutation of selected letters changes by an even permutation. Consequently, the sum (37) is multiplied by x_j^r bringing the value of (37) to

$$x_0^{n_0} \cdots x_{j-1}^{n_{j-1}} x_j^{r+1} \cdots x_{2k}. \tag{38}$$

Let $c^1, \cdots, c^s$ be the other children of j, the ones with an even number of descendants. Let $z_1, \cdots, z_s$ denote the indices of the letter a^j after which these letters are inserted, eg, c^1 is inserted after $a^j_{z_1}$. Take the sum:

$$\sum_z \operatorname{sgn}(\sigma) \sum \operatorname{sgn}_\sigma(a^0_{i_0} \cdots a^{2k}_{i_{2k}}) \tag{39}$$

over all $(x_j)^s$ insertion points $z = (z_1, \cdots, z_s)$ for all of the children c^i together with their descendants. The question is: How do the terms in this sum compare to the term in which all of the z_i are maximal (equal to x_j)?

Case 1 s is odd (Then there are an odd number of vertices in T minus j and its descendants. So $n_j = r$.) In this case we claim that the sum (39) is equal to $\frac{1}{x_j}$ times the summand in which each z_i is maximal. This is the first case of Lemma 8.1. To see this consider what happens when we decrease by one the insertion point z_i of c^i and its descendants. This will change the sign of σ by $(-1)^{m+1}$ where m is the number of other b_p which are transposed with c^i. But the selected sign $\operatorname{sgn}_\sigma(a^0_{i_0} \cdots a^{2k}_{i_{2k}})$ also changes by $(-1)^m$ so the net effect is to change the sign of the oriented sign sum. Since x_j is odd and the sign changes $x_j - z_i$ times, this gives a sign factor of

$$(-1)^{s+z_1+\cdots+z_s} = -(-1)^{z_1+\cdots+z_s}. \tag{40}$$

For each value of the index i_j of a^j, the selected sign only changes when some z_i goes below i_j. Taking the sum over all values of i_j we get a factor of

$$A(z) - B(z)$$

instead of x_j where $A(z), B(z)$ are as defined in Lemma 8.1. This factor, together with (40), adds up to 1 by the lemma. This is instead of the factor of x_j which we get in the case when each z_i is maximal. So, the sum (37) for $j+1$ is equal to

$$x_0^{n_0} \cdots x_{j-1}^{n_{j-1}} x_j^r \cdots x_{2k}$$

which is correct since $r = n_j$.

Case 2 s is even (Then there are an even number of vertices in T minus j and its descendants making $n_j = r+1$.) In this case we claim that the sum (39) is equal to the term in which all the z_i are maximal. The proof is the same as in Case 1, using the second case of Lemma 8.1. This leaves the product (38) unchanged. But this is correct since $n_j = r+1$. □

9 Recursion for $\widetilde{T}_k$

We will show that the reduced tree polynomial $\widetilde{T}_k$ (in variables $x_0, \cdots, x_{2k}$) satisfies a recursion which we can express in terms of an exponential generating function. First we need to generalize the reduced tree polynomial.

Definition 9.1 For $k, n \geq 0$ let L_k^n be the polynomial in generators $x_0, \cdots, x_{2k}$ given by

$$L_k^n = \sum_T \frac{x^T}{x_{2k+1} \cdots x_{2k+2n}}$$

where the sum is taken over all increasing trees T with vertices 0 through $2k+2n$ of which the last $2n$ are leaves. To simplify notation we write the summand above as $\hat{x}^T$ (ie, this is x^T with $x_{2k+1}, \cdots, x_{2k+2n}$ set equal to zero. If we delete the last $2n$ vertices from T we get what we call the *base tree* T_0 which is an arbitrary increasing tree with vertices $0, \cdots, 2k$.

We make some trivial observations about this polynomial.

Proposition 9.2 (1) *$L_k^0 = \widetilde{T}_k$ is the reduced tree polynomial.*

(2) $L_0^n(x_0) = 1$

(3) *The polynomial L_k^n has nonnegative integer coefficients adding up to*

$$L_k^n(1, \cdots, 1) = (2k)!(2k+1)^{2n}.$$

Let $g_k(t)$ be the exponential generating function:

$$g_k(t)=\sum_{n=0}^{\infty} L_k^n \frac{t^{2n}}{(2n)!}$$

Then $g_k(0)=L_k^0=\widetilde{T}_k$. So it suffices to compute $g_k(t)$ for all k. When $k=0$ we have $L_0^n=1$ so

$$g_0(t)=\sum \frac{t^{2n}}{(2n)!}=\cosh t.$$

Theorem 9.3 *The generating function $g_k(t)$ which gives $g_k(0)=\widetilde{T}_k$ is given recursively as follows.*

(1) $g_0(t)=\cosh t$

(2) $g_{k+1}(t)=g_k(t)\left(z_{2k}z_{2k+2}\sinh^2 t+z_{2k}y_2\right)$

$$+g_k'(t)z_{2k+1}(y_1+y_2)\sinh t\cosh t+g_k''(t)y_1y_2\cosh^2 t$$

where we use the notation: $z_j=x_0+\cdots+x_j, y_i=x_{2k+i}$.

We will obtain a recursive formula to compute the polynomials L_k^n and use the recursion to show the theorem. We begin with the first nontrivial case $k=1$.

For $k=1$ there are two possibilities for the base tree (consisting of the vertices $0,1,2$). They are connected either as $1-0-2$ or $0-1-2$. In each case we attach $2n$ leaves in all 3^{2n} possible ways.

Let α,β,γ be the number of leaves attached to $1,2,0$ respectively. We note that there are

$$\sum_{j=1}^{n}\binom{2n}{2j}2^{2j-1}=\frac{3^{2n}-1}{4}$$

ways for $\alpha/\beta/\gamma$ to be odd/odd/even and similarly for the cases odd/even/odd and even/odd/odd. This leaves

$$\frac{3^{2n}+3}{4}$$

ways for α,β,γ to be all even. We determine the monomials $\hat{x}^T$ in each case.

(1) Base $1-0-2$ with α,β,γ all even. In this case the monomial is $\hat{x}^T=x_1x_2$. So the contribution is

$$\left(\frac{3^{2n}+3}{4}\right)x_1x_2.$$

(2) Base $1-0-2$ with α,β both odd (and γ even). Then the monomial is $\hat{x}^T = x_0^2$. So the contribution is

$$\left(\frac{3^{2n}-1}{4}\right) x_0^2.$$

(3) Base $1-0-2$ with γ odd. Then the monomial is $\hat{x}^T = x_0 x_i$ where $i=1,2$ with equal probability. So the contribution is

$$\left(\frac{3^{2n}-1}{4}\right) x_0(x_1+x_2).$$

Adding these three together we get

$$\frac{3^{2n}}{4}(x_0(x_0+x_1+x_2)+x_1x_2)+\frac{1}{4}(3x_1x_2-x_0(x_0+x_1+x_2))$$

If the base tree is $0-1-2$ then we just switch x_0 and x_1 in the above expression. Adding these two cases gives

$$L_1^n = \frac{3^{2n}}{4}(x_0+x_1)(2x_2+x_0+x_1)+\frac{1}{4}(x_0+x_1)(2x_2-x_0-x_1). \tag{41}$$

Note that n occurs only in the exponent of 3. More generally, we have the following.

Lemma 9.4

$$L_k^n = \sum_{s=0}^{k} 4^{-k}(2s+1)^{2n} P_k^{2s+1}$$

where P_k^{2s+1} is a polynomial in $x_0,\cdots,x_{2k}$ with integer coefficients depending only on k,s.

Remark 9.5 This lemma can be rephrased in terms of the exponential generating function $g_k(t)$ as follows.

$$g_k(t) = \sum_{n\geq 0} L_k^n \frac{t^{2n}}{(2n)!} = \sum_{n,c} \frac{P_k^c}{4^k}\frac{c^{2n}t^{2n}}{(2n)!} = \sum_{s=0}^{k} 4^{-k} P_k^{2s+1}\cosh((2s+1)t).$$

We will prove Lemma 9.5 and find a recursion for L_k^n at the same time. Suppose we know the polynomial L_k^n for all n and we wish to compute L_{k+1}^n. This is a sum of monomials $\hat{x}^T$. There are again two cases for the base tree T_0. Either $2k+1, 2k+2$ are leaves of the base tree or $2k+2$ is attached to $2k+1$. In both cases we attach $2n$ leaves to T_0, α on $2k+1$, β on $2k+2$ and γ on T_- where T_- is T_0 with the vertices $2k+1, 2k+2$ removed.

Case 1 $2k+1, 2k+2$ are leaves of the base tree T_0.

(1) α, β, γ all even with $\gamma = 2m$. In this case the vertices $2k+1, 2k+2$ act like leaves and T looks like T_- with $2m+2$ leaves. The monomials in this case add up to

$$L_k^{m+1} x_{2k+1} x_{2k+2}.$$

We need to multiply this with the number of choices for the α, β, γ leaves which is

$$\binom{2n}{2m} 2^{2n-2m-1}$$

if $0 \le m < n$ and 1 if $m = n$. This gives a contribution of

$$L_k^{n+1} x_{2k+1} x_{2k+2} + \sum_{m=0}^{n-1} L_k^{m+1} \binom{2n}{2m} 2^{2n-2m-1} x_{2k+1} x_{2k+2}. \tag{42}$$

(2) α, β both odd with $\gamma = 2m$. In this case the vertices $2k+1, 2k+2$ simply add a factor of $x_i x_j$ to $\hat{x}^T$ if they are attached to vertices $i, j \le 2k$. Taking the sum over all i, j we get a factor of z_{2k}^2 where

$$z_{2k} = x_0 + x_1 + \cdots + x_{2k}.$$

The contribution to L_{k+1}^n is thus

$$\sum_{m=0}^{n-1} L_k^m \binom{2n}{2m} 2^{2n-2m-1} z_{2k}^2. \tag{43}$$

(3) $\gamma = 2m-1$. In this case one of α, β is odd and the other is even. This gives a factor of $z_{2k}(x_{2k+1} + x_{2k+2})$ for a contribution of

$$\sum_{m=1}^{n} L_k^m \binom{2n}{2m-1} 2^{2n-2m} z_{2k}(x_{2k+1} + x_{2k+2}). \tag{44}$$

Case 2 $2k+2$ is attached on $2k+1$.

(1) α, β, γ all even with $\gamma = 2m$. Then the tree consisting of vertices $2k+1, 2k+2$ and $\alpha+\beta$ leaves has an even number of vertices and contributes a factor of $z_{2k} x_{2k+2}$. As in Case 1(1) we get a contribution to L_{k+1}^n of

$$L_k^n z_{2k} x_{2k+2} + \sum_{m=0}^{n-1} L_k^m \binom{2n}{2m} 2^{2n-2m-1} z_{2k} x_{2k+2}. \tag{45}$$

(2) α, β both odd with $\gamma = 2m$. This time we get a factor of $z_{2k} x_{2k+1}$ so the contribution is

$$\sum_{m=0}^{n-1} L_k^m \binom{2n}{2m} 2^{2n-2m-1} z_{2k} x_{2k+1}. \tag{46}$$

(3) $\gamma = 2m-1$. This is just like Case 1(3). The tree with vertices $2k+1, 2k+2$ and $\alpha+\beta$ leaves acts like one leaf. We get a factor of x_{2k+1}^2 or $x_{2k+1}x_{2k+2}$ depending on whether α or β is even. Thus the contribution is

$$\sum_{m=1}^{n} L_k^m \binom{2n}{2m-1} 2^{2n-2m} x_{2k+1}(x_{2k+1} + x_{2k+2}). \tag{47}$$

The value of L_{k+1}^n is given by adding these six terms:

$$L_{k+1}^n = (42) + (43) + (44) + (45) + (46) + (47).$$

To simplify the computation we need to use Lemma 9.4 and the following two formulas.

$$\sum_{m=0}^{n-1} \binom{2n}{2m} c^{2m} 2^{2n-2m} = \frac{(c+2)^{2n} + (c-2)^{2n}}{2} - c^{2n}$$

$$\sum_{m=1}^{n} \binom{2n}{2m-1} c^{2m} 2^{2n-2m} = \frac{c}{2}\left(\frac{(c+2)^{2n} - (c-2)^{2n}}{2}\right)$$

Proof of Lemma 9.4 We know that the lemma holds for $k = 0, 1$ so suppose that $k \geq 1$ and the lemma holds for k. Substituting the expression c^{2m} for L_k^m and letting $y_i = x_{2k+i}$ we get the following.

$$\begin{aligned}
\text{expression}(42) &= c^{2n+2} y_1 y_2 + \frac{c^2}{2}\left(\frac{(c+2)^{2n} + (c-2)^{2n}}{2} - c^{2n}\right) y_1 y_2 \\
&= c^2 \left(\frac{(c+2)^{2n} + (c-2)^{2n}}{4} + \frac{c^{2n}}{2}\right) y_1 y_2 \\
\text{expression}(43) &= \left(\frac{(c+2)^{2n} + (c-2)^{2n}}{4} - \frac{c^{2n}}{2}\right) z_{2k}^2 \\
\text{expression}(44) &= c\left(\frac{(c+2)^{2n} - (c-2)^{2n}}{4}\right) z_{2k}(y_1 + y_2) \\
\text{expression}(45) &= c^{2n} z_{2k} y_2 + \frac{1}{2}\left(\frac{(c+2)^{2n} + (c-2)^{2n}}{2} - c^{2n}\right) z_{2k} y_2 \\
&= \left(\frac{(c+2)^{2n} + (c-2)^{2n}}{4} + \frac{c^{2n}}{2}\right) z_{2k} y_2 \\
\text{expression}(46) &= \left(\frac{(c+2)^{2n} + (c-2)^{2n}}{4} - \frac{c^{2n}}{2}\right) z_{2k} y_1 \\
\text{expression}(47) &= c\left(\frac{(c+2)^{2n} - (c-2)^{2n}}{4}\right) y_1(y_1 + y_2)
\end{aligned}$$

Collect together the terms with $c^{2n}/4, (c\pm 2)^{2n}/4$. Then, for every c^{2n} term which occurs in L_k^n we get the following three terms in L_{k+1}^n.

$$\frac{c^{2n}}{4}\left(2c^2y_1y_2 - 2z_{2k}^2 + 2z_{2k}y_2 - 2z_{2k}y_1\right) = \frac{c^{2n}}{4}\left(2c^2y_1y_2 - 2z_{2k}(z_{2k+1} - y_2)\right) \tag{48}$$

$$\begin{aligned}\frac{(c+2)^{2n}}{4}&\left(c^2y_1y_2 + z_{2k}^2 + cz_{2k}(y_1+y_2) + z_{2k}y_2 + z_{2k}y_1 + cy_1(y_1+y_2)\right)\\ &= \frac{(c+2)^{2n}}{4}\left(c^2y_1y_2 + cz_{2k+1}(y_1+y_2) + z_{2k}z_{2k+2}\right)\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{(c-2)^{2n}}{4}&\left(c^2y_1y_2 + z_{2k}^2 - cz_{2k}(y_1+y_2) + z_{2k}y_2 + z_{2k}y_1 - cy_1(y_1+y_2)\right)\\ &= \frac{(c-2)^{2n}}{4}\left(c^2y_1y_2 - cz_{2k+1}(y_1+y_2) + z_{2k}z_{2k+2}\right)\end{aligned} \tag{50}$$

If L_k^n is a linear combination of $c^{2n}/4^k$ for $c = 1, 3, \cdots, 2k+1$ then L_{k+1}^n is a linear combination of the above three expressions which in turn are linear combinations of $c^{2n}/4^{k+1}$ for $c = 1, 3, \cdots, 2k+3$. This proves the lemma. □

If we change the sign of c then (49), (50) are interchanged and (48) remains the same. Consequently, these three expressions directly translate into the following recursion for the coefficients P_k^c.

Theorem 9.6 *$L_k^n = \sum_{s=0}^k 4^{-k} P_k^{2s+1}(2s+1)^{2n}$ where $P_k^c = P_k^{-c}$ is given for all odd integers c as follows.*

$$P_0^1 = P_0^{-1} = 1,\ P_0^c = 0 \text{ if } |c| > 1$$

$$\begin{aligned}P_{k+1}^c = P_k^c&(2c^2y_1y_2 - 2z_{2k}(z_{2k+1} - y_2))\\ &+ P_k^{c-2}\left((c-2)^2y_1y_2 + (c-2)z_{2k+1}(y_1+y_2) + z_{2k}z_{2k+2}\right)\\ &+ P_k^{c+2}\left((c+2)^2y_1y_2 - (c+2)z_{2k+1}(y_1+y_2) + z_{2k}z_{2k+2}\right)\end{aligned}$$

where $z_j = x_0 + \cdots + x_j$ and $y_i = x_{2k+i}$.

Corollary 9.7 *The reduced tree polynomial is given by*

$$\widetilde{T}_k = L_k^0 = \sum_{s=0}^k 4^{-k} P_k^{2s+1}.$$

Proof of Theorem 9.3 The exponential generating function for L_k^n is

$$g_k(t) = \sum_{n,c} \frac{P_k^c}{4^k} c^{2n} \frac{t^{2n}}{(2n)!} = \sum_c \frac{P_k^c}{4^k} \cosh ct.$$

Using the hypertrigonometric identity

$$\cosh(ct \pm 2t) = \cosh ct \cosh 2t \pm \sinh ct \sinh 2t$$

we get:

$$g_k \cosh 2t = \sum_c \frac{P_k^c}{4^k} \frac{1}{2}(\cosh(ct+2t) + \cosh(ct-2t))$$
$$g_k' \sinh 2t = \sum_c \frac{P_k^c}{4^k} \frac{c}{2}(\cosh(ct+2t) - \cosh(ct-2t))$$
$$g_k'' \cosh 2t = \sum_c \frac{P_k^c}{4^k} \frac{c^2}{2}(\cosh(ct+2t) + \cosh(ct-2t)).$$

So, the recursion in Theorem 9.6 gives us:

$$g_{k+1} = \frac{g_k''}{2} y_1 y_2 - \frac{g_k}{2} z_{2k}(z_{2k+1} - y_2)$$
$$+ \frac{g_k''}{2}(\cosh 2t) y_1 y_2 + \frac{g_k'}{2}(\sinh 2t) z_{2k+1}(y_1 + y_2) + \frac{g_k}{2}(\cosh 2t) z_{2k} z_{2k+2}$$

Simplify this to get the theorem. □

10 Examples of $\widetilde{T}_k$

We will use the following version of Theorem 9.6 to compute the reduced tree polynomial $\widetilde{T}_k$ for small k. By Proposition 8.8 it suffices to consider the case when $x_0 = 0$. We use the following version of the recurrence.

$$P_{k+1}^c = P_k^c(2c^2 x_{2k+1} x_{2k+2} - 2z_{2k}(z_{2k+1} - x_{2k+2}))$$
$$+ P_k^{c-2}(z_{2k} + (c-2)x_{2k+1})(z_{2k+1} + (c-1)x_{2k+2})$$
$$+ P_k^{c+2}(z_{2k} - (c+2)x_{2k+1})(z_{2k+1} - (c+1)x_{2k+2})$$
$$P_0^1 = 1 \qquad \widetilde{T}_0 = P_0^1 = 1$$

Since $z_0 = x_0 = 0$ we get:

$$P_1^3 = P_0^1 x_1(x_1 + 2x_2) = x_1^2 + 2x_1 x_2$$

$$P_1^1 = P_0^1(2x_1x_2 - x_1(x_1)) = -x_1^2 + 2x_1x_2$$

$$\widetilde{T}_1 = \frac{1}{4}(P_1^1 + P_1^3) = x_1x_2$$

When $k = 2$ the polynomials P_k^c and $\widetilde{T}_2$ are still manageable:

$$\begin{aligned}
P_2^5 &= P_1^3(z_2 + 3x_3)(z_3 + 4x_4) \\
&= x_1(x_1 + 2x_2)(z_2 + 3x_3)(z_3 + 4x_4) \\
P_2^3 &= P_1^3(18x_3x_4 - 2z_2(z_3 - x_4)) + P_1^1(z_2 + x_3)(z_3 + 2x_4) \\
&= x_1(x_1 + 2x_2)(18x_3x_4 - 2z_2(z_3 - x_4)) + x_1(-x_1 + 2x_2)(z_2 + x_3)(z_3 + 2x_4) \\
P_2^1 &= P_1^1(2x_3x_4 - 2z_2(z_3 - x_4) + (z_2 - x_3)z_3) + P_1^3(z_2 - 3x_3)(z_3 - 2x_4) \\
&= x_1(-x_1 + 2x_2)(2x_3x_4 - 2z_2(z_3 - x_4) + (z_2 - x_3)z_3) + x_1(x_1 + 2x_2) \\
&\qquad (z_2 - 3x_3)(z_3 - 2x_4)
\end{aligned}$$

$$\widetilde{T}_2 = \frac{1}{4^2}(P_2^1 + P_2^3 + P_2^5) = x_1^2x_2x_4 + x_1x_2^2x_4 + 2x_1^2x_3x_4 + 5x_1x_2x_3x_4$$

For $k \geq 3$ both P_k^c and $\widetilde{T}_k$ become more complex (except for P_k^{2k+1}):

$$\begin{aligned}
P_3^7 &= x_1(x_1 + 2x_2)(z_2 + 3x_3)(z_3 + 4x_4)(z_4 + 5x_5)(z_5 + 6x_6) \\
\widetilde{T}_3 &= \frac{1}{4^3}(P_3^1 + P_3^3 + P_3^5 + P_3^7) \\
&= 8x_1^2x_2x_3x_4x_6 + 16x_1^2x_2x_3x_5x_6 + x_1^3x_2x_4x_6 + 2x_1^2x_2^2x_4x_6 + x_1^2x_2x_4^2x_6 \\
&\quad +23x_1^2x_2x_4x_5x_6 + 6x_1x_2^2x_3x_4x_6 + 12x_1x_2^2x_3x_5x_6 + 2x_1^3x_3x_4x_6 + 2x_1^2x_3^2x_4x_6 \\
&\quad +2x_1^2x_3x_4^2x_6 + 28x_1^2x_3x_4x_5x_6 + 5x_1x_2x_3^2x_4x_6 + 10x_1x_2x_3^2x_5x_6 + 2x_1^3x_2x_5x_6 \\
&\quad +4x_1^2x_2^2x_5x_6 + 6x_1^3x_4x_5x_6 + 4x_1^2x_3^2x_5x_6 + 5x_1x_2x_3x_4^2x_6 + 61x_1x_2x_3x_4x_5x_6 \\
&\quad +x_1x_2^3x_4x_6 + x_1x_2^2x_4^2x_6 + 2x_1x_2^3x_5x_6 + 4x_1^3x_3x_5x_6 + 17x_1x_2^2x_4x_5x_6
\end{aligned}$$

The coefficients of $\widetilde{T}_k$ tell us something about increasing trees. For example, 61 (the coefficient of $x_1x_2x_3x_4x_5x_6$) is the number of increasing trees in which each node has an even number of children.

Summary of algorithm

First we obtain the reduced tree polynomial by substituting x_0+x_1 for x_1. For example $\widetilde{T}_2$ is given by:

$$\widetilde{T}_2(x_0, \cdots, x_4) = (x_0+x_1)^2x_2x_4 + (x_0 + x_1)x_2^2x_4 + 2(x_0 + x_1)^2x_3x_4 + 5(x_0 + x_1)x_2x_3x_4$$

Next, we need to find Q_k which is given in general by

$$Q_k(x_0,\cdots,x_{2k})=\frac{\widetilde{T}_k(x_0,\cdots,x_{2k})}{z_1z_2\cdots z_{2k-1}}.$$

For $k=2$ this is

$$Q_k(x_0,\cdots,x_4)=\frac{(x_0+x_1+x_2+x_3)x_2x_4+2(x_0+x_1+x_2)x_3x_4+2x_2x_3x_4}{(x_0+x_1+x_2)(x_0+x_1+x_2+x_3)}.$$

Take any partition μ of m with at most $2k+1$ parts. Write the parts in any order and insert 0 at the end:

$$\mu=(m_0,m_1,\cdots,m_{2k}),\qquad \sum m_i=m.$$

The simplest example has only one part: $\mu=m0^{2k}$. Let

$$R_k(\mu):=\frac{2m_0+1}{2m_0+3}Q_k(2m_0+3,2m_1+1,\cdots,2m_{2k}+1).$$

Let $S_k(\mu)$ be the symmetrized version of R_k:

$$S_k(\mu):=\frac{1}{\mathrm{Sym}(\mu)}\sum_\sigma R_k(m_{\sigma(0)},m_{\sigma(1)},\cdots,m_{\sigma(2k)}),$$

where the sum is over all permutations σ of the letters $0,\cdots,2k$ and $\mathrm{Sym}(\mu)$ is the number of σ which leave μ fixed. (Or equivalently, we take the sum over all distinct permutations of the numbers m_i.) For example:

$$S_2(m)=R_2(m,0^4)+R_2(0,m,0^3)+R_2(0^2,m,0^2)+R_2(0^3,m,0)+R_2(0^4,m)$$

$$=\frac{2m+1}{2m+3}Q_2(2m+3,1^4)+\sum_{q=0}^{2k-1}\frac{1}{3}Q_2(1^{2k-q-1},m,1^q)$$

$$=\frac{2m+7}{5}-\frac{6}{(2m+5)(2m+3)}$$

If λ is any partition of m then Theorem 5.5 says that

$$b^{m+k}_{\lambda,k}=\sum_\mu\frac{b^\mu_\lambda S_k(\mu)}{(-2)^{k+1}(2k-1)!!}$$

where the sum is over all partitions μ of m with at most $2k+1$ parts. This gives a recursive formula for b^m_λ. The coefficients b^μ_λ are then given by the sum of products formula (Lemma 1.4).

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