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On Axially Symmetric Flows in $\mathbf{R}^3$

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Abstract. We prove in a simpler as ususal way global-in-time existence of regular solutions to three-dimensional Navier-Stokes equations under the assumption that the flow is axially symmetric.

Keywords: *Navier-Stokes equations, axially symmetric flows, regularity of solution, Cauchy problem in three dimensions*

AMS subject classification: 35 Q 35, 35 J 65

1. Introduction

We investigate the evolutionary Navier-Stokes equations under the assumptions that the analyzed flow is axially symmetric and that the fluid fills in the whole three-dimensional space.

It is not known generally whether the 3-dimensional Navier-Stokes equations have a uniquely defined solution without restricting to the size of data or to the length of the time interval. However, if only axially symmetric flows are permitted, then it is possible to show global-in-time existence of a regular solution (see Theorem 1). As well-known, this solution is unique even in the class of all weak solutions subjected to axially symmetric data only (see Theorem 2). In addition, one can show (see [5]) that this global axially symmetric solution to the Navier-Stokes equations is stable for small (not necessarily axially symmetric) perturbations of axially symmetric forces and initial conditions.

The aim of this paper is to present a proof of Theorem 1, by an elementary and clear method. We take the advantage that we consider the flow in the whole space. We build our proof on known (and nowadays standard) results on existence, uniqueness

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and regularity of a weak solution to the evolutionary Stokes system. Starting from this, we present first local-in-time existence and uniqueness of a smooth axially symmetric solution to the Navier-Stokes system. Further we derive some global estimates, which allow us to extend the smooth solution to an arbitrary time interval.

The presented result has been already proved by Ladyzhenskaya in [3] (the result was announced in [4]) and simultaneously by Uchovskii and Yudovich in [7]. The key observation which is common to both the papers [3] and [7], and also to this paper is the following: the convective term in the equation for the vorticity, it means the term $[\nabla \mathrm{curl}\, \mathbf{v}]\mathbf{v}$, is *formally* orthogonal in $L^2(\mathbb{R}^3)$ to $\frac{\mathrm{curl}\, \mathbf{v}}{r^2}$ (r is the distance from the axes of symmetry). The main problem is to justify this step. While in [3] and [7] the proofs are based on the use of the Galerkin method in bounded domains and require a construction of special bases in spaces with weights (which we have attempted to avoid), here we directly 'test' the vorticity equation by $\frac{\mathrm{curl}\, \mathbf{v}}{r^{2-\varepsilon}}$ and derive the estimates in which we are allowed to pass $\varepsilon \to 0$.

Let us finally remark that a similar situation concerns the inviscid fluid described by the Euler equations: in general it is not known whether the system admits smooth global solutions in 3 dimensions; however, for axially symmetric smooth data one can show the existence of a sufficiently regular global-in-time solution. For the proof we can refer to Uchovskii and Yudovich (see [7], but also Remark at the end of this paper for more details) or to Beale, Kato and Majda (see [1]).

Throughout the whole paper we use the summation convention and standard notations for function spaces.

2. Navier-Stokes equations in cylindrical coordinates

The Navier-Stokes equations in $\mathbb{R}^3$, written in Cartesian coordinates x_1, x_2, x_3, have a non-dimensional form

$$\left.\begin{aligned} \mathrm{div}\, \mathbf{v} = 0 \\ \frac{\partial \mathbf{v}}{\partial t} + v_k \frac{\partial \mathbf{v}}{\partial x_k} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f} \end{aligned}\right\} \tag{1}$$

where $\mathbf{v} = (v_1, v_2, v_3) : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ and $p : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ are unknowns and $\mathbf{f} = (f_1, f_2, f_3) : (0, \infty) \to \mathbb{R}^3$ is prescribed. System (1) is completed by an initial condition

$$\mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \quad (\mathbf{x} \in \mathbb{R}^3) \quad \text{where} \quad \mathrm{div}\, \mathbf{v}_0 = 0.$$

In cylindrical coordinates given by

$$\left.\begin{aligned} x_1 &= r \cos\theta \\ x_2 &= r \sin\theta \\ x_3 &= z \end{aligned}\right\}$$

equations (1) are transformed into the system

$$\begin{aligned}
&\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{1}{r}v_\theta^2 + \frac{\partial p}{\partial r} \\
&-\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] = f_r \\
&\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{1}{r}v_\theta v_r + \frac{1}{r}\frac{\partial p}{\partial \theta} \\
&-\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] = f_\theta \\
&\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} + \frac{\partial p}{\partial z} \\
&-\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] = f_z \\
&\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0.
\end{aligned} \tag{2}$$

If ξ stands instead of $\mathbf{v}$, $\mathbf{v}_0$ or $\mathbf{f}$ above, then by $(\xi_r, \xi_\theta, \xi_z)$ we mean the vector $(\xi_1\cos\theta + \xi_2\sin\theta, -\xi_1\sin\theta + \xi_2\cos\theta, \xi_3)$.

Definition 1. A scalar function φ written in cylindrical coordinates is called *axially symmetric* if it is independent of θ, i.e. $\varphi = \varphi(r, z)$.

A vector function $\xi = (\xi_r, \xi_\theta, \xi_z)$ is called *axially symmetric* if $\xi_\theta \equiv 0$ and ξ_r and ξ_z are axially symmetric.

3. Local-in-time existence of an axially symmetric solution

We first want to show that if $\mathbf{v}_0$ and $\mathbf{f}$ are axially symmetric, then there exist a $t > 0$ and an axially symmetric solution $(\mathbf{v}, p)$ of equation (1) defined on $(0, t)$ satisfying the initial condition.

Let us recall some elementary results on the evolutionary Stokes system.

Lemma 1. *Let $T \in (0, \infty)$, $I = (0, T)$ and let $1 \le k \in \mathbb{N}$. Let us assume that $\mathbf{v}_0 \in W^{k,2}(\mathbb{R}^3)$ and $\mathbf{F} \in L^2(I; W^{k-1,2}(\mathbb{R}^3))$ are axially symmetric. Then there exists exactly one solution to the Stokes problem*

$$\left.\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{F} \\ \operatorname{div}\mathbf{v} = 0 \end{aligned}\right\} \quad \textit{in } I \times \mathbb{R}^3$$

$$\mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \quad \textit{in } \mathbb{R}^3$$

such that

$$\begin{aligned}
&\mathbf{v} \in L^\infty(I; W^{k,2}(\mathbb{R}^3)) \cap L^2(I; W^{k+1}(\mathbb{R}^3)) \\
&\frac{\partial \mathbf{v}}{\partial t} \in L^2(I; W^{k-1}(\mathbb{R}^3)) \\
&\nabla p \in L^2(I; W^{k-1,2}(\mathbb{R}^3))
\end{aligned}$$

$(k \geq 2)$. *Moreover,* $\mathbf{v}$ *and* p *are axially symmetric.*

Proof. The existence, uniqueness as well as the energy estimates are standard. The fact that the solution is axially symmetric can be seen by transforming the Stokes system into the cylindrical coordinates (see (2) with zero convective (i.e. nonlinear) term and $f_\theta = 0$), constructing the axially symmetric solution, and applying the uniqueness argument ∎

We now construct the axially symmetric solution to the full Navier-Stokes system. Let $t > 0$ and $J = (0, t)$. We set

$$X = X(t) \equiv \Big\{ \mathbf{u} \in L^\infty(J; W^{2,2}(\mathbb{R}^3)) \cap L^2(J; W^{3,2}(\mathbb{R}^3)) : \mathbf{u} \ \text{ axially symmetric} \Big\}.$$

Further, let $\mathbf{v}_0 \in W^{2,2}(\mathbb{R}^3)$ and $\mathbf{f} \in L^2(0, \infty; W^{1,2}(\mathbb{R}^3))$ be axially symmetric, $\operatorname{div} \mathbf{v}_0 = 0$. Take $\mathbf{v} \in X$ and define an operator $S : X \mapsto X$ in such a way that $\mathbf{u} \equiv S(\mathbf{v})$ solves the evolutionary Stokes system with the initial value $\mathbf{v}_0$ and the right-hand side $\mathbf{f} - v_k \frac{\partial \mathbf{v}}{\partial x_k}$. Notice that $\mathbf{f} - v_k \frac{\partial \mathbf{v}}{\partial x_k}$ is axially symmetric. Consequently, by Lemma 1, we observe that $\mathbf{u} \in X(t)$ for all $t \in (0, \infty)$ and p is axially symmetric.

In the sequel we will frequently use the classical interpolation inequality

$$\|\mathbf{z}\|_4 \leq \|\mathbf{z}\|_2^{\frac{1}{4}} \|\mathbf{z}\|_6^{\frac{3}{4}} \leq c \, \|\mathbf{z}\|_2^{\frac{1}{4}} \|D\mathbf{z}\|_2^{\frac{3}{4}}$$

holding for any $\mathbf{z} \in W^{1,2}(\mathbb{R}^3)$, and also two inequalities of Agmon's type

$$\begin{aligned} \|\mathbf{z}\|_\infty &\leq c \, \|\mathbf{z}\|_2^{\frac{1}{4}} \|\Delta \mathbf{z}\|_2^{\frac{3}{4}} \\ \|\mathbf{z}\|_\infty &\leq c \, \|D\mathbf{z}\|_2^{\frac{1}{2}} \|D^2 \mathbf{z}\|_2^{\frac{1}{2}} \end{aligned} \tag{3}$$

holding for any $\mathbf{z} \in W^{2,2}(\mathbb{R}^3)$ which can be shown using the Fourier transform.

Lemma 2. *Let* $\mathbf{f} \in L^2(0, \infty; W^{1,2}(\mathbb{R}^3))$ *and* $\mathbf{v}_0 \in W^{2,2}(\mathbb{R}^3)$ (*divergence free*) *be axially symmetric. Then there exists exactly one solution* $(\mathbf{v}, p)$ *such that*

$$\begin{aligned} &\mathbf{v} \in L^\infty(J; W^{2,2}(\mathbb{R}^3)) \cap L^2(J; W^{3,2}(\mathbb{R}^3)) \\ &\frac{\partial \mathbf{v}}{\partial t} \in L^2(J; W^{1,2}(\mathbb{R}^3)) \\ &Dp \in L^\infty(J; L^2(\mathbb{R}^3)) \end{aligned}$$

solving the Navier-Stokes equations on (*possibly short*) *time interval* $J = (0, t)$. *Moreover,* $\mathbf{v}$ *and* p *are axially symmetric.*

Proof. It is easy to see, with the help of Lemma 1, that S maps X into X. Then it remains to verify that $S : X \mapsto X$ is a contraction. Let us denote by $\mathbf{g}$ the difference of the convective terms, i.e. $\mathbf{g} = v_k^1 \frac{\partial \mathbf{v}^1}{\partial x_k} - v_k^2 \frac{\partial \mathbf{v}^2}{\partial x_k}$. It is sufficient to estimate $\mathbf{g}$ and $D\mathbf{g}$ in $L^2(J; L^2(\mathbb{R}^3))$. We have (with the help of (3))

$$\begin{aligned} \int_0^t \|D\mathbf{g}(\tau)\|_2^2 d\tau \leq & \int_0^t \|D(\mathbf{v}^2 - \mathbf{v}^1)(\tau)\|_{1,2}^2 \big(\|D\mathbf{v}^2(\tau)\|_{1,2}^2 + \|D\mathbf{v}^1(\tau)\|_{1,2}^2 \big) \, d\tau \\ & + \int_0^t \|(\mathbf{v}^2 - \mathbf{v}^1)(\tau)\|_{2,2}^2 \|D^2 \mathbf{v}^2(\tau)\|_2^2 d\tau \\ & + \int_0^t \|D^2(\mathbf{v}^2 - \mathbf{v}^1)(\tau)\|_2^2 \|\mathbf{v}^1(\tau)\|_{2,2}^2 d\tau \\ \leq & \, K(C) \, t \|\mathbf{v}^2 - \mathbf{v}^1\|_X^2. \end{aligned}$$

Analogously we get the L^2-estimates of $\mathbf{g}$ and again it is an easy matter to see that for $t > 0$ sufficiently small ($t \le t_0$) we obtain an estimate of the type

$$\|S(\mathbf{v}^2) - S(\mathbf{v}^1)\|_X \le \varrho \|\mathbf{v}^2 - \mathbf{v}^1\|_X$$

with $\varrho < 1$. The Banach fixed point theorem then gives the existence and uniqueness of a $\mathbf{v} \in X$ solving the Navier-Stokes system. The regularity for the time derivative and the pressure can be obtained in a standard way ∎

4. Global-in-time existence of an axially symmetric solution

4.1 Preliminaries. We denote by t^* the supremum of all $t > 0$ for which Lemma 2 holds, i.e.

$$t^* = \sup\Big\{t : \text{ there is an axially symmetric solution to (1) on } (0,t) \text{ in } X\Big\}.$$

Then either $t^* = \infty$ or $t^* < \infty$. The aim is to exclude the latter. Let us assume that $t^* < \infty$. Then necessarily

$$\limsup_{t \to t^*-} \|\mathbf{v}(t)\|_{W^{1,2}(\mathbb{R}^3)} = \infty. \tag{4}$$

Otherwise we could easily show that $\mathbf{v} \in C([0,t]; W^{2,2}(\mathbb{R}^3))$ for all $t \in (0, t^*)$ (more precisely, $\mathbf{v} \in L^2(0,t; W^{3,2}(\mathbb{R}^3))$ and $\frac{\partial \mathbf{v}}{\partial t} \in L^2(0,t; W^{1,2}(\mathbb{R}^3))$) and by Lemma 2, we could extend $(\mathbf{v}, p)$ behind t^*, which would contradict to the definition of t^*.

Let $t < t^*$ be arbitrary, $I \equiv (0,t)$ and $(\mathbf{v}, p)$ be a solution on I given by Lemma 2. Because of regularity we can take curl of (1). Thanks to axial symmetry the vector $\mathbf{w} \equiv \operatorname{curl} \mathbf{v}$ has the only non-zero component w_θ given by $w_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$. For lucidity, we denote w_θ by ω. We see that ω solves

$$\frac{\partial \omega}{\partial t} + v_r \frac{\partial \omega}{\partial r} + v_z \frac{\partial \omega}{\partial z} - \frac{v_r}{r}\omega - \nu\left[\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} - \frac{\omega}{r^2}\right] = g \tag{5}$$

where $g \equiv (\operatorname{curl} \mathbf{f})_\theta$.

We will need the following lemma on equivalence of norms for ω and $\mathbf{v}$.

Lemma 3. *Let $\mathbf{v}$ be a smooth, divergence free, axially symmetric vector field and $\omega \equiv (\operatorname{curl} \mathbf{v})_\theta$. Then:* [1)]

(i) $\|\omega\|_2$ *is equivalent to* $\|D\mathbf{v}\|_2$.

(ii) $\|\nabla\omega\|_2 + \|\frac{\omega}{r}\|_2$ *is equivalent to* $\|D^2\mathbf{v}\|_2$.

(iii) $\|\nabla^2\omega\|_2 + \|\nabla\left(\frac{\omega}{r}\right)\|_2 \le C\,\|D^3\mathbf{v}\|_2$.

[1)] By $\nabla\omega$ we understand $(\frac{\partial \omega}{\partial r}, \frac{\partial \omega}{\partial z})$, while $D\mathbf{v}$ denotes $(\frac{\partial \mathbf{v}}{\partial x_1}, \frac{\partial \mathbf{v}}{\partial x_2}, \frac{\partial \mathbf{v}}{\partial x_3})$.

Proof. It is based on the fact that there exist constants K_i $(i = 1, \dots, 5)$ such that for any smooth axially symmetric vector $\mathbf{v}$ with $\operatorname{curl} \mathbf{v} = \mathbf{w}$ and any $\mathbf{a} \in \mathbb{R}^3$, $a_1^2 + a_2^2 > 0$, we have

$$K_1|\mathbf{w}(\mathbf{a})| \le |\omega(\mathbf{a})| \le K_2|\mathbf{w}(\mathbf{a})|$$

$$K_3|D\mathbf{w}(\mathbf{a})| \le \Big|\frac{\partial\omega(\mathbf{a})}{\partial r}\Big| + \Big|\frac{\partial\omega(\mathbf{a})}{\partial z}\Big| + \Big|\frac{\omega(\mathbf{a})}{r}\Big| \le K_4|D\mathbf{w}(\mathbf{a})|$$

$$\Big|\frac{\partial^2\omega(\mathbf{a})}{\partial r^2}\Big| + \Big|\frac{\partial^2\omega(\mathbf{a})}{\partial z^2}\Big| + \Big|\frac{\partial}{\partial r}\frac{\omega(\mathbf{a})}{r}\Big| + \Big|\frac{\partial}{\partial z}\frac{\omega(\mathbf{a})}{r}\Big| \le K_5|D^2\mathbf{w}(\mathbf{a})|$$

where $r = \sqrt{a_1^2 + a_2^2}$. Now it is enough to use the well known fact that for divergence free vector fields the L^2-norms of gradients of $\mathbf{v}$ are equivalent to the L^2-norms of $\operatorname{curl} \mathbf{v}$ and its gradients, which can be shown by means of the Parseval equality ∎

We would like to multiply (5) by $\frac{\omega}{r^2}$ and ω, integrate over $\mathbb{R}^3$ with the aim to derive a priori estimates for ω which, combined with Lemma 3/(i), would yield a contradiction to (4). Although the multiplication by $\frac{\omega}{r^2}$ is the key step in the proof of Ladyzhenskaya, we do not know if $\frac{\omega}{r^2} \in L^2(I; L^2(\mathbb{R}^3)$ here. However, we can multiply (5) by $\frac{\omega}{r^{2-\varepsilon}}$ with $\varepsilon > 0$ arbitrarily small, as follows from the next lemma.

Lemma 4. *Let* $\omega = \operatorname{curl} \mathbf{v}$ *with* $\mathbf{v} \in X$. *Then:*

(i) $\frac{\omega}{r^{2-\varepsilon}}$ *and* $\frac{1}{r^{1-\varepsilon}}\frac{\partial\omega}{\partial r}$ *belong to* $L^2(I; L^2(\mathbb{R}^3))$ *for all* $\varepsilon > 0$.

(ii) *Let* $g_1(\eta) \equiv \int_{-\infty}^{\infty} (\eta^\delta |\frac{\omega}{\eta}|^2)(\eta, z)\, dz$ *and* $g_2(\eta) \equiv \int_{-\infty}^{\infty} (\eta^\delta |\frac{\partial\omega}{\partial r}|^2)(\eta, z)\, dz$. *Then* g_1 *and* g_2 *are bounded for any* $\delta \in (0, 2)$.

Proof. To prove statement (i) we first observe that, by Lemma 3, $\frac{\omega}{r}$ and $\frac{\partial}{\partial r}\left(\frac{\omega}{r}\right)$ belong to $L^2(I; L^2(\mathbb{R}^3))$. We then define $g \in W^{1,2}(\mathbb{R}^3)$ in such a way that $g = \frac{\omega}{r}$ for $r < 1$, $g = 0$ for $r > 2$ and $\|g\|_{1,2} \le C\|\frac{\omega}{r}\|_{1,2}$. For $\varepsilon > 0$ fixed, the Hardy inequality yields

$$\begin{aligned}\int_{-\infty}^{\infty}\int_0^1 \frac{\omega}{r^{2-\varepsilon}}\, r\, dr dz &\le \int_{-\infty}^{\infty}\int_0^{\infty} r^{-1+2\varepsilon}|g|^2 dr dz \\ &\le \left(\frac{2}{\varepsilon}\right)^2 \int_{-\infty}^{\infty}\int_0^{\infty} r^{1+2\varepsilon}\Big|\frac{\partial g}{\partial r}\Big|^2 dr dz \\ &\le C(\varepsilon)\Big\|\nabla\frac{\omega}{r}\Big\|_2^2.\end{aligned}$$

Moreover, for $r > 1$ we have

$$\int_{-\infty}^{\infty}\int_1^{\infty} \Big|\frac{\omega}{r^{2-\varepsilon}}\Big|^2 r\, dr dz \le \int_{-\infty}^{\infty}\int_1^{\infty} \Big|\frac{\omega}{r}\Big|^2 r\, dr dz.$$

In a very similar way we can show that

$$\Big\|\frac{1}{r^{1-\varepsilon}}\frac{\partial\omega}{\partial r}\Big\|_2 \le C(\varepsilon)\Big\|\frac{\partial\omega}{\partial r}\Big\|_{1,2}.$$

Thus statement (i) is proved.

To verify statement (ii), let $\eta > 0$ and $\delta \in (0,2)$. Then

$$\begin{aligned} g_1(\eta) &= -\int_{-\infty}^{\infty}\int_{\eta}^{\infty} \frac{\partial}{\partial r}\Big(r^\delta\Big|\frac{\omega}{r}\Big|^2\Big)drdz \\ &\le \int_{-\infty}^{\infty}\int_{\eta}^{\infty}\Big(\delta r^{-1+\delta}\Big|\frac{\omega}{r}\Big|^2 + 2r^\delta\Big|\frac{\omega}{r}\Big|\Big|\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)\Big|\Big)drdz \\ &\le C\Big\|\frac{\omega}{r^{2-\frac{\delta}{2}}}\Big\|_2^2 + C\Big\|\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)\Big\|_2\Big\|\frac{\omega}{r^{2-\delta}}\Big\|_2. \end{aligned}$$

Thus $g_1(\eta)$ is bounded for all η due to statement (i) and Lemma 3/(iii). The boundedness of g_2 is proved analogously starting from $-\int_{-\infty}^{\infty}\int_{\eta}^{\infty}\frac{\partial}{\partial r}\big(r^\delta|\frac{\partial\omega}{\partial r}|^2\big)\,drdz$ and using Lemma 3/(iii) ∎

Corollary 1. *For every $\varepsilon > 0$,*

$$\lim_{\eta\to 0+}\int_{-\infty}^{\infty}\Big(\frac{\partial\omega}{\partial r}\frac{\omega}{\eta^{1-\varepsilon}}\Big)(\eta,z)\,dz = 0.$$

Proof. For fixed $\varepsilon > 0$, we have

$$\begin{aligned} &\int_{-\infty}^{\infty}\Big(\frac{\partial\omega}{\partial r}\frac{\omega}{\eta^{1-\varepsilon}}\Big)(\eta,z)\,dz \\ &\quad\le \Big(\int_{-\infty}^{\infty}\Big(\Big|\frac{\partial\omega}{\partial r}\Big|^2\eta^{\frac{\varepsilon}{2}}\Big)(\eta,z)\,dz\Big)^{\frac12}\Big(\int_{-\infty}^{\infty}\Big(\Big|\frac{\omega}{\eta}\Big|^2\eta^{\frac{\varepsilon}{2}}\Big)(\eta,z)\,dz\Big)^{\frac12}\eta^{\frac{\varepsilon}{2}} \end{aligned}$$

which gives the assertion thanks to Lemma 4/(ii) ∎

4.2 Global estimates. Now, we are going to multiply (5) by $\frac{\omega}{r^{2-\varepsilon}}$ and integrate over $\mathbb{R}^3$ with the aim to let finally $\varepsilon \to 0+$. The integration over $\mathbb{R}^3$ is clearly allowed as all integrals are finite; for example (by $\int$ we mean $\int_{-\infty}^{\infty}\int_0^{\infty}$ in what follows) it holds

$$\Big|\int \frac{1}{r}\frac{\partial\omega}{\partial r}\frac{\omega}{r^{2-\varepsilon}}r\,drdz\Big| \le \Big\|\frac{1}{r^{1-\frac{\varepsilon}{2}}}\frac{\partial\omega}{\partial r}\Big\|_2\Big\|\frac{\omega}{r^{2-\frac{\varepsilon}{2}}}\Big\|_2$$

and the right-hand side is finite due to Lemma 4/(i).

In the following lemma we obtain the fundamental apriori estimates which allow us to exclude the possibility of the blow-up (4).

Lemma 5 (Key step). *Let $\mathbf{v}_0 \in W^{2,2}(\mathbb{R}^3)$ and $\mathbf{f} \in L^2(0,\infty;W^{1,2}(\mathbb{R}^3))$ be axially symmetric and let $t < t^*$. Then*

$$\Big\|\frac{\omega(t)}{r}\Big\|_2^2 \le C(\mathbf{v}_0,\mathbf{f}) \tag{6}$$

$$\|\omega(t)\|_2^2 + \nu\int_0^t\Big(\|\nabla\omega(\tau)\|_2^2 + \Big\|\frac{\omega(\tau)}{r}\Big\|_2^2\Big)d\tau \le C(\mathbf{v}_0,\mathbf{f}) \tag{7}$$

where $C(\mathbf{v}_0, \mathbf{f})$ *denotes a quantity depending on* $\|\mathbf{v}_0\|_{2,2}$ *and* $\int_0^\infty \|\mathbf{f}(t)\|_{1,2}^2 dt$.

Proof. In order to prove (6) we multiply (5) by $\frac{\omega}{r^{2-\varepsilon}}$ with $\varepsilon > 0$ small, which is allowed due to Lemma 4/(i) (see also the note above), and integrate over $\mathbb{R}^3$ with respect to the measure $r\,drdz$. We will obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \Big|\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\Big|^2 r\,drdz + \nu \int \Big(\Big|\nabla\Big(\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\Big)\Big|^2 + \Big(\varepsilon - \frac{\varepsilon^2}{4}\Big)\Big|\frac{\omega}{r^{2-\frac{\varepsilon}{2}}}\Big|^2\Big) r\,drdz \\ = \int g\frac{\omega}{r^{2-\varepsilon}} r\,drdz + \frac{\varepsilon}{2}\int \frac{v_r}{r}\frac{\omega^2}{r^{2-\varepsilon}} r\,drdz.\end{aligned} \tag{8}$$

Indeed, the term including $\frac{\partial \omega}{\partial t}$ is elementary. The convective term gives

$$\begin{aligned}&\int \Big(v_r\frac{\partial \omega}{\partial r} + v_z\frac{\partial \omega}{\partial z} - \frac{v_r\omega}{r}\Big)\frac{\omega}{r^{2-\varepsilon}}\, r\,drdz \\ &\quad = -\frac{1}{2}\int \Big(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\Big)\frac{\omega^2}{r^{1-\varepsilon}}\,drdz - \frac{\varepsilon}{2}\int \frac{v_r}{r}\frac{\omega^2}{r^{1-\varepsilon}}\,drdz \\ &\quad = -\frac{\varepsilon}{2}\int \frac{v_r}{r}\frac{\omega^2}{r^{1-\varepsilon}}\,drdz \qquad \text{(due to } (2)_4).\end{aligned}$$

The elliptic term requires precise investigations

$$\begin{aligned}&-\nu\int \Big(\frac{\partial^2\omega}{\partial r^2} + \frac{\partial^2\omega}{\partial z^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} - \frac{\omega}{r^2}\Big)\frac{\omega}{r^{2-\varepsilon}}\, r\,drdz \\ &\quad = -\nu\Bigg[\int_{-\infty}^{\infty} \frac{\partial\omega}{\partial r}\frac{\omega}{r^{1-\varepsilon}}\,dz\Bigg]_{r=0}^{r=\infty} \\ &\qquad + \nu\int \Big(\Big|\frac{\partial}{\partial r}\Big(\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\Big)\Big|^2 + \Big|\frac{\partial}{\partial z}\Big(\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\Big)\Big|^2 + \varepsilon\Big(1 - \frac{\varepsilon}{4}\Big)\Big|\frac{\omega}{r^{2-\frac{\varepsilon}{2}}}\Big|^2\Big) r\,drdz\end{aligned}$$

and the boundary term vanishes due to Corollary 1.

Now we can estimate the right-hand side of (8). Since

$$\begin{aligned}&\int \Big(\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r}\Big)\frac{\omega}{r^{2-\varepsilon}}\, r\,drdz \\ &\quad = -\int \Big(f_r\frac{\partial}{\partial z}\Big(\frac{\omega}{r^{1-\varepsilon}}\Big) - f_z\frac{\partial}{\partial r}\Big(\frac{\omega}{r^{1-\varepsilon}}\Big)\Big)\,drdz \\ &\quad = -\int \Big(f_r r^{\frac{\varepsilon}{2}}\frac{\partial}{\partial z}\Big(\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\Big) - f_z r^{\frac{\varepsilon}{2}}\frac{\partial}{\partial r}\Big(\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\Big) - \frac{\varepsilon}{2} f_z r^{\frac{\varepsilon}{2}}\frac{\omega}{r^{2-\frac{\varepsilon}{2}}}\Big)\,drdz\end{aligned}$$

we have by the Hardy and Young inequalities

$$\begin{aligned}\Big|\int g\frac{\omega}{r^{2-\varepsilon}}\, r\,drdz\Big| &\le \Big\|\frac{\mathbf{f}}{r^{1-\frac{\varepsilon}{2}}}\Big\|_2\Big(\Big\|\nabla\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\Big\|_2 + \frac{\varepsilon}{2}\Big\|\frac{\omega}{r^{2-\frac{\varepsilon}{2}}}\Big\|_2\Big) \\ &\le \frac{\nu}{2}\Big\|\nabla\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\Big\|_2^2 + \frac{\varepsilon}{4}\Big\|\frac{\omega}{r^{2-\frac{\varepsilon}{2}}}\Big\|_2^2 + c\,\|\mathbf{f}\|_{1,2}^2.\end{aligned}$$

Further, by means of (3) we have

$$\varepsilon\left|\int \frac{v_r}{r}\frac{\omega^2}{r^{2-\varepsilon}}\, r\, drdz\right| \le \varepsilon\|v_r\|_\infty \left\|\frac{\omega}{r^{2-\frac{\varepsilon}{2}}}\right\|_2 \left\|\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\right\|_2$$
$$\le \frac{\varepsilon}{4}\left\|\frac{\omega}{r^{2-\frac{\varepsilon}{2}}}\right\|_2^2 + \varepsilon\, c\, \|D\mathbf{v}\|_2\|D^2\mathbf{v}\|_2 \left\|\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\right\|_2^2.$$

Putting all calculations together and integrating the result with respect to time we obtain for all $\tau \in (0,t)$

$$\left\|\frac{\omega(\tau)}{r^{1-\frac{\varepsilon}{2}}}\right\|_2^2 \le c(\mathbf{f},\mathbf{v}_0) + \varepsilon \int_0^\tau \|D\mathbf{v}(s)\|_{1,2}^2 \left\|\frac{\omega(s)}{r^{1-\frac{\varepsilon}{2}}}\right\|_2^2 ds.$$

The Gronwall Lemma then implies

$$\left\|\frac{\omega(\tau)}{r^{1-\varepsilon}}\right\|_2^2 \le c(\mathbf{f},\mathbf{v}_0)\exp\left(\varepsilon\int_0^t \|D\mathbf{v}(\tau)\|_{1,2}^2 d\tau\right).$$

The right-hand side is finite by the assumption on $\mathbf{v}$, which allows to pass to the limit as $\varepsilon \to 0$ at the right-hand side. As $\left|\frac{\omega}{r^{1-\frac{\varepsilon}{2}}}\right|$ is bounded by $\left|\frac{\omega}{r}\right|$ for $r \in (0,1)$, and by $|\omega|$ for $r \ge 1$, we can let ε tend to 0 at the left-hand side by the Lebesgue dominated theorem, and we obtain (6).

The next estimate (7) is obtained by multiplying (5) by ω (and integrating over $\mathbb{R}^3$). The elliptic term gives

$$-\nu\int\left(\frac{\partial^2\omega}{\partial r^2} + \frac{\partial^2\omega}{\partial z^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} - \frac{\omega}{r^2}\right)\omega r\, drdz$$
$$= \nu\int\left[\left(\frac{\partial\omega}{\partial r}\right)^2 + \frac{\omega^2}{r^2} + \left(\frac{\partial\omega}{\partial z}\right)^2\right] r\, drdz$$
$$= \nu\left(\|\nabla\omega\|_2^2 + \left\|\frac{\omega}{r}\right\|_2^2\right).$$

Since

$$\int \frac{\partial\omega}{\partial t}\,\omega r\, drdz = \frac{1}{2}\frac{d}{dt}\|\omega\|_2^2$$

and

$$\int g\omega r\, drdz \le 2\int |\mathbf{f}|\left(|\nabla\omega| + \left|\frac{\omega}{r}\right|\right) r\, drdz \le 2\|\mathbf{f}\|_2\left(\|\nabla\omega\|_2 + \left\|\frac{\omega}{r}\right\|_2\right),$$

we can concentrate on the estimate of the convective term. We have

$$\int\left(v_r\frac{\partial\omega}{\partial r}\omega + v_z\frac{\partial\omega}{\partial z}\omega - \frac{v_r\omega^2}{r}\right) r\, drdz$$
$$= \int\frac{\omega^2}{2}\left(-\frac{\partial v_r}{\partial r} - \frac{\partial v_z}{\partial z} - 2\frac{v_r}{r} - \frac{v_r}{r}\right) r\, drdz$$
$$= -\int \frac{v_r\omega^2}{r}\, r\, drdz.$$

Adding all computations, integrating over $(0,t)$, using the Agmon inequality (3), (6) and Lemma 4 we obtain

$$\|\omega(t)\|_2^2 + C\int_0^t \|D^2\mathbf{v}(\tau)\|_2^2 d\tau + \frac{\nu}{4}\int_0^t \left(\|\nabla\omega(\tau)\|_2^2 + \left\|\frac{\omega}{r}\right\|_2^2 d\tau\right)$$

$$\le C\int_0^t\int |\omega(\tau)|^2|v_r|\, dr dz + C(\mathbf{v}_0,\mathbf{f})$$

$$\le \int_0^t \|\mathbf{v}(\tau)\|_\infty \left\|\frac{\omega(\tau)}{r}\right\|_2 \|\omega(\tau)\|_2 d\tau + C(\mathbf{v}_0,\mathbf{f})$$

$$\le C(\mathbf{v}_0,\mathbf{f})\int_0^t \|D\mathbf{v}\|_2^{\frac{1}{2}}\|D^2\mathbf{v}\|_2^{\frac{1}{2}}\|\omega\|_2 d\tau + C(\mathbf{v}_0,\mathbf{f})$$

$$\le \frac{C}{8}\int_0^t \|D^2\mathbf{v}\|_2^2 d\tau + C(\mathbf{v}_0,\mathbf{f})$$

where we use (at the last step) the classical first energy estimate

$$\|\mathbf{v}(t)\|_2^2 + \int_0^t \|D\mathbf{v}(\tau)\|_2^2 d\tau \le \|\mathbf{v}_0\|_2^2 + c\int_0^\infty \|\mathbf{f}(\tau)\|_2^2 d\tau. \tag{9}$$

Lemma 5 is proved ∎

4.3 Main theorems. The task to exclude (4) is now very easy. By the equivalence of the norm (cf. Lemma 3/(i)-(ii)) we see that (7) and (9) can be rewritten as

$$\|\mathbf{v}(t)\|_{1,2}^2 \le C\left(\|\mathbf{v}_0\|_2^2, \int_0^t \|\mathbf{f}\|_{1,2}^2 d\tau\right)$$

valid for all $t < t^*$. Passing to the limsup at the left-hand side we obtain

$$\limsup_{t\to t^*-} \|\mathbf{v}(t)\|_{1,2}^2 < \infty.$$

Thus (4) does not hold and consequently $t^* = \infty$.

We have proved

Theorem 1. *Let $T \in (0,\infty)$ be arbitrary, and let $\mathbf{v}_0 \in W^{2,2}(\mathbb{R}^3)$, $\operatorname{div}\mathbf{v}_0 = 0$, and $\mathbf{f} \in L^2(0,T;W^{1,2}(\mathbb{R}^3))$ be axially symmetric. Then there exists a (global) axially symmetric solution to the Navier-Stokes equations (1) satisfying*

$$\mathbf{v} \in L^\infty(0,T;W^{2,2}(\mathbb{R}^3)) \cap L^2(0,T;W^{3,2}(\mathbb{R}^3))$$

$$\frac{\partial\mathbf{v}}{\partial t} \in L^2(0,T;W^{1,2}(\mathbb{R}^3)).$$

An easy consequence of Theorem 1 is the following statement.

Theorem 2. *Let* $\mathbf{v}_0$ *and* $\mathbf{f}$ *be as in Theorem* 1. *Then the global axially symmetric solution to* (1) *given by Theorem* 1 *is unique in the class of weak solutions to* (1).

Proof. It is standard, compare with [2: Chapter 10] or [6] ∎

Remark. This remark is devoted to the Euler equations. Using a very similar method as in [7] we could show that for axially symmetric and smooth data there exists a uniquely determined axially symmetric solution to the incompressible Euler equations in the whole three-dimensional space. The idea of the proof is the following. We first construct sufficiently regular solution to the Euler equations on a sufficiently short time interval as limit of solutions to the Navier-Stokes equations by passing with the viscosity to zero. Next we obtain some new apriori estimates by multiplying the curl of the Euler equations successively by $|\frac{\omega}{r}|^{p-1}r$ and $|\omega|^{p-1}$ and integrating $r\,drdz$, $p \in (2,\infty)$. Passing with p to ∞ we then obtain an estimate of ω in $L^\infty(I; L^\infty(\mathbb{R}^3))$ which excludes the possibility of the blow up (compare with [1]).

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