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Nondiffeomorphic Symplectic 4–Manifolds with the same Seiberg–Witten Invariants

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Abstract The goal of this paper is to demonstrate that, at least for nonsimply connected 4–manifolds, the Seiberg–Witten invariant alone does not determine diffeomorphism type within the same homeomorphism type.

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Dedicated to Robion C Kirby on the occasion of his 60^{th} birthday

1 Introduction

The goal of this paper is to demonstrate that, at least for nonsimply connected 4–manifolds, the Seiberg–Witten invariant alone does not determine diffeomorphism type within the same homeomorphism type. The first examples which demonstrate this phenomenon were constructed by Shuguang Wang [13]. These are examples of two homeomorphic 4–manifolds with $\pi_1 = \mathbf{Z}_2$ and trivial Seiberg–Witten invariants. One of these manifolds is irreducible and the other splits as a connected sum. It is our goal here to exhibit examples among symplectic 4–manifolds, where the Seiberg–Witten invariants are known to be nontrivial. We shall construct symplectic 4–manifolds with $\pi_1 = \mathbf{Z}_p$ which have the same nontrivial Seiberg–Witten invariant but whose universal covers have different Seiberg–Witten invariants. Thus, at the very least, in order to determine diffeomorphism type, one needs to consider the Seiberg–Witten invariants of finite covers.

Recall that the Seiberg–Witten invariant of a smooth closed oriented 4–manifold X with $b_2^+(X) > 1$ is an integer-valued function which is defined on the set of $spin^c$ structures over X (cf [14]). In case $H_1(X,\mathbf{Z})$ has no 2–torsion there is a natural identification of the $spin^c$ structures of X with the characteristic elements of $H_2(X,\mathbf{Z})$ (ie, those elements k whose Poincaré duals $\hat{k}$ reduce mod 2 to $w_2(X)$). In this case we view the Seiberg–Witten invariant as

$$\mathrm{SW}_X\colon \{k \in H_2(X,\mathbf{Z}) | \hat{k} \equiv w_2(TX) \pmod 2)\} \to \mathbf{Z}.$$

The sign of $\mathcal{SW}_X$ depends on an orientation of $H^0(X,\mathbf{R}) \otimes \det H^2_+(X,\mathbf{R}) \otimes \det H^1(X,\mathbf{R})$. If $\mathrm{SW}_X(\beta) \neq 0$, then β is called a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. Furthermore, if β is a basic class, then so is $-\beta$ with $\mathrm{SW}_X(-\beta) = (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\,\mathrm{SW}_X(\beta)$ where $\mathrm{e}(X)$ is the Euler number and $\mathrm{sign}(X)$ is the signature of X.

Now let $\{\pm\beta_1,\dots,\pm\beta_n\}$ be the set of nonzero basic classes for X. Consider variables $t_\beta = \exp(\beta)$ for each $\beta \in H^2(X;\mathbf{Z})$ which satisfy the relations $t_{\alpha+\beta} = t_\alpha t_\beta$. We may then view the Seiberg–Witten invariant of X as the Laurent polynomial

$$\mathcal{SW}_X = \mathrm{SW}_X(0) + \sum_{j=1}^{n} \mathrm{SW}_X(\beta_j)\cdot(t_{\beta_j} + (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\,t_{\beta_j}^{-1}).$$

2 The Knot and Link Surgery Construction

We shall need the knot surgery construction of [3]: Suppose that we are given a smooth simply connected oriented 4–manifold X with $b^+ > 1$ containing an essential smoothly embedded torus T of self-intersection 0. Suppose further that $\pi_1(X\backslash T) = 1$ and that T is contained in a cusp neighborhood. Let $K \subset S^3$ be a smooth knot and M_K the 3–manifold obtained from 0–framed surgery on K. The meridional loop m to K defines a 1–dimensional homology class $[m]$ both in $S^3 \setminus K$ and in M_K. Denote by T_m the torus $S^1 \times m \subset S^1 \times M_K$. Then X_K is defined to be the fiber sum

$$X_K = X \#_{T=T_m} S^1 \times M_K = (X \setminus N(T)) \cup (S^1 \times (S^3 \setminus N(K)),$$

where $N(T) \cong D^2 \times T^2$ is a tubular neighborhood of T in X and $N(K)$ is a neighborhood of K in S^3. If λ denotes the longitude of K (λ bounds a surface in $S^3 \setminus K$) then the gluing of this fiber sum identifies $\{\mathrm{pt}\} \times \lambda$ with a normal circle to T in X. The main theorem of [3] is:

Theorem [3] *With the assumptions above, X_K is homeomorphic to X, and*

$$\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t)$$

where Δ_K is the symmetrized Alexander polynomial of K and $t = \exp(2[T])$.

In case the knot K is fibered, the 3–manifold M_K is a surface bundle over the circle; hence $S^1 \times M_K$ is a surface bundle over T^2. It follows from [12] that $S^1 \times M_K$ admits a symplectic structure and T_m is a symplectic submanifold. Hence, if $T \subset X$ is a torus satisfying the conditions above, and if in addition X is a symplectic 4–manifold and T is a symplectic submanifold, then the fiber sum $X_K = X\#_{T=T_m} S^1 \times M_K$ carries a symplectic structure [4]. Since K is a fibered knot, its Alexander polynomial is the characteristic polynomial of its monodromy φ; in particular, $M_K = S^1 \times_\varphi \Sigma$ for some surface Σ and $\Delta_K(t) = \det(\varphi_* - tI)$, where φ_* is the induced map on H_1.

There is a generalization of the above theorem in this case due to Ionel and Parker [7] and to Lorek [8].

Theorem [7, 8] *Let X be a symplectic 4–manifold with $b^+ > 1$, and let T be a symplectic self-intersection 0 torus in X which is contained in a cusp neighborhood. Also, let Σ be a symplectic 2–manifold with a symplectomorphism $\varphi\colon \Sigma \to \Sigma$ which has a fixed point $\varphi(x_0) = x_0$. Let $m_0 = S^1 \times_\varphi \{x_0\}$ and $T_0 = S^1 \times m_0 \subset S^1 \times (S^1 \times_\varphi \Sigma)$. Then $X_\varphi = X\#_{T=T_0} S^1 \times (S^1 \times_\varphi \Sigma)$ is a symplectic manifold whose Seiberg–Witten invariant is*

$$\mathcal{SW}_{X_\varphi} = \mathcal{SW}_X \cdot \Delta(t)$$

where $t = \exp(2[T])$ and $\Delta(t)$ is the obvious symmetrization of $\det(\varphi_* - tI)$.

Note that in case K is a fibered knot and $M_K = S^1 \times_\varphi \Sigma$, Moser's theorem [9] guarantees that the monodromy map φ can be chosen to be a symplectomorphism with a fixed point.

There is a related link surgery construction which starts with an oriented n–component link $L = \{K_1, \dots, K_n\}$ in S^3 and n pairs (X_i, T_i) of smoothly embedded self-intersection 0 tori in simply connected 4–manifolds as above. Let

$$\alpha_L\colon\ \pi_1(S^3 \setminus L) \to \mathbf{Z}$$

denote the homomorphism characterized by the property that it send the meridian m_i of each component K_i to 1. Let $N(L)$ be a tubular neighborhood of L. Then if ℓ_i denotes the longitude of the component K_i, the curves $\gamma_i = \ell_i + \alpha_L(\ell_i) m_i$ on $\partial N(L)$ given by the $\alpha_L(\ell_i)$ framing of K_i form the boundary of a Seifert surface for the link. In $S^1 \times (S^3 \setminus N(L))$ let $T_{m_i} = S^1 \times m_i$ and define the 4–manifold $X(X_1, \dots X_n; L)$ by

$$X(X_1, \dots X_n; L) = (S^1 \times (S^3 \setminus N(L)) \cup \bigcup_{i=1}^n (X_i \setminus (T_i \times D^2))$$

where $S^1 \times \partial N(K_i)$ is identified with $\partial N(T_i)$ so that for each i:

$$[T_{m_i}] = [T_i], \quad \text{and} \quad [\gamma_i] = [\text{pt} \times \partial D^2].$$

Theorem [3] *If each T_i is homologically essential and contained in a cusp neighborhood in X_i and if each $\pi_1(X \setminus T_i) = 1$, then $X(X_1, \dots X_n; L)$ is simply connected and its Seiberg–Witten invariant is*

$$\mathcal{SW}_{X(X_1,\dots X_n;L)} = \Delta_L(t_1, \dots, t_n) \cdot \prod_{j=1}^{n} \mathcal{SW}_{E(1)\#_{F=T_j}X_j}$$

where $t_j = \exp(2[T_j])$ and $\Delta_L(t_1, \dots, t_n)$ is the symmetric multivariable Alexander polynomial.

3 2–bridge knots

Recall that 2–bridge knots, K, are classified by the double covers of S^3 branched over K, which are lens spaces. Let $K(p/q)$ denote the 2–bridge knot whose double branched cover is the lens space $L(p,q)$. Here, p is odd and q is relatively prime to p. Notice that $L(p,q) \cong L(p, q-p)$; so we may assume at will that either q is even or odd. We are first interested in finding a pair of distinct fibered 2–bridge knots $K(p/q_i)$, $i = 1,2$ with the same Alexander polynomial. Since 2–bridge knots are alternating, they are fibered if and only if their Alexander polynomials are monic [2]. There is a simple combinatorial scheme for calculating the Alexander polynomial of a 2–bridge knot $K(p/q)$; it is described as follows in [10]. Assume that q is even and let $\mathbf{b}(p/q) = (b_1, \dots, b_n)$ where p/q is written as a continued fraction:

$$\frac{p}{q} = 2b_1 + \cfrac{1}{-2b_2 + \cfrac{1}{2b_3 + \cfrac{1}{\ddots + \cfrac{1}{\pm 2b_n}}}}$$

There is then a Seifert surface for $K(p/q)$ whose corresponding Seifert matrix is:

$$V(p/q) = \begin{pmatrix} b_1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & b_2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & b_3 & 0 & 0 & \cdots \\ 0 & 0 & 1 & b_4 & 1 & \cdots \\ .. & .. & .. & .. & .. & \cdots \end{pmatrix}$$

Thus the Alexander polynomial for $K(p/q)$ is

$$\Delta_{K(p/q)}(t) = \det(t \cdot V(p/q) - V(p/q)^{\mathrm{tr}}).$$

Using this technique we calculate:

Proposition 3.1 *The 2–bridge knots $K(105/64)$ and $K(105/76)$ share the Alexander polynomial*

$$\Delta(t) = t^4 - 5t^3 + 13t^2 - 21t + 25 - 21t^{-1} + 13t^{-2} - 5t^{-3} + t^{-4}.$$

In particular, these knots are fibered.

Proof The knots $K(105/64)$ and $K(105/76)$ correspond to the vectors

$$\mathbf{b}(105/64) = (1, 1, -1, -1, -1, -1, 1, 1)$$
$$\mathbf{b}(105/76) = (1, 1, 1, -1, -1, 1, 1, 1).$$ □

4 The examples

Consider any pair of inequivalent fibered 2–bridge knots $K_i = K(p/q_i)$, $i = 1, 2$, with the same Alexander polynomial $\Delta(t)$. Let $\tilde{K}_i = \pi_i^{-1}(K_i)$ denote the branch knot in the 2–fold branched covering space $\pi_i\colon L(p, q_i) \to S^3$, and let $\tilde{m}_i = \pi_i^{-1}(m_i)$, with m_i the meridian of K_i. Then $M_{K_i} = S^1 \times_{\varphi_i} \Sigma$ with double cover $\tilde{M}_{K_i} = S^1 \times_{\varphi_i^2} \Sigma$.

Let X be the K3–surface and let F denote a smooth torus of self-intersection 0 which is a fiber of an elliptic fibration on X. Our examples are

$$X_{K_i} = X \#_{F = T_{\tilde{m}_i}} (S^1 \times \tilde{M}_{K_i}).$$

The gluing is chosen so that the boundary of a normal disk to F is matched with the lift $\tilde{\ell}_i$ of a longitude to K_i. A simple calculation and our above discussion implies that X_{K_1} and X_{K_2} are homeomorphic [5] and have the same Seiberg–Witten invariant:

Theorem 4.1 *The manifolds X_{K_i} are homeomorphic symplectic rational homology K3–surfaces with fundamental groups $\pi_1(X_{K_i}) = \mathbf{Z}_p$. Their Seiberg–Witten invariants are*

$$\mathcal{SW}_{X_{K_i}} = \det(\varphi_{i,*}^2 - \tau^2 I) = \Delta(\tau) \cdot \Delta(-\tau)$$

where $\tau = \exp([F])$.

5 Their universal covers

The purpose of this final section is to prove our main theorem.

Theorem 5.1 *$X_{K(105/64)}$ and $X_{K(105/76)}$ are homeomorphic but not diffeomorphic symplectic 4–manifolds with the same Seiberg–Witten invariant.*

Let $K_1 = K(105/64)$ and $K_2 = K(105/76)$. We have already shown that X_{K_1} and X_{K_2} are homeomorphic symplectic 4–manifolds with the same Seiberg–Witten invariant. Suppose that $f\colon X_{K_1} \to X_{K_2}$ is a diffeomorphism. It then satisfies $f_*(\mathcal{SW}_{X_{K_1}}) = \mathcal{SW}_{X_{K_2}}$. Since these are both Laurent polynomials in the single variable $\tau = \exp([F])$, and $[F] = [T_{\tilde{m}_i}]$ in X_{K_i}, after appropriately orienting $T_{\tilde{m}_2}$, we must have

$$f_*[T_{\tilde{m}_1}] = [T_{\tilde{m}_2}].$$

We study the induced diffeomorphism $\hat{f}\colon \hat{X}_{K_1} \to \hat{X}_{K_2}$ of universal covers. The universal cover $\hat{X}_{K_i}$ of X_{K_i} is obtained as follows. Let $\vartheta_i\colon S^3 \to L(p,q_i)$ be the universal covering ($p = 105$, $q_1 = 64$, $q_2 = 76$) which induces the universal covering $\hat{\vartheta}_i\colon \hat{X}_{K_i} \to X_{K_i}$, and let $\hat{L}_i$ be the p–component link $\hat{L}_i = \vartheta_i^{-1}(\tilde{K}_i)$. The composition of the maps $\varphi \circ \vartheta_i\colon S^3 \to S^3$ is a dihedral covering space branched over K_i, and the link $\hat{L}_i = \hat{L}(p/q_i)$ is classically known as the 'dihedral covering link' of $K(p/q_i)$. This is a symmetric link, and in fact, the deck transformations $\tau_{i,k}$ of the cover $\vartheta_i\colon S^3 \to L(p,q_i)$ permute the link components. The collection of linking numbers of $\hat{L}_i$ (the dihedral linking numbers of $K(p/q_i)$) classify the 2–bridge knots [2]. The universal cover $\hat{X}_{K_i}$ is obtained via the construction $\hat{X}_{K_i} = X(X_1, \dots X_p; L_i)$ of section 2, where each $(X_i, T_i) = (K3, F)$. Hence it follows from section 2 that

$$\mathcal{SW}_{\hat{X}_{K_i}} = \Delta_{\hat{L}_i}(t_{i,1}, \dots, t_{i,p}) \cdot \prod_{j=1}^{p} \mathcal{SW}_{E(1)\#_F K3} = \Delta_{\hat{L}_i}(t_{i,1}, \dots, t_{i,p}) \cdot \prod_{j=1}^{p} (t_{i,j}^{1/2} - t_{i,j}^{-1/2})$$

where $t_{i,j} = \exp([2T_{i,j}])$ and $T_{i,j}$ is the fiber F in the jth copy of K3. Let $L_{i,1}, \dots, L_{i,p}$ denote the components of the covering link $\hat{L}_i$ in S^3, and let $m_{i,j}$ denote a meridian to $L_{i,j}$. Then $[T_{i,j}] = [S^1 \times m_{i,j}]$ in $H_2(\hat{X}_{K_i}; \mathbf{Z})$, and so $\hat{\vartheta}_{i*}[T_{i,j}] = [T_i]$.

Now we have $\hat{f}_*(\mathcal{SW}_{\hat{X}_{K_1}}) = \mathcal{SW}_{\hat{X}_{K_2}}$ as elements of the integral group ring of $H_2(\hat{X}_{K_2}; \mathbf{Z})$. The formula given for $\mathcal{SW}_{\hat{X}_{K_i}}$ shows that each basic class may be written in the form $\beta = \sum_{j=1}^{p} a_j[T_{i,j}]$. Thus if β is a basic class of $\hat{X}_{K_1}$, then

$$\hat{f}_*(\beta) = \hat{f}_*(\sum_{j=1}^{p} a_j[T_{1,j}]) = \sum_{j=1}^{p} b_j[T_{2,j}]$$

for some integers, $b_1, \dots, b_p$. But since $f_*[T_1] = [T_2]$ in $H_2(X_{K_2}; \mathbf{Z})$ we have

$$(\sum_{j=1}^{p} a_j)[T_2] = f_*(\sum_{j=1}^{p} a_j[T_1]) = f_*\hat{\vartheta}_{1*}(\beta) = \hat{\vartheta}_{2*}\hat{f}_*(\beta) = \hat{\vartheta}_{2*}(\sum_{j=1}^{p} b_j[T_{2,j}]) = \sum_{j=1}^{p} b_j[T_2].$$

Hence $\sum_{j=1}^{p} a_j = \sum_{j=1}^{p} b_j$.

Form the 1–variable Laurent polynomials $P_i(t) = \Delta_{\hat{L}_i}(t, \dots, t) \cdot (t^{1/2} - t^{-1/2})^p$ by equating all the variables $t_{i,j}$ in $\mathcal{SW}_{\hat{X}_{K_i}}$. The coefficient of a fixed term t^k in $P_i(t)$ is

$$\sum \{ \mathrm{SW}_{\hat{X}_{K_i}}(\sum_{j=1}^{p} a_j[T_{i,j}]) \mid \sum_{j=1}^{p} a_j = k \}.$$

Our argument above (and the invariance of the Seiberg–Witten invariant under diffeomorphisms) shows that $\hat{f}_*$ takes $P_1(t)$ to $P_2(t)$; ie, $P_1(t) = P_2(t)$ as Laurent polynomials.

The reduced Alexander polynomials $\Delta_{\hat{L}_i}(t, \dots, t)$ have the form

$$\Delta_{\hat{L}_i}(t, \dots, t) = (t^{1/2} - t^{-1/2})^{p-2} \cdot \nabla_{\hat{L}_i}(t),$$

where the polynomial $\nabla_{\hat{L}_i}(t)$ is called the Hosokawa polynomial [6]. Consider the matrix:

$$\Lambda(p/q) = \begin{pmatrix} \sigma & \varepsilon_1 & \cdots & \varepsilon_{p-1} \\ \varepsilon_{p-1} & \sigma & \cdots & \varepsilon_{p-2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \varepsilon_1 & \varepsilon_2 & \cdots & \sigma \end{pmatrix}$$

(Burde has shown that this is the linking matrix of $\hat{L}(p/q)$.)

It is a theorem of Hosokawa [6] that $\nabla_{\hat{L}(p/q)}(1)$ can be calculated as the determinant of any $(p-1)$ by $(p-1)$ minor $\Lambda'(p/q)$ of $\Lambda(p/q)$. In particular, we have the following Mathematica calculations. (Note that $K(105/64) = K(105/-41)$ and $K(105/76) = K(105/-29)$.)

$$\det(\Lambda'(105/-41))/105 = 13^2 \cdot 61^2 \cdot 127^2 \cdot 463^2 \cdot 631^4 \cdot 1358281^4$$
$$\det(\Lambda'(105/-29))/105 = 139^4 \cdot 211^4 \cdot 491^2 \cdot 8761^2 \cdot 10005451^4.$$

This means that $\nabla_{\hat{L_1}}(1) \neq \nabla_{\hat{L_2}}(1)$. However, if we let $Q(t) = (t^{1/2} - t^{-1/2})^{2p-2}$, then $P_i(t) = \nabla_{\hat{L_i}}(t) \cdot Q(t)$. For $|u-1|$ small enough, $P_1(u)/Q(u) \neq P_2(u)/Q(u)$. Hence for $u \neq 1$ in this range, $P_1(u) \neq P_2(u)$. This contradicts the existence of the diffeomorphism f and completes the proof of Theorem 5.1.

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