ISSN 1464-8997 (on line) 1464-8989 (printed) **303**

Geometry & Topology Monographs
Volume 4: Invariants of knots and 3-manifolds (Kyoto 2001)
Pages 303–311

On the potential functions for the hyperbolic structures of a knot complement

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Abstract We explain how to construct certain potential functions for the hyperbolic structures of a knot complement, which are closely related to the analytic functions on the deformation space of hyperbolic structures.

AMS Classification 57M50; 57M25, 57M27

Keywords Potential function, hyperbolicity equations, volume, Chern–Simons invariant

Dedicated to Professor Mitsuyoshi Kato for his 60th birthday

1 Introduction

Let M be the complement of a hyperbolic knot K in S^3. Through the study of *Kashaev's conjecture*, we have found a complex function which gives the *volume* and the *Chern–Simons invariant* of the complete hyperbolic structure of M at the critical point corresponding to the promised solution to the hyperbolicity equations for M, see [2, 4] for details.

The purpose of this article is to explain how to construct such complex functions for the non-complete hyperbolic structures of M. Such functions are closely related to the analytic functions on the deformation space of the hyperbolic structures of M, parametrized by the eigenvalue of the holonomy representation of the meridian of K, which reveal a complex-analytic relation between the volumes and the Chern–Simons invariants of the hyperbolic structures of M, see [3, 5] for details.

In this note, we suppose K is 5_2 for simplicity which is represented by the diagram D depicted in Figure 1.

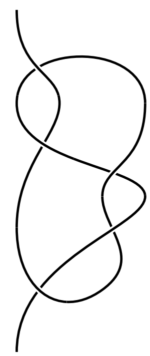

Figure 1

2 Geometry of a knot complement

2.1 Ideal triangulations

We first review an ideal triangulation of M due to D. Thurston. Let $\dot{M}$ denote M with two poles $\pm\infty$ of S^3 removed. Then, $\dot{M}$ decomposes into 5 ideal octahedra corresponding to the 5 crossings of D, each of which further decomposes into 4 ideal tetrahedra around an axis, as shown in Figure 2.

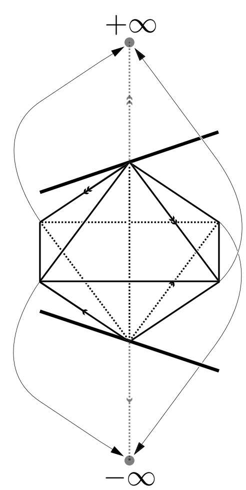

Figure 2

In fact, we can reocover $\dot{M}$ by glueing adjacent tetrahedra as shown in Figure 3.

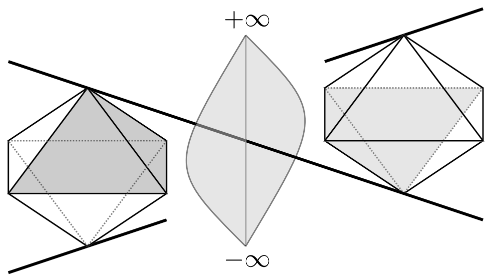

Figure 3

As usual, we put a hyperbolic structure on each tetrahedron by assigning a complex number, called *modulus*, to the edge corresponding to the axis as shown in Figure 4. In what follows, we denote the tetrahedron with modulus z by $T(z)$.

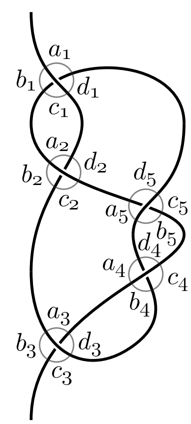

Figure 4

Let B be the intersection between $T(a_1) \cup T(b_1)$ and $T(b_3) \cup T(c_3)$. Then, each of

$$T(a_1), T(b_1), T(b_3), T(c_3)$$

intersects $\partial N(B \cup K)$ in two triangles, and they are essentially one-dimensional objects in $S^3 \setminus N(B \cup K)$. On the other hand, each of

$$T(c_1), T(d_1), T(a_2), T(b_2), T(d_2), T(a_3), T(d_3), T(a_4), T(b_4), T(c_4), T(c_5)$$

intersects $\partial N(B \cup K)$ in two triangles and one quadrangle, and they are essentially two-dimensional objects in $S^3 \setminus N(B \cup K)$. Thus, by contracting these

15 tetrahedra, we obtain an ideal triangulation $\mathcal{S}$ of M with

$$T(c_2), T(d_4), T(a_5), T(b_5), T(d_5).$$

Figure 5 exhibits the triangulation of $\partial N(B \cup K)$ induced by $\mathcal{S}$, where each couple of edges labeled with the same number are identified.

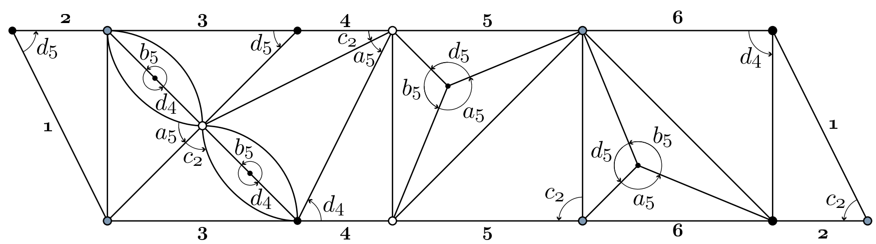

Figure 5

2.2 Hyperbolicity equations

If c_2, d_4, a_5, b_5, d_5 above give a hyperbolic structure of M, the product of the moduli around each edge in $\mathcal{S}$ should be 1, which is called the *hyperbolicity equations* and can be read from Figure 5 as follows.

$$\begin{gathered}
d_4 b_5 = a_5 b_5 d_5 = 1, \\
\frac{c_2 a_5 (1-1/d_4)}{1-d_4} \cdot \frac{(1-1/d_5)(1-1/c_2)(1-1/b_5)}{(1-a_5)(1-b_5)} = 1, \\
\frac{c_2(1-1/a_5)}{(1-d_5)(1-c_2)} \cdot \frac{(1-1/d_5)(1-1/b_5)}{(1-d_5)(1-a_5)(1-d_4)} = 1, \\
\frac{d_4(1-1/a_5)(1-1/d_4)}{1-b_5} \cdot \frac{d_5(1-1/c_2)}{1-c_2} = 1.
\end{gathered}$$

It is easy to observe that these equations are generated by

$$d_4 b_5 = a_5 b_5 d_5 = 1, \ \frac{c_2 a_5}{d_4} = \frac{1-1/a_5}{(1-1/c_2)(1-d_5)} = \frac{(1-1/a_5)(1-d_4)}{1-b_5} = \frac{c_2}{d_5},$$

which suggests to put

$$c_2 = y\xi, \ d_4 = x/\xi, \ a_5 = x/y, \ b_5 = \xi/x, \ d_5 = y/\xi$$

and to rewrite the hyperbolicity equations as follows.

$$\frac{(1-y/x)(1-x/\xi)}{1-\xi/x} = \frac{1-y/x}{(1-y/\xi)(1-1/y\xi)} = \xi^2.$$

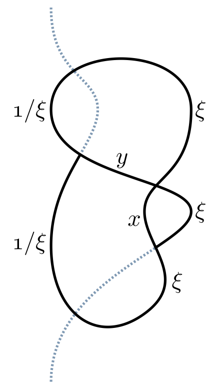

Figure 6

Note that the variables x, y correspond to the interior edges of a graph depicted in Figure 6, which is D with some edges deleted.

A solution to the equations above determines a hyperbolic structure of M, where ξ is nothing but the eigenvalue of the holonomy representation of the meridian of K. The set $\mathcal{D}$ of such solutions is called the *deformation space* of the hyperbolic structures of M and can be parametrized by ξ or the eigenvalue η of the holonomy representation of the longitude of K. In our example, η is given by

$$\eta = \frac{y\xi^6}{x} \cdot (1 - 1/y\xi) = \frac{y\xi^6}{x} \cdot \frac{1 - \xi/x}{(1 - x/\xi)(1 - y/\xi)}.$$

Note that the factors $1 - x/\xi, 1 - y/\xi, 1 - \xi/x$ and $1 - 1/y\xi$ correspond to the corners of D which touch the unbounded regions.

3 Potential functions

Curious to say, we can always construct a *potential function* for the hyperbolicity equations and η combinatorially by using Euler's dilogarithm function

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1 - w)}{w} dw,$$

where we remark that the volume of a tetrahedron with modulus z is given by

$$D(z) = \operatorname{Im}\mathrm{Li}_2(z) + \log|z| \arg(1 - z).$$

3.1 Neumann–Zagier's functions

In fact, we define $V(x, y, \xi)$ by

$$-\mathrm{Li}_2(1/y\xi) + \mathrm{Li}_2(y/\xi) - \mathrm{Li}_2(y/x) + \mathrm{Li}_2(\xi/x) + \mathrm{Li}_2(x/\xi) + \log\xi \log\frac{x^2}{y^2\xi^6} - \frac{\pi^2}{6},$$

the principal part of which is nothing but the sum of dilogarithm functions associated to the corners of the graph as shown in Figure 7.

Figure 7

Then, we have

$$x\frac{\partial V}{\partial x} = \log\frac{\xi^2(1-\xi/x)}{(1-y/x)(1-x/\xi)}, \quad y\frac{\partial V}{\partial y} = \log\frac{1-y/x}{\xi^2(1-y/\xi)(1-1/y\xi)},$$

both of which vanish on $\mathcal{D}$, and

$$\begin{aligned} \xi\frac{\partial V}{\partial \xi} &= \log\frac{x^2(1-x/\xi)(1-y/\xi)}{y^2\xi^{12}(1-\xi/x)(1-1/y\xi)} \\ &= \log\left\{\frac{x}{y\xi^6}\cdot\frac{1}{1-1/y\xi}\right\}^2 - x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y} \\ &= \log\left\{\frac{x}{y\xi^6}\cdot\frac{(1-x/\xi)(1-y/\xi)}{1-\xi/x}\right\}^2 + x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}, \end{aligned}$$

that is,

$$\xi\frac{\partial V}{\partial \xi} = -\log\eta^2$$

on $\mathcal{D}$, which shows $V(x, y, e^u)$ coincides with $\Phi(u)$ given in [3, Theorem 3].

3.2 Dehn fillings

Furthermore, for a *slope* $\alpha \in \mathbb{Q}$, we put

$$V_\alpha(x, y, \xi) = V(x, y, \xi) + \frac{\log\xi(2\pi\sqrt{-1} - p\log\xi)}{q},$$

where $p, q \in \mathbb{Z}$ denote the numerator and the denominator of α. Then, we have

$$\xi \frac{\partial V_\alpha}{\partial \xi} = \xi \frac{\partial V}{\partial \xi} + \frac{2\pi\sqrt{-1} - p \log \xi^2}{q} = \frac{2\pi\sqrt{-1} - p \log \xi^2 - q \log \eta^2}{q},$$

and so a solution $(x_\alpha, y_\alpha, \xi_\alpha)$ to the equations

$$dV_\alpha(x, y, \xi) = 0$$

determines the complete hyperbolic structure of the closed 3-manifold M_α obtained from M by α Dehn filling. Note that, by choosing $r, s \in \mathbb{Z}$ such that $ps - qr = 1$, we can compute the logarithm of the eigenvalue of the holonomy representation of the core geodesic γ_α of M_α which is related to the *length* and the *torsion* of γ_α as follows, see [3, Lemma 4.2].

$$\log \xi^r \eta^s = \frac{s\pi\sqrt{-1} - \log \xi}{q} = \frac{\mathrm{length}(\gamma_\alpha) + \sqrt{-1} \cdot \mathrm{torsion}(\gamma_\alpha)}{2}.$$

Volumes and Chern–Simons invariants

3.3 Yoshida's functions

As in [4], we can observe

$$\begin{aligned}\operatorname{Im} V_\alpha(x, y, \xi) \;&=\; -D(1/y\xi) + D(y/\xi) - D(y/x) + D(\xi/x) + D(x/\xi) \\ &\quad + \log|x| \cdot \operatorname{Im} x \frac{\partial V_\alpha}{\partial x} + \log|y| \cdot \operatorname{Im} y \frac{\partial V_\alpha}{\partial y} + \log|\xi| \cdot \operatorname{Im} \xi \frac{\partial V_\alpha}{\partial \xi},\end{aligned}$$

and so

$$\operatorname{Im} V_\alpha(x_\alpha, y_\alpha, \xi_\alpha) = \mathrm{vol}(M_\alpha).$$

To detect $\operatorname{Re} V_\alpha(x_\alpha, y_\alpha, \xi_\alpha)$, we shall consider

$$R(x, y, \xi) = -R(1/y\xi) + R(y/\xi) - R(y/x) + R(\xi/x) + R(x/\xi) - \frac{\pi^2}{6},$$

where $R(z)$ denotes Roger's dilogarithm function defined by

$$R(z) = \mathrm{Li}_2(z) + \log z \log(1 - z)/2.$$

Then, $R(x, y, \xi)$ can be expressed as

$$\begin{aligned}&-\mathrm{Li}_2(1/y\xi) + \mathrm{Li}_2(y/\xi) - \mathrm{Li}_2(y/x) + \mathrm{Li}_2(\xi/x) + \mathrm{Li}_2(x/\xi) \\ &-\frac{\log x}{2}\left(x\frac{\partial V}{\partial x} - \log \xi^2\right) - \frac{\log y}{2}\left(y\frac{\partial V}{\partial y} + \log \xi^2\right) - \frac{\log \xi}{2}\left(\xi \frac{\partial V}{\partial \xi} - \log \frac{x^2}{y^2 \xi^{12}}\right),\end{aligned}$$

and so $R(x, y, \xi)$ agrees with

$$V(x, y; \xi) + \log \xi \, \log \eta$$

on $\mathcal{D}$ and with

$$V_\alpha(x,y,\xi) - \frac{\log\xi(2\pi\sqrt{-1} - p\log\xi)}{q} + \log\xi\ \log\eta = V_\alpha(x,y,\xi) - \frac{\pi\sqrt{-1}\cdot\log\xi}{q}$$

at $(x_\alpha, y_\alpha, \xi_\alpha) \in \mathcal{D}$. Therefore, we have

$$\begin{aligned} R(x_\alpha,y_\alpha,\xi_\alpha) &= V_\alpha(x_\alpha,y_\alpha,\xi_\alpha) - \frac{s\pi^2 + \pi\sqrt{-1}\cdot\log\xi_\alpha}{q} + \frac{s\pi^2}{q} \\ &= V_\alpha(x_\alpha,y_\alpha,\xi_\alpha) + \frac{\pi\sqrt{-1}}{2}\cdot\{\text{length}(\gamma_\alpha) + \sqrt{-1}\cdot\text{torsion}(\gamma_\alpha)\} + \frac{s\pi^2}{q}. \end{aligned}$$

In particular,

$$\text{Im}\,\frac{2}{\pi}\cdot R(x_\alpha,y_\alpha,\xi_\alpha) = \text{Im}\,\frac{2}{\pi}\cdot V_\alpha(x_\alpha,y_\alpha,\xi_\alpha) + \frac{2\log|\xi_\alpha|}{q} = \frac{2}{\pi}\cdot\text{vol}(M_\alpha) + \text{length}(\gamma_\alpha),$$

which shows that, up to a pure imaginary constant,

$$\frac{2}{\pi\sqrt{-1}}R(x,y,e^u)$$

must coincide with $2\pi f(u)$ of [3, Theorem 2], and that

$$\text{Re}\,\frac{2}{\pi}\cdot R(x_\alpha,y_\alpha,\xi_\alpha) = \text{Re}\,\frac{2}{\pi}\cdot\left\{V_\alpha(x_\alpha,y_\alpha,\xi_\alpha) + \frac{s\pi^2}{q}\right\} - \text{torsion}(\gamma_\alpha)$$

must coincide with $-4\pi CS(M_\alpha) - \text{torsion}(\gamma_\alpha)$. Consequently, up to some constant which is independent of α, we have

$$\text{Re}\left\{V_\alpha(x_\alpha,y_\alpha,\xi_\alpha) + \frac{s\pi^2}{q}\right\} = -2\pi^2 CS(M_\alpha).$$

4 Concluding remarks

We redefine $V_\alpha(x,y,\xi)$ as follows.

$$V_\alpha(x,y,\xi) = V(x,y,\xi) + \frac{\log\xi(2\pi\sqrt{-1} - p\log\xi) + s\pi^2}{q}.$$

Then, $dV_\alpha(x,y,\xi) = 0$ gives the hyperbolicity equations for M_α, and

$$V_\alpha(x_\alpha,y_\alpha,\xi_\alpha) = -2\pi^2 CS(M_\alpha) + \text{vol}(M_\alpha)\sqrt{-1}$$

up to a real constant, where $(x_\alpha,y_\alpha,\xi_\alpha)$ is a solution to the equations above.

We finally remark that such a construction always works, even for a *link*, and the analytic functions in [3, 5] are now combinatorially constructed up to a constant. For the figure-eight knot and $\alpha \in \mathbb{Z}$, our potential function coincides with the function in [1] which appears in the "optimistic" limit of the quantum SU(2) invariants of M_α.

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Received: 5 December 2001 Revised: 19 February 2002