# Weighted $L^2$–cohomology of Coxeter groups based on barycentric subdivisons

BORIS L OKUN

*Department of Mathematical Sciences*
*University of Wisconsin–Milwaukee*
*Milwaukee, WI 53201, USA*

Email:  `okun@uwm.edu`

**Abstract**

Associated to any finite flag complex $L$ there is a right-angled Coxeter group $W_L$ and a contractible cubical complex $\Sigma_L$ (the Davis complex) on which $W_L$ acts properly and cocompactly, and such that the link of each vertex is $L$. It follows that if $L$ is a generalized homology sphere, then $\Sigma_L$ is a contractible homology manifold. We prove a generalized version of the Singer Conjecture (on the vanishing of the reduced weighted $L^2_{\mathbf{q}}$–cohomology above the middle dimension) for the right-angled Coxeter groups based on barycentric subdivisions in even dimensions. We also prove this conjecture for the groups based on the barycentric subdivision of the boundary complex of a simplex.

# 1   Introduction

A construction of Davis ([1], [2], [4]), associates to any finite flag complex $L$, a "right-angled" Coxeter group $W_L$ and a contractible cubical cell complex $\Sigma_L$ on which $W_L$ acts properly and cocompactly. $W_L$ has the following presentation: the generators are the vertices of $L$, each generator has order 2, and two generators commute if the span an edge in $L$. The most important feature of this construction is that the link of each vertex of $\Sigma_L$ is isomorphic to $L$. A simplicial complex $L$ is a *generalized homology $m$–sphere* (for short, a GHS$^m$) if it is a homology $m$–manifold having the same homology as a standard sphere $\mathbb{S}^m$ (the homology is with real coefficients.) It follows, that if $L$ is a GHS$^{n-1}$, then $\Sigma_L$ is a homology $n$–manifold.

If $L$ is a simplicial complex, $bL$ will denote the barycentric subdivision of $L$. $bL$ is a flag simplicial complex. Let $\partial\Delta^n$ denote the boundary complex of the standard $n$–dimensional simplex.

We study a certain weighted $L^2$–cohomology theory $L_{\mathbf{q}}^2 \mathcal{H}^*$, described in [7], [5]. Suppose, for each vertex of $v \in L$ we are given a positive real number $q_v$, and let $\mathbf{q}$ denote the vector with components $q_v$. Given a minimal word $w = v_1 \ldots v_n \in W_L$, let $\mathbf{q}^w = q_{v_1} \ldots q_{v_n}$. For each $W_L$–orbit of cubes pick a representative $\sigma_0$ and let $w(\sigma) = w$ if $\sigma = w\sigma_0$. (The ambiguity in the choices will not matter in our discussion.) Let $L_{\mathbf{q}}^2 C^i(\Sigma_L) = \{\Sigma c_\sigma \sigma \mid \Sigma c_\sigma^2 \mathbf{q}^{w(\sigma)} < \infty\}$ be the Hilbert space of infinite $i$–cochains, which are square-summable with respect to the weight $\mathbf{q}^w$. The usual coboundary operator $d$ is then a bounded operator, and we define the reduced weighted $L_{\mathbf{q}}^2$–cohomology to be $L_{\mathbf{q}}^2 \mathcal{H}^i(\Sigma_L) = \mathrm{Ker}(d^i)/\overline{\mathrm{Im}(d^{i-1})}$. Similarly, one can define the reduced weighted $L_{\mathbf{q}}^2$–homology, except, instead of the usual boundary operator one uses the adjoint of $d$. It follows from the Hodge decomposition that the resulting homology and cohomology spaces are naturally isomorphic. These spaces are Hilbert modules over the Hecke–von Neumann algebra $\mathcal{N}_{\mathbf{q}}$ (an appropriately completed Hecke algebra of $W_L$.) This allows us to introduce the weighted $L_{\mathbf{q}}^2$ Betti numbers — the dimension of $L_{\mathbf{q}}^2 \mathcal{H}^i$ over $\mathcal{N}_{\mathbf{q}}$. If $\mathbf{q} = \mathbf{1} = (1, \ldots, 1)$, we obtain the usual reduced $L^2$–cohomology, and we omit the index $\mathbf{q}$. We write $\mathbf{q} \le \mathbf{1}$, if each component of $\mathbf{q}$ is $\le 1$.

The following conjecture, attributed to Singer, goes back to 1970's.

**The Singer Conjecture**   If $M^n$ is a closed aspherical manifold, then
$$L^2 \mathcal{H}^i(\widetilde{M^n}) = 0 \text{ for all } i \ne n/2.$$

As explained in [5, Section 14], the appropriate generalization of the Singer Conjecture to the weighted case is the following conjecture:

**The Generalized Singer Conjecture**  Suppose $L$ is a flag $\mathrm{GHS}^{n-1}$. Then $L_{\mathbf{q}}^2 \mathcal{H}^i(\Sigma_L) = 0$ for $i > n/2$ and $\mathbf{q} \leq \mathbf{1}$.

(Poincaré duality shows that for $\mathbf{q} = \mathbf{1}$ this conjecture implies the Singer Conjecture for $\Sigma$.)

This conjecture holds true for $n \leq 4$ by [5]. One of the main results of this paper is a proof of this conjecture for barycentric subdivisions in even dimensions. The proof uses a reduction to a very special case $L = b\partial\Delta^{2k-1}$.

We prove this case as Theorem 5.2. It turns out (Theorem 5.3), that this result implies the vanishing of the $L_{\mathbf{q}}^2$–cohomology in a certain range for arbitrary right-angled Coxeter groups based on barycentric subdivisions. (For $\mathbf{q} = \mathbf{1}$, this implication is proved in [6].) In particular, it follows that the Generalized Singer Conjecture is true for all barycentric subdivisions in even dimensions (Theorem 5.4), and for $b\partial\Delta^n$ in all dimensions (Theorem 5.6).

This paper relies very heavily on [5]. In the inductive proofs we mostly omit the first steps, they are easy exercises using [5].

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## 2   Vanishing conjectures

We will follow the notation from [5]. Given a flag complex $L$ and a full subcomplex $A$, set:

$$\mathfrak{h}_i^{\mathbf{q}}(L) = L^2 \mathcal{H}_i(\Sigma_L)$$
$$\mathfrak{h}_i^{\mathbf{q}}(A) = L^2 \mathcal{H}_i(W_L \Sigma_A)$$
$$\mathfrak{h}_i^{\mathbf{q}}(L, A) = L^2 \mathcal{H}_i(\Sigma_L, W_L \Sigma_A)$$
$$\mathrm{b}_{\mathbf{q}}^i(L) = \dim_{\mathcal{N}_{\mathbf{q}}}(\mathfrak{h}_i^{\mathbf{q}}(L))$$
$$\mathrm{b}_{\mathbf{q}}^i(L, A) = \dim_{\mathcal{N}_{\mathbf{q}}}(\mathfrak{h}_i^{\mathbf{q}}(L, A))$$

The dimension of $\Sigma_L$ is one greater than the dimension of $L$. Hence, $\mathrm{b}_{\mathbf{q}}^i(L) = 0$ for $i > \dim L + 1$.

We will use the following three properties of $L_{\mathbf{q}}^2$–homology.

**Proposition 2.1**  (See [5, Section 15])

**The Mayer–Vietoris sequence**  *If $L = L_1 \cup L_2$ and $A = L_1 \cap L_2$, where $L_1$ and $L_2$ (and therefore, $A$) are full subcomplexes of $L$, then*

$$\to \mathfrak{h}_i^{\mathbf{q}}(A) \to \mathfrak{h}_i^{\mathbf{q}}(L_1) \oplus \mathfrak{h}_i^{\mathbf{q}}(L_2) \to \mathfrak{h}_i^{\mathbf{q}}(L) \to$$

*is weakly exact.*

**The Künneth Formula**  *The Betti numbers of the join of two complexes are given by:*

$$\mathrm{b}_{\mathbf{q}}^k(L_1 * L_2) = \sum_{i+j=k} \mathrm{b}_{\mathbf{q}}^i(L_1)\, \mathrm{b}_{\mathbf{q}}^j(L_2).$$

**Poincaré Duality**  *If $L$ is a flag $\mathrm{GHS}^{n-1}$, then $\mathrm{b}_{\mathbf{q}}^i(L) = \mathrm{b}_{\mathbf{q}^{-1}}^{n-i}(L)$.*

If $\sigma$ is a simplex in $L$, let $L_\sigma$ denote the link of $\sigma$ in $L$. To simplify notation we will write $bL_v$ instead of $(bL)_v$ to denote the link of the vertex $v$ in $bL$. Let $\mathcal{C}$ be a class of GHS's closed under the operation of taking link of vertices, i.e. if $S \in \mathcal{C}$ and $v$ is a vertex of $S$ then $S_v \in \mathcal{C}$. Following Section 15 of [5] we consider several variations of the Generalized Singer Conjecture for the class $\mathcal{C}$.

**I**$(n)$  If $S \in \mathcal{C}$ and $\dim S = n - 1$, then $\mathrm{b}_{\mathbf{q}}^i(S) = 0$ for $i > n/2$ and $\mathbf{q} \leq \mathbf{1}$.

**III$'$**$(2k + 1)$  Let $S \in \mathcal{C}$ and $\dim S = 2k$. Let $v$ be a vertex of $S$. Then the map $i_* \colon \mathfrak{h}_k^{\mathbf{q}}(S_v) \to \mathfrak{h}_k^{\mathbf{q}}(S)$, induced by the inclusion, is the zero homomorphism for $\mathbf{q} \geq \mathbf{1}$.

**V**$(n)$  Let $S \in \mathcal{C}$ and $\dim S = n - 1$. Let $A$ be a full subcomplex of $S$.

- If $n = 2k$ is even, then $\mathrm{b}_{\mathbf{q}}^i(S, A) = 0$ for all $i > k$ and $\mathbf{q} \leq \mathbf{1}$.
- If $n = 2k + 1$ is odd, then $\mathrm{b}_{\mathbf{q}}^i(A) = 0$ for all $i > k$ and $\mathbf{q} \leq \mathbf{1}$.

The argument in Section 16 of [5] goes through without changes if we consider only GHS's from a class $\mathcal{C}$ to give the following:

**Theorem 2.2**  (Compare [5, Section 16])  *If we only consider GHS's from a class $\mathcal{C}$, then the following implications hold.*

(1)  **I**$(2k + 1) \implies$ **III$'$**$(2k + 1)$.

(2)  **V**$(n) \implies$ **I**$(n)$.

(3)  **V**$(2k - 1) \implies$ **V**$(2k)$.

(4)  [**V**$(2k)$ and **III$'$**$(2k + 1)$] $\implies$ **V**$(2k + 1)$.

Let $\mathcal{JD}$ denote the class of finite joins of the barycentric subdivisions of the boundary complexes of standard simplices:

$$\mathcal{JD} = \{b\partial\Delta^{n_1} * \cdots * b\partial\Delta^{n_i}\}.$$

**Lemma 2.3** *The class $\mathcal{JD}$ is closed under the operation of taking link of vertices.*

**Proof** Let $S = b\partial\Delta^{n_1} * \cdots * b\partial\Delta^{n_j}$ and $v \in S$. We can assume that $v \in b\partial\Delta^{n_1}$. Then $S_v = b\partial\Delta_v^{n_1} * b\partial\Delta^{n_2} * \cdots * b\partial\Delta^{n_j} = b\partial\Delta^{\dim(v)} * b\partial\Delta^{n_1 - \dim(v) - 1} * b\partial\Delta^{n_2} * \cdots * b\partial\Delta^{n_j}$, and therefore $S \in \mathcal{JD}$.                                   $\square$

Next, consider the following statement:

**III''$(2k+1)$** *Let $v$ be a vertex of $b\partial\Delta^{2k+1}$. Then the map $i_* \colon \mathfrak{h}_k^{\mathbf{q}}(b\partial\Delta_v^{2k+1}) \to \mathfrak{h}_k^{\mathbf{q}}(b\partial\Delta^{2k+1})$, induced by the inclusion, is the zero homomorphism for $\mathbf{q} \geq \mathbf{1}$.*

**Lemma 2.4** **III''$(2k+1)$** $\implies$ **III'$(2k+1)$** *for the class $\mathcal{JD}$.*

**Proof** By induction, we can assume that the lemma holds for all odd numbers $< 2k+1$. Then it follows from the Theorem 2.2 that $\mathbf{V}(m)$ and therefore $\mathbf{I}(m)$ hold for all $m < 2k+1$.

Let $S = b\partial\Delta^{n_1} * \cdots * b\partial\Delta^{n_j}$ with $n_1 + \cdots + n_j = 2k+1$ and $v \in S$. We assume that $v \in b\partial\Delta^{n_1}$. Then $S_v = b\partial\Delta_v^{n_1} * b\partial\Delta^{n_2} * \cdots * b\partial\Delta^{n_j}$ and, by the Künneth formula, the map in question decomposes as the direct sum of maps of the form

$$(\mathfrak{h}_{k_1}^{\mathbf{q}}(b\partial\Delta_v^{n_1}) \to \mathfrak{h}_{k_1}^{\mathbf{q}}(b\partial\Delta^{n_1})) \otimes \bigotimes_{i=2}^{j}(\mathfrak{h}_{k_i}^{\mathbf{q}}(b\partial\Delta^{n_i}) \to \mathfrak{h}_{k_i}^{\mathbf{q}}(b\partial\Delta^{n_i}))$$

where $k_1 + \cdots + k_j = k$. Since $n_1 + \cdots + n_j = 2k+1$ it follows that $k_i < n_i/2$ for some index $i$. If $n_i < 2k+1$, then the range of the corresponding map in the above tensor product is 0 by $\mathbf{I}(n_i)$ and Poincaré duality, and therefore the tensor product map is 0. If $n_i = 2k+1$ then, in fact, $i = 1$ (the join is a trivial join) and the result follows from **III''$(2k+1)$**.                    $\square$

Thus, it follows from Theorem 2.2, Lemmas 2.3 and 2.4, and induction on dimension, that in order to prove the Generalized Singer Conjecture for the class $\mathcal{JD}$ all we need is to prove **III''$(2k+1)$**.

# 3 Removal of an odd-dimensional vertex

Let $L$ be a simplicial complex and $bL$ be its barycentric subdivision. The vertices of $bL$ are naturally graded by "dimension": each vertex $v$ of $bL$ is the barycenter of a unique cell (which we still denote $v$) of the complex $L$, and we call the dimension of this cell the *dimension* of the vertex $v$. Let $E_L$ denote the subcomplex of $bL$ spanned by the even dimensional vertices. Let $\mathcal{A}_L$ denote the set of full subcomplexes $A$ of $L$ containing $E_L$, which have the following property: if $A$ contains a vertex of odd dimension $2j + 1$, then $A$ contains all vertices of $bL$ of dimensions $\leq 2j$. In other words, any such $A$ can be obtained inductively from $bL$ by repeated removal of an odd-dimensional vertex of the highest dimension.

If $L = \partial \Delta^n$ we will use the notation $E_n = E_L$ and $\mathcal{A}_n = \mathcal{A}_L$.

**Lemma 3.1** *Assume* $\mathbf{III}''(2m + 1)$ *holds for* $2m + 1 < n$*. Then for any* $(n - 1)$*–dimensional simplicial complex* $L$ *and any complex* $A \in \mathcal{A}_L$ *we have:*

$$\mathrm{b}_{\mathbf{q}}^i(A) = \mathrm{b}_{\mathbf{q}}^i(bL) = 0 \text{ for } i > (n + 1)/2 \text{ and } \mathbf{q} \leq \mathbf{1}.$$

**Proof** By induction, we can assume that the lemma holds for all $m < n$. First, we claim that removal of odd-dimensional vertices does not change the homology above $(n+1)/2$. Let $A \in \mathcal{A}_L$ and let $B = A - v$ where $v$ is a vertex of the highest odd dimension of $A$. We let $\dim(v) = 2d - 1$, $1 \leq d \leq k$. We want to prove that $\mathrm{b}_{\mathbf{q}}^i(A) = \mathrm{b}_{\mathbf{q}}^i(B)$ for $i > (n + 1)/2$. Consider the Mayer–Vietoris sequence of the union $A = B \cup_{A_v} CA_v$:

$$\to \mathfrak{h}_i^{\mathbf{q}}(A_v) \to \mathfrak{h}_i^{\mathbf{q}}(B) \oplus \mathfrak{h}_i^{\mathbf{q}}(CA_v) \to \mathfrak{h}_i^{\mathbf{q}}(A) \to \mathfrak{h}_{i-1}^{\mathbf{q}}(A_v)$$

Suppose $i > (n + 1)/2$. Since $A_v = B \cap bL_v = B \cap (b\partial \Delta^{2d-1} * b(L_v))$, and since $B \in \mathcal{A}_L$, it follows, by construction, that $A_v$ splits as the join:

$$A_v = b\partial \Delta^{2d-1} * A_1,$$

with $A_1 \in \mathcal{A}_{(L_v)}$. By inductive assumption the lemma holds for $L_v$, i.e. $\mathrm{b}_{\mathbf{q}}^i(A_1) = \mathrm{b}_{\mathbf{q}}^i(b(L_v)) = 0$ for $i > (n + 1)/2 - d$.

Since $\mathbf{III}''(2d-1)$ holds by hypothesis, by Lemma 2.4 and Theorem 2.2, $\mathbf{I}(2d-1)$ holds for the class $\mathcal{JD}$, and, thus, $\mathrm{b}_{\mathbf{q}}^i(b\partial \Delta^{2d-1}) = 0$ for $i \geq d$.

Then, by the Künneth formula, $\mathrm{b}_{\mathbf{q}}^{i-1}(A_v) = 0$ for $i - 1 \geq (n + 1)/2$, i.e. for $i > (n + 1)/2$. By [5, Proposition 15.2(d)], $\mathrm{b}_{\mathbf{q}}^i(CA_v) = \frac{1}{q_v+1} \mathrm{b}_{\mathbf{q}}^i(A_v)$. Therefore in the above sequence the terms corresponding to $A_v$ and $CA_v$ are 0, and

the claim follows. Then it follows by induction, that $b_{\mathbf{q}}^i(A) = b_{\mathbf{q}}^i(bL)$ for all $A \in \mathcal{A}_L$ and $i > (n+1)/2$.

To prove the vanishing we note that, in particular, $b_{\mathbf{q}}^i(E_L) = b_{\mathbf{q}}^i(bL)$ for $i > (n+1)/2$. Since $E_L$ is spanned by the even-dimensional vertices of $bL$ and since a simplex in $bL$ has vertices of pairwise different dimensions, we have $\dim(E_L) = [(n+1)/2] - 1$. Therefore, $b_{\mathbf{q}}^i(E_L) = 0$ for $i > (n+1)/2$ and we have proved the lemma. $\square$

In the special case $L = \Delta^{2k+1}$ this lemma admits the following strengthening:

**Lemma 3.2** *Let* $n = 2k+1$. *Assume* $\mathbf{III}''(2m+1)$ *holds for* $2m+1 < n$. *Then for any complex* $A \in \mathcal{A}_n$, $A \subset b\partial\Delta^n$, *we have:*

$$b_{\mathbf{q}}^i(A) = b_{\mathbf{q}}^i(b\partial\Delta^n) \text{ for } i > k \text{ and } \mathbf{q} \leq \mathbf{1}.$$

**Proof** We proceed as in the previous proof. As before, we have $B = A - v$, $\dim(v) = 2d - 1$ and $A_v = b\partial\Delta^{2d-1} * A_1$, where now $A_1 \subset b\partial\Delta^{2k+1-2d}$. Therefore, the inductive assumption and the hypothesis on $\mathbf{III}''(2d-1)$ imply that $b_i(A_1) = 0$ for $i > k + d$. The lemma follows as before. $\square$

As explained in [6], when $\mathbf{q} = \mathbf{1}$, the removal of the odd-dimensional vertex does not change homology in *all* dimensions. We record this result below.

**Lemma 3.3** *Assume* $\mathbf{III}''(2m+1)$ *holds for* $2m+1 < n$ *and* $\mathbf{q} = \mathbf{1}$. *Then for any* $(n-1)$–*dimensional simplicial complex* $L$ *and for any complex* $A \in \mathcal{A}_L$, *obtained by the repeated removal of highest odd-dimensional vertices, we have:*

$$b^*(A) = b^*(bL).$$

**Proof** Again we repeat the proof of Lemma 3.1. As before, we have the splitting $A_v = b\partial\Delta^{2d-1} * A_1$. The point now is that for $\mathbf{q} = \mathbf{1}$, $\mathbf{I}(2d-1)$ and Poincaré duality imply $b^*(b\partial\Delta^{2d-1}) = 0$ and therefore $b^*(A_v) = 0$ by the Künneth formula. $\square$

# 4 Intersection form

**Lemma 4.1** *Let* $L$ *be a* $\mathrm{GHS}^{2k}$ *and let* $v$ *be a vertex of* $L$. *Then the image of the restriction map on* $L^2$–*cohomology* $i^*\colon L^2\mathcal{H}^k(\Sigma_L) \to L^2\mathcal{H}^k(\Sigma_{L_v})$ *is an isotropic subspace of the intersection form of* $\Sigma_{L_v}$.

**Proof** Note that the cup product of two $L^2$–cocycles is an $L^1$–cocycle. The intersection form is the result of evaluation of the cup product of two middle-dimensional cocycles on the fundamental class, which is $L^\infty$. Since $\Sigma_{L_v}$ bounds a half-space in $\Sigma_L$, $i_*([\Sigma_{L_v}]) = 0$ in $L^\infty$–homology of $\Sigma_L$. Thus, if $a, b \in L^2\mathcal{H}^n(\Sigma_L)$, then $\langle i^*(a) \cup i^*(b), [\Sigma_{L_v}] \rangle = \langle a \cup b, i_*([\Sigma_{L_v}]) \rangle = 0$. $\qquad\square$

**Lemma 4.2** *Let $G$ be a group and let $A$ be a bounded $G$–invariant (with respect to the diagonal action) non-degenerate bilinear form on a Hilbert submodule $M \subset \ell^2(G)$. Then $A$ has no nontrivial $G$–invariant isotropic subspaces.*

**Proof** Let us consider the case $M = \ell^2(G)$ first. $G$–invariance and continuity of the form $A$ implies that $A$ is completely determined by it values $a_g = (g \, A \, 1)$, $g \in G$. It is convenient to think of the form as given by $(x \, A \, y) = \langle x, Ay \rangle$, where $\langle \, , \, \rangle$ is the inner product and $A = \Sigma_{g \in G} a_g g$ is a bounded $G$–equivariant operator on $\ell^2(G)$. Non-degeneracy of $A$ means that $Ax = 0$ only if $x = 0$. $A$ is the limit of the group ring elements, and $Ax$ is the limit of the corresponding linear combinations of $G$–translates of $x$, i.e. $Ax = \lim \Sigma_{g \in G_n} a_g(gx)$, where $G_n$ is some exhaustion of $G$ by finite sets. It follows that if $x$ belongs to $G$–invariant isotropic subspace, then $Ax$ belongs to the closure of this subspace. Thus, we have $\langle Ax, Ax \rangle = (Ax \, A \, x) = 0$ by isotropy and continuity, therefore $x = 0$.

The case of general submodule $M \subset \ell^2(G)$ reduces to the above, since the bilinear form $A$ can be extended to $\ell^2(G)$, for example, by taking the orthogonal sum $A \oplus \langle \, , \, \rangle$ of $A$ on $M$ and the inner product on the orthogonal complement of $M$. $\qquad\square$

# 5 Vanishing theorems

Our main technical results are the following two theorems.

**Theorem 5.1** $\mathbf{III}''(2k+1)$ *is true for all $k > 0$ and $\mathbf{q} = \mathbf{1}$.*

**Proof** The proof is by induction on $k$. Suppose the theorem is true for all $m < k$.

Let $v$ be a vertex of $b\partial\Delta^{2k+1}$. We need to show that the restriction map $i^* \colon \mathfrak{h}^k(b\partial\Delta^{2k+1}) \to \mathfrak{h}^k(b\partial\Delta_v^{2k+1})$ is the 0–map.

First let us suppose that $v$ is a vertex of dimension 0, i.e. a vertex of $\Delta^{2k+1}$.

Consider the action of the symmetric group $\mathbf{S}_{2k+1}$ on $\Delta^{2k+1}$ which fixes the vertex $v$ and permutes other vertices. This action gives a simplicial action of $\mathbf{S}_{2k+1}$ on $b\partial\Delta^{2k+1}$ and therefore, after choosing a base point, lifts to a cubical action of $\mathbf{S}_{2k+1}$ on $\Sigma_{b\partial\Delta^{2k+1}}$ stabilizing $\Sigma_{b\partial\Delta^{2k+1}_v}$. Let $G'$ be the group of cubical automorphisms of $\Sigma_{b\partial\Delta^{2k+1}}$ generated by this action and the standard action of $W_{b\partial\Delta^{2k+1}}$, and let $G$ be the orientation-preserving subgroup of $G'$. Similarly, let $G'_v$ be the group of cubical automorphisms of $\Sigma_{b\partial\Delta^{2k+1}_v}$ generated by this action and the standard action of $W_{b\partial\Delta^{2k+1}_v}$, and let $G_v$ be the orientation-preserving subgroup of $G'_v$.

We claim that, as a Hilbert $G_v$–module $L^2\mathcal{H}^k(\Sigma_{b\partial\Delta^{2k+1}_v})$, is a submodule of $\ell^2(G_v)$. Note that $b\partial\Delta^{2k+1}_v$ is naturally isomorphic to $b\partial\Delta^{2k}$.

Using the inductive assumption and Lemma 3.3, we can remove from $b\partial\Delta^{2k}$ all odd-dimensional vertices without changing the $L^2$-cohomology: $\mathfrak{h}^*(E_{2k}) = \mathfrak{h}^*(b\partial\Delta^{2k})$. Since the action of $\mathbf{S}_{2k+1}$ on $b\partial\Delta^{2k+1}_v = b\partial\Delta^{2k}$ preserves the dimension of the vertices, we have isomorphism $L^2\mathcal{H}^*(G_v\Sigma_{E_{2k}}) = L^2\mathcal{H}^*(\Sigma_{b\partial\Delta^{2k}})$ as Hilbert $G_v$–modules.

The complex $E_{2k}$ is spanned by the even-dimensional vertices of $b\partial\Delta^{2k}$, which correspond to the proper subsets of vertices of $\Delta^{2k}$ of odd cardinality. Thus, the dimension of $E_{2k}$ is $k-1$, and its top-dimensional simplices are chains $v_0 < v_0v_1v_2 < ... < v_0...v_{2k-2}$ of length $k$ of distinct vertices of $\Delta^{2k}$. Therefore $\mathbf{S}_{2k+1}$ acts transitively on $(k-1)$–dimensional simplices of $E_{2k}$ and it follows that $G_v$ acts transitively on $k$–dimensional cubes of $G_v\Sigma_{E_{2k}}$. Therefore the space of $k$–cochains is a Hilbert $G_v$–submodule of $\ell^2(G_v)$, and the claim follows from the Hodge decomposition.

We have, by construction, $G_v = \mathrm{Stab}_G(\Sigma_{b\partial\Delta^{2k+1}_v})$. Then the restriction map $i^*\colon L^2\mathcal{H}^k(\Sigma_{b\partial\Delta^{2k+1}}) \to L^2\mathcal{H}^k(\Sigma_{b\partial\Delta^{2k+1}_v})$ is $G_v$–equivariant and therefore its image is a $G_v$–invariant subspace of $L^2\mathcal{H}^k(\Sigma_{b\partial\Delta^{2k+1}_v})$. Since $G_v$ acts preserving orientation, the intersection form is $G_v$–invariant. By Lemma 4.1 the image is isotropic, thus by Lemma 4.2 it is 0. Thus, the map $i^*\colon \mathfrak{h}^k(b\partial\Delta^{2k+1}) \to \mathfrak{h}^k(b\partial\Delta^{2k+1}_v) = L^2\mathcal{H}^k(W_{b\partial\Delta^{2k+1}}\Sigma_{b\partial\Delta^{2k+1}_v})$ is the 0–map.

For vertices of the other even dimensions the argument is similar. If $\dim(v) = 2d$, then its link is $b\partial\Delta^{2d} * b\partial\Delta^{2k-2d}$. Again, using Lemma 3.3, we remove, without changing the $L^2$–cohomology, all odd-dimensional vertices from each factor to obtain $E_{2d} * E_{2k-2d}$. The group $\mathbf{S}_{2d+1} \times \mathbf{S}_{2k-2d+1}$ acts naturally on $b\partial\Delta^{2k+1}$ fixing the vertex $v$ and stabilizing both the link and $E_{2d}*E_{2k-2d}$. This action is again transitive on the top-dimensional simplices of $E_{2d} * E_{2k-2d}$, and the rest of the argument goes through.

Finally, if $v$ is an odd-dimensional vertex, $\dim(v) = 2d+1$, then we have $b\partial\Delta_v^{2k+1} = b\partial\Delta^{2d+1} * b\partial\Delta^{2k-2d-1}$. The hypothesis on $\mathbf{III}''$ and Theorem 2.2 and Lemma 2.4 imply that both $\mathbf{I}(2d+1)$ and $\mathbf{I}(2k-2d-1)$ hold. Therefore, by the Künneth formula $\mathfrak{h}^k(b\partial\Delta_v^{2k+1}) = 0$ in this case. $\qquad\square$

**Theorem 5.2** *The Generalized Singer Conjecture holds true for $b\partial\Delta^{2k+1}$:*
$$\mathrm{b}_{\mathbf{q}}^i(b\partial\Delta^{2k+1}) = 0 \text{ for } i > k \text{ and } \mathbf{q} \leq \mathbf{1}.$$

**Proof** We proceed by induction on $k$. Using the inductive assumption and Lemma 3.2, we can remove all odd-dimensional vertices without changing the weighted $L_{\mathbf{q}}^2$–homology above $k$. Thus, since the remaining part $E_{2k+1}$ is $k$–dimensional, the problem reduces to showing that $\mathfrak{h}_{k+1}^{\mathbf{q}}(E_{2k+1}) = 0$ for $\mathbf{q} \leq \mathbf{1}$. Since $E_{2k+1}$ is $k$–dimensional, the natural map $\mathfrak{h}_{k+1}(E_{2k+1}) \to \mathfrak{h}_{k+1}^{\mathbf{q}}(E_{2k+1})$ is injective and the result follows from the Theorem 5.1. $\qquad\square$

Next, we list some consequences. Lemma 3.1 implies:

**Theorem 5.3** *Let $bL$ be the barycentric subdivision of an $(n-1)$–dimensional simplicial complex $L$. Then*
$$\mathrm{b}_{\mathbf{q}}^i(bL) = 0 \text{ for } i > (n+1)/2 \text{ and } \mathbf{q} \leq \mathbf{1}.$$

Taking $L$ to be a $\mathrm{GHS}^{2k-1}$, we obtain:

**Theorem 5.4** *The Generalized Singer Conjecture holds true for the barycentric subdivision of a $\mathrm{GHS}^{n-1}$ for all even $n$.*

For odd $n$ we obtain a weaker statement:

**Theorem 5.5** *Let $bL$ be the barycentric subdivision of a $\mathrm{GHS}^{2k}$. Then*
$$\mathrm{b}_{\mathbf{q}}^i(bL) = 0 \text{ for } i > k+1 \text{ and } \mathbf{q} \leq \mathbf{1}.$$
*In particular,*
$$\mathrm{b}^i(bL) = 0 \text{ for } i \neq k,\ k+1.$$

Specializing further, and combining with Theorem 5.2, we obtain:

**Theorem 5.6** *The Generalized Singer Conjecture holds true for $b\partial\Delta^n$:*
$$\mathrm{b}_{\mathbf{q}}^i(b\partial\Delta^n) = 0 \text{ for } i > n/2 \text{ and } \mathbf{q} \leq \mathbf{1},$$
*and, therefore, for the class $\mathcal{JD}$.*

Finally, let us mention an application of the above result to a more analytic object. Let $T_n$ denote the space of all symmetric tridiagonal $(n+1) \times (n+1)$–matrices with fixed generic eigenvalues, the so-called Tomei manifold. It is proved in [8] that $T_n$ is an $n$–dimensional closed aspherical manifold.

**Theorem 5.7** *The Singer Conjecture holds true for Tomei manifolds $T_n$.*

**Proof** The space $T_n$ can be identified with a natural finite index orbifoldal cover of $\Sigma_{b\partial\Delta^n}/W_{b\partial\Delta^n}$ [3]. Thus $\Sigma_{b\partial\Delta^n}$ is the universal cover of $T^n$, and the claim follows from the previous theorem. $\square$

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