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The Generalized Riemann-Hilbert Boundary Value Problem for Non-Homogeneous Polyanalytic Differential Equation of Order n in the Sobolev Space $W_{n,p}(D)$

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Abstract. Given is a nonlinear non-homogeneous polyanalytic differential equation of order n in a simply-connected domain D in the complex plane. Initially we prove (under certain conditions) the existence of its general solution in $W_{n,p}(D)$ by first transforming it into a system of integro-differential equations. Next we prove the solvability of a generalized Riemann-Hilbert problem for the differential equation. This is effected by first reducing the boundary value problem posed to a corresponding one for a polyanalytic function. The latter is then transformed into n classical Riemann-Hilbert problems for holomorphic functions, whose solutions are known in the literature.

Keywords: *Polyanalytic functions, generalized Cauchy-Pompeiu integral operators of higher order, Riemann-Hilbert problem*

AMS subject classification: 30 G 30, 35 J 40, 47 G 10

1. Introduction

We consider the following non-homogeneous polyanalytic differential equation of order n in a given simply-connected bounded domain D in the complex plane $\mathbb{C}$:

$$\frac{\partial^n w}{\partial \bar{z}^n} = F\left(z, w, \left\{\frac{\partial^{m+k} w}{\partial z^m \partial \bar{z}^k}\right\}\right) \tag{1}$$
$$n \geq m, k \in \mathbb{N}_0, m + k \leq n, (0,0) \neq (m,k) \neq (0,n), n \in \mathbb{N}.$$

The right-hand side is a continuous function of its variables $z \in D$, w and the partial derivatives of w of order not exceeding n and excluding $\frac{\partial^n w}{\partial \bar{z}^n}$, which are denoted here by $\left\{\frac{\partial^{m+k} w}{\partial z^m \partial \bar{z}^k}\right\}$. Following [5, 6] the general solution of equation (1) may be expressed in

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the form

$$\begin{aligned} w(z) &= \Phi(z) + T_{0,n,D}F\Big(\zeta, w(\zeta), \Big\{\frac{\partial^{m+k}w}{\partial\zeta^m\partial\overline{\zeta}^k}\Big\}\Big)(z) \\ &= \Phi(z) + \iint_D K_{0,n}(z-\zeta)F\Big(\zeta, w(\zeta), \Big\{\frac{\partial^{m+k}w}{\partial\zeta^m\partial\overline{\zeta}^k}\Big\}\Big)d\xi d\eta \end{aligned} \tag{2}$$

where Φ is a polyanalytic function of order n in D, and $T_{0,n,D}$ is a generalized Cauchy-Pompeiu type singular integral operator:

$$K_{m,n}(z) = \begin{cases} \dfrac{(-m)!(-1)^m}{(n-1)!\pi}\, z^{m-1}\overline{z}^{n-1} & \text{if } m \le 0 \\ \dfrac{(-n)!(-1)^n}{(m-1)!\pi}\, z^{m-1}\overline{z}^{n-1} & \text{if } n \le 0 \\ \dfrac{z^{m-1}\overline{z}^{n-1}}{(m-1)!(n-1)!\pi}\Big(\log|z|^2 - \sum_{r=1}^{m-1}\dfrac{1}{r} - \sum_{s=1}^{n-1}\dfrac{1}{s}\Big) & \text{if } m,n \in \mathbb{N}. \end{cases} \tag{3}$$

When $m = 1$ or $n = 1$, the corresponding summation in the formula is dropped. The kernel $K_{m,n}$ of the integral operator $T_{m,n,D}$ has no singularity on D, except possibly at the origin. Moreover, it follows from the properties of the operators $T_{m,k,D}$ $(m+k \le n)$ that $T_{0,n,D}f \in W_{n,p}(D)$, if $f \in L_p(D)$ $(1 < p < \infty)$ (cf. [2, 5, 6]).

Suppose $w \in W_{n,p}(D)$ is a solution of equation (1). Thus w may be expressed in the form (2), and hence we obtain

$$\frac{\partial w}{\partial z} = \Phi_z + T_{-1,n,D}F\,, \qquad \frac{\partial w}{\partial \overline{z}} = \Phi_{\overline{z}} + T_{0,n-1,D}F$$
$$\frac{\partial^k w}{\partial z^k} = \frac{\partial^k \Phi}{\partial z^k} + T_{-k,n,D}F\,, \qquad \frac{\partial^k w}{\partial \overline{z}^k} = \frac{\partial^k \Phi}{\partial \overline{z}^k} + T_{0,n-k,D}F \quad (0 \le k \le n)$$

and, in general,

$$\frac{\partial^{m+k} w}{\partial z^m \partial\overline{z}^k} = \frac{\partial^{m+k} \Phi}{\partial z^m \partial\overline{z}^k} + T_{-m,n-k,D}F \qquad (n \ge m,k;\ m+k \le n).$$

Consequently, we arrive at the following result (cf. [12, 15, 21])).

Theorem 1. *The function $w \in W_{n,p}(D)$ $(2 < p < \infty)$ defined by equation (2) is a general solution of the non-homogeneous polyanalytic equation (1) if and only if for a given in the domain D polyanalytic function $\Phi \in W_{n,p}(D)$ of order n, $(w, \{h_{m,k}\})$ is a solution of the system*

$$\left.\begin{aligned} &w(z) = \Phi(z) + T_{0,n,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z) \\ &h_{m,k}(z) = \frac{\partial^{m+k}\Phi}{\partial z^m \partial\overline{z}^k} + T_{-m,n-k,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z) \\ &n \ge m, k \in \mathbb{N}_0, m+k \le n, (0,0) \ne (m,k) \ne (0,n), n \in \mathbb{N}. \end{aligned}\right\} \tag{4}$$

We note in passing that the integral operators $T_{m,k,D}$ $(m+k=0<m^2+k^2)$ are of singular Calderon-Zygmund type, and may be viewed as analogues of Vekua-type integral operators Π_D and $\overline{\Pi}_D$ defined by

$$\Pi_D f(z) = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{(\zeta - z)^2}\, d\xi d\eta$$

$$\overline{\Pi}_D f(z) = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{(\overline{\zeta} - \overline{z})^2}\, d\xi d\eta$$

(cf. [8, 11, 17, 18, 22]). They are singular and must be understood in the sense of Cauchy's principal value. Moreover, they satisfy the Calderon-Zygmund inequality (cf. [5, 6, 8, 17, 18, 22])

$$\|T_{m,n,D} f\|_{p,D} \le A_p \|f\|_{p,D} \tag{5}$$

where

$$A_p = \|T_{m,n,D}\|_p, \quad A_p \ge A_2 = 1 \quad (1<p<\infty).$$

On the other hand, if $m+k>0$, then $T_{m,k,D}$ are regular or weakly singular integral operators, and they may be viewed as generalizations of the Cauchy-Pompeiu integral operators T_D, $\overline{T}_D$, T_D^* and the potential operator P_D given by

$$T_D f(z) = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{\zeta - z}\, d\xi d\eta, \qquad \overline{T}_D f(z) = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{\overline{\zeta} - \overline{z}}\, d\xi d\eta$$

$$T_D^* f(z) = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{|\zeta - z|}\, d\xi d\eta, \qquad P_D f(z) = \frac{2}{\pi}\iint_D f(\zeta)\log|\zeta - z|\, d\xi d\eta.$$

Moreover, since $\|K_{m,n}\|_{1,D} \le C(m,n,D)$ =const, it follows from the convolution theorem of W. H. Young (see [17], for instance) that $T_{m,k,D}$ maps the Banach space $L_p(D)$ $(1 \le p \le \infty)$ into itself, and the estimate

$$\|T_{m,k,D} f\|_{p,D} \le C(m,k,D)\, \|f\|_{p,D} \qquad (1 \le p \le \infty, m+k>0) \tag{6}$$

holds.

2. Existence of the general solution

We make the following assumptions on the right-hand side of equation (1):

(A1) $F(z,w,\{h_{m,k}\})$ is a continuous function of its variables $z \in D$, w and the partial derivatives of w of order not exceeding n and excluding $\frac{\partial^n w}{\partial \overline{z}^n}$, which are denoted here by $\{h_{m,k}\}$.

(A2) There exists a tupel $(w^*, \{h_{m,k}^*\})$ $(w^*, h_{m,k}^* \in L_p(D), 2<p<\infty)$ such that $F(z, w^*, \{h_{m,k}^*\}) \in L_p(D)$.

(A3) $F(z,w,\{h_{m,k}\})$ satisfies a Lipschitz condition of the form

$$\begin{aligned}&\Big|F(z,w(z),\{h_{m,k}(z)\}) - F(z,\widetilde{w}(z),\{\widetilde{h}_{m,k}(z)\}\Big|\\ &\quad \le L_1 \max\Big\{\max_{m+k<n} |h_{m,k}(z) - \widetilde{h}_{m,k}(z)|,\ |w(z) - \widetilde{w}(z)|\Big\}\\ &\qquad + L_2 \max_{m+k=n} \big|h_{m,k}(z) - \widetilde{h}_{m,k}(z)\big|\end{aligned}$$

almost everywhere on D. While $0<L_2<1$, L_1 may take any positive value.

Note. It follows from assumptions (A2) and (A3) that $F(z,w,\{h_{m,k}\}) \in L_p(D)$ $(2<p<\infty)$ if w and all the elements of $\{h_{m,k}\}$ belong to $L_p(D)$.

We introduce the following Banach space $\mathcal{L}_p(D)$ $(2<p<\infty)$:

$$\mathcal{L}_p(D) = \left\{(w,\{h_{m,k}\}) \,\middle|\, w, h_{m,k} \in L_p(D)\right\}$$

$$n \geq m,k \in \mathbb{N}_0, m+k \leq n, (0,0) \neq (m,k) \neq (0,n), n \in \mathbb{N}.$$

$$\|(w,\{h_{m,k}\})\| = \max\left\{\gamma\|w\|_{p,D},\, \gamma \max_{m+k<n} \|h_{m,k}\|_{p,D},\, \max_{m+k=n} \|h_{m,k}\|_{p,D}\right\} \quad (\gamma > 0).$$

Next we define a mapping $\mathbb{P}$ in $\mathcal{L}_p(D)$ through the right-hand side of (4). For any tuple $(w,\{h_{m,k}\}) \in \mathcal{L}_p(D)$ we set

$$(W,\{H_{m,k}\}) = \mathbb{P}(w,\{h_{m,k}\})$$

$$W(z) = \Phi(z) + T_{0,n,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z)$$

$$H_{m,k}(z) = \frac{\partial^{m+k}}{\partial z^m \partial \overline{z}^k}\Phi + T_{-m,n-k,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z) \tag{7}$$

$$n \geq m,k \in \mathbb{N}_0, m+k \leq n, (0,0) \neq (m,k) \neq (0,n), n \in \mathbb{N}$$

$$\Phi \in W_{n,p}(D) \quad (2<p<\infty) \text{ a polyanalytic function in } D.$$

It follows immediately from the preceding discussion that $\mathbb{P}$ maps $\mathcal{L}_p(D)$ $(2<p<\infty)$ into itself.

We next show that, under certain conditions, $\mathbb{P}$ is contractive in $\mathcal{L}_p(D)$, so that we can apply the Banach fixed point theorem. To this end we consider the images $(W,\{H_{m,k}\})$, $(\widetilde{W},\{\widetilde{H}_{m,k}\})$ of $(w,\{h_{m,k}\})$, $(\widetilde{w},\{\widetilde{h}_{m,k}\}) \in \mathcal{L}_p(D)$, respectively, under the mapping $\mathbb{P}$. We then have

$$\begin{aligned}
\gamma\|W-\widetilde{W}\|_p &\leq \gamma\|T_{0,n,D}\|_p \big\|F(z,w,\{h_{m,k}\}) - F(z,\widetilde{w},\{\widetilde{h}_{m,k}\})\big\|_{p,D} \\
&\leq \gamma\|T_{0,n,D}\|_p \bigg(L_1 \max\left\{\max_{0<m+k<n} \|h_{m,k}-\widetilde{h}_{m,k}\|_{p,D}, \|w-\widetilde{w}\|_{p,D}\right\} \\
&\qquad + L_2 \max_{m+k=n} \|h_{m,k}-\widetilde{h}_{m,k}\|_{p,D}\bigg) \\
&\leq \|T_{0,n,D}\|_p (L_1+\gamma L_2)\big\|(w,\{h_{m,k}\}) - (\widetilde{w},\{\widetilde{h}_{m,k}\})\big\|.
\end{aligned}$$

Similarly we obtain

$$\gamma\|H_{m,k}-\widetilde{H}_{m,k}\|_{p,D} \leq \|T_{-m,n-k,D}\|_p (L_1+\gamma L_2)\big\|(w,\{h_{m,k}\}) - (\widetilde{w},\{\widetilde{h}_{m,k}\})\big\|$$

$$\|H_{\alpha,\beta}-\widetilde{H}_{\alpha,\beta}\|_p \leq \|T_{-\alpha,n-\beta,D}\|_p \Big(\frac{1}{\gamma}L_1 + L_2\Big)\big\|(w,\{h_{m,k}\}) - (\widetilde{w},\{\widetilde{h}_{m,k}\})\big\|$$

for $0<m+k<n$ and $\alpha+\beta=n$ with $(\alpha,\beta) \neq (0,n)$. On account of the relations

$$\|T_{-m,n-k,D}\| = \begin{cases} C(m,k,D) & \text{for } 0<m+k<n \\ \|\Pi_D\|_p & \text{for } m+k=n \end{cases} \qquad (1<p<\infty), \tag{8}$$

where Π_D is the strongly singular Vekua-type integral operator, we arrive at the estimate

$$\begin{aligned}&\big\|(W,\{H_{m,k}\})-(\widetilde{W},\{\widetilde{H}_{m,k}\})\big\|\\ &\quad\le\Big(\frac{1}{\gamma}L_1+L_2\Big)\max\Big\{\gamma\|T_{0,n,D}\|_p,\gamma\max_{m+k<n}\|T_{-m,n-k,D}\|_p,\|\Pi_D\|_p\Big\}\\ &\qquad\times\big\|(w,\{h_{m,k}\})-(\widetilde{w},\{\widetilde{h}_{m,k}\})\big\|\end{aligned}$$

and $\mathbb{P}$ is contractive in $\mathcal{L}_P(D)$ $(2<p<\infty)$ if

$$\Big(\frac{1}{\gamma}L_1+L_2\Big)\max\Big\{\gamma\|T_{0,n,D}\|_p,\ \gamma\max_{m+k<n}\|T_{-m,n-k,D}\|_p,\ \|\Pi_D\|_p\Big\}<1. \tag{9}$$

This condition may be satisfied if the constants L_1, L_2 and γ can be chosen properly and the domain D made sufficiently small. It is known (cf. [5, 6, 8, 12, 17, 18], for instance) that

$$\|\Pi_D\|_p\ge 1\quad(1<p<\infty)\qquad\text{and}\qquad\|\Pi_D\|_2=1.$$

Thus for a chosen $\mathcal{L}_p(D)$ $(2<p<\infty)$ we need L_2, $0<L_2<1$, such that $L_2\|\Pi_D\|_p<1$. Next we choose the constant $\gamma>0$ large enough so that, for the given $L_1>0$, $(\frac{1}{\gamma}L_1+L_2)\|\Pi_D\|_p<1$ also holds. Finally, since $\|T_{0,n,D}\|_p$ and $\|T_{-m,n-k,D}\|_p$, $0<m+k<n$, vary directly with the area of the domain D, we may satisfy estimate (9) eventually be reducing the size of D.

If estimate (9) is realized, then $\mathbb{P}$ has a unique fixed element $(\dot{w},\{\dot{h}_{m,k}\})\in\mathcal{L}_p(D)$ $(2<p<\infty)$ and $\dot{w}$ is the general solution of equation (1) corresponding to the given in D polyanalytic function $\Phi\in W_{n,p}(D)$ of order n. Moveover, $\dot{w}\in W_{n,p}(D)$ $(2<p<\infty)$:

$$\dot{w}(z)=\Phi(z)+T_{0,n,D}F\big(\zeta,\dot{w}(\zeta),\{\dot{h}_{m,k}(\zeta)\}\big)(z)$$

$$\dot{h}_{m,k}(z)=\frac{\partial^{m+k}\Phi}{\partial z^m\partial\overline{z}^k}+T_{-m,n-k,D}F\big(\zeta,\dot{w}(\zeta),\{\dot{h}_{m,k}(\zeta)\}\big)(z)$$

$$n\ge m,k\in\mathbb{N}_0, m+k\le n,(0,0)\ne(m,k)\ne(0,n),n\in\mathbb{N}.$$

Theorem 2. *Under assumptions* (A1) - (A3) *and* (9) *the non-homogeneous polyanalytic differential equation* (1) *admits a uniquely defined solution* $w\in W_{n,p}(D)$ $(2<p<\infty)$ *given by equation* (2) *for every prescribed in the domain* D *polyanalytic function* $\Phi\in W_{n,p}(D)$. *This solution defines a mapping from* $\Phi\longrightarrow w=R(\Phi)$.

3. The generalized Riemann-Hilbert problem for polyanalytic functions

We consider the following boundary value problem for a polyanalytic function Φ of order n:

$$\begin{aligned} \frac{\partial^n \Phi}{\partial \overline{z}^n} &= 0 \qquad \text{on } D = \{z : \ |z| < 1\} \\ \operatorname{Re}\left[(a_k + ib_k)\frac{\partial^{n-1}\Phi}{\partial x^{n-k}\partial y^{k-1}}\right](t) &= c_k(t) \qquad \text{on } \partial D \quad (k = 1, \dots, n) \end{aligned} \tag{10}$$

where $a_k, b_k, c_k \in W_{1-\frac{1}{p},p}(\partial D)$ $(2 < p < \infty)$ are prescribed real-valued functions on ∂D. Moreover, $(a_k + ib_k)(t) \neq 0$ for all $t \in \partial D$.

A polyanalytic function Φ of order n may be expressed as

$$\Phi = \Phi(z, \overline{z}) = \sum_{\rho=0}^{n-1} \overline{z}^{\rho} \varphi_\rho(z) \qquad (\varphi_\rho \ \text{ holomorphic}).$$

Thus

$$\begin{aligned} \frac{\partial^{n-1}\Phi}{\partial x^{n-k}\partial y^{k-1}} &= i^{k-1}\Big(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\Big)^{n-k}\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\Big)^{k-1} \sum_{\rho=0}^{n-1} \overline{z}^{\rho}\varphi_\rho(z) \\ &= i^{k-1} \sum_{\alpha=0}^{n-k}\sum_{\beta=0}^{k-1} (-1)^{\beta} \binom{n-k}{\alpha}\binom{k-1}{\beta} \\ &\quad \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho-\alpha-\beta)!}\, \overline{z}^{\rho-\alpha-\beta} \frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}} \varphi_\rho(z). \end{aligned}$$

Since on $\partial D\, \overline{t} = \frac{1}{t}$ we shall replace $\overline{z}$ by $\frac{1}{z}$ in the expression above and then reduce the given boundary conditions for the polyanalytic function Φ to n eqivalent Riemann-Hilbert boundary value problems for some holomorphic functions G_k $(k = 1, \dots, n)$ which are defined in terms of the holomorphic functions φ_ρ $(\rho = 0, \dots, n-1)$. Thus

$$\begin{aligned} \frac{\partial^{n-1}\Phi}{\partial x^{n-k}\partial y^{k-1}} &= i^{k-1} \sum_{\alpha=0}^{n-k}\sum_{\beta=0}^{k-1} (-1)^{\beta} \binom{n-k}{\alpha}\binom{k-1}{\beta} \\ &\quad \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho-\alpha-\beta)!}\, z^{\alpha+\beta-\rho} \frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}} \varphi_\rho(z). \end{aligned}$$

Hence

$$\begin{aligned} &\operatorname{Re}\left[(a_k + ib_k)\frac{\partial^{n-1}\Phi}{\partial x^{n-k}\partial y^{k-1}}\right](t) \\ &\quad = \operatorname{Re}\Bigg[(a_k + b_k)(t)\, t^{1-n} i^{k-1} \sum_{\alpha=0}^{n-k}\sum_{\beta=0}^{k-1} (-1)^{\beta} \binom{n-k}{\alpha}\binom{k-1}{\beta} \\ &\qquad \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho-\alpha-\beta)!}\, t^{n+\alpha+\beta-\rho-1} \frac{d^{n-\alpha-\beta-1}}{dt^{n-\alpha-\beta-1}}\, \varphi_\rho(t)\Bigg], \end{aligned}$$

i.e.

$$\mathrm{Re}\left[(a_k+ib_k)(t)\, i^{k-1}t^{1-n}G_k(t)\right]=c_k(t)\qquad (k=1,\ldots,n)\tag{11}$$

where

$$\begin{aligned}G_k(z)=&\sum_{\alpha=0}^{n-k}\sum_{\beta=0}^{k-1}(-1)^{\beta}\binom{n-k}{\alpha}\binom{k-1}{\beta}\\&\times\sum_{\rho=\alpha+\beta}^{n-1}\frac{\rho!}{(\rho-\alpha-\beta)!}\,z^{n+\alpha+\beta-\rho-1}\frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}}\varphi_\rho(z).\end{aligned}\tag{12}$$

The solution of the Riemann-Hilbert problem (11) is known (see [9, 11, 14, 16]). If $\kappa_k := \mathrm{index}\,[(a_k-ib_k),\partial D]\ge 0$, then the general solution G_k is given with the aid of the Schwarz integral as

$$z^{1-n}G_k(z)=\frac{X_k(z)}{2\pi i}\left[\int_{\partial D}\frac{i^{1-k}c_k(t)}{(a_k+ib_k)(t)\,X_k^+(t)}\,\frac{t+z}{(t-z)t}\,dt+P_{\kappa_k}(z)\right]\tag{13}$$

where P_{κ_k} is a polynomial of degree not exceeding κ_k and X_κ is the canonical solution of the corresponding homogeneous problem

$$\begin{aligned}X_k(z)&=z^{\kappa_k}\exp\Gamma_k(z)\\ \Gamma_k(z)&=\frac{1}{4\pi i}\int_{\partial D}\log\left((-1)^{k-2}t^{-2\kappa_k}\frac{(a_k-ib_k)(t)}{(a_k+ib_k)(t)}\right)\frac{t+z}{(t-z)t}\,dt\,.\end{aligned}$$

If any of the κ_k is negative, then the corresponding Riemann-Hilbert problem has a unique solution, bounded at infinity for instance, if and only if the conditions

$$\int_{\partial D}\frac{c_k(t)}{(a_k+ib_k)(t)\,X_k^+(t)}\,t^j\,dt=0\qquad (j=0,\ldots,-2\kappa_k-2)\tag{14}$$

are fulfilled, and in that case the solution is given by (13) as well, with the obvious modification that we set $P_{\kappa_k}(z)\equiv 0$ (cf. [9, 11, 16]).

We investigate the possibility for the satisfaction of the solvability conditions (14). For this purpose we consider the modified Riemann-Hilbert problem (cf. [4, 25])

$$\mathrm{Re}\left[i^{k-1}t^{1-n}(a_k+ib_k)(t)\,G_k(t)\right]=c_k(t)-\sum_{s=\kappa_k+1}^{-\kappa_k-1}\lambda_s t^s\qquad\text{on }\partial D\tag{11$'$}$$

where $\lambda_{-s}=\overline{\lambda_s}$ are constants yet to be determined appropriately. The modified problem is uniquely solvable for $\kappa_k<0$, and the solution G_k to the original Riemann-Hilbert problem (11) has the representation

$$z^{1-n}G_k(z)=\frac{X_k(z)}{2\pi i}\int_{\partial D}\frac{i^{1-k}}{(a_k+ib_k)(t)X_k^+(t)}\left(c_k(t-\sum_{s=\kappa_k+1}^{-\kappa_k-1}\lambda_s t^s\right)\frac{t+z}{(t-z)t}\,dt\tag{13$'$}$$

(cf. [4, 25]).

In order that a Riemann-Hilbert problem with non-negative index to be uniquely solvable $2\kappa_k + 1$ point conditions need to be imposed on the solution G_k. These conditions can be expressed in terms of the solution Φ of the given polyanalytic equation (10). Suppose r among the n Riemann-Hilbert problems (11) have non-negative indices, whose sum is N. Then, we demand that

$$\begin{gathered}\operatorname{Im}\left[i^{k-1}(a_k + ib_k)(\tau_j)\,\tau_j^{1-n} G_k(\tau_j)\right] = dj \quad (j = 1, 2, \ldots, N + r) \\ \tau_j \in \partial D, \tau_m \neq \tau_n \text{ for } m \neq n, d_j \in \mathbb{R}.\end{gathered} \tag{15}$$

It can be shown that $z^{1-n}G_k(z) \in W_{1,p}(0)$ $(2 < p < \infty, k = 1, \ldots, n)$ and estimates of the form

$$\left.\begin{aligned}\|z^{1-n}G_k\|_{p,D} &\leq C_k(a_k, b_k, p, D)\,\|c_k\|_{p,\partial D} \\ \|z^{1-n}G_k\|_{1,p,D} &\leq K_k(a_k, b_k, p, D)\,\|c_k\|_{1-\frac{1}{p},p,\partial D}\end{aligned}\right\} \quad (k = 1, \ldots, n; 2 < p < \infty) \tag{16}$$

hold (cf. [12, 14, 15]).

Suppose we have determined all n holomorphic functions G_k $(k = 1, \ldots, n)$ uniquely.■ We proceed to compute the required polyanalytic function Φ by expressing the holomorphic functions φ_p $(p = 0, \ldots, n-1)$ in terms of G_k. We shall make use of the following three facts:

1. Derivatives of Φ with respect to x, y can easily be expressed by the holomorphic functions G_k. It follows from (12) that

$$\begin{aligned}\frac{\partial^{n-j}\Phi}{\partial x^{n-q-\nu-j}\partial y^{q+\nu}} &= i^{q+\nu}\Big(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\Big)^{n-q-\nu-j}\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\Big)^{q+\nu}\sum_{\rho=0}^{n-1}\overline{z}^{\rho}\varphi_\rho(z) \\ &= i^{q+\nu}\sum_{\alpha=0}^{n-q-\nu-j}\sum_{\beta=0}^{q+\nu}(-1)^{\beta}\binom{n-q-\nu-j}{\alpha}\binom{q+\nu}{\beta} \\ &\qquad \times \sum_{\rho=\alpha+\beta}^{n-1}\frac{\rho!}{(\rho-\alpha-\beta)!}\,\overline{z}^{\rho-\alpha-\beta}\,\frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}}\,\varphi_\rho(z) \\ &= i^{q+\nu}z^{1-n}G_{q+\nu+j}(z)\end{aligned}$$

on ∂D (i.e. $\overline{z} = \frac{1}{z}$).

2. Derivatives of Φ with respect to $z, \overline{z}$ can be expressed by the derivatives with respect to x, y, and hence in terms of G_k. Indeed, on ∂D we have

$$\begin{aligned}\frac{\partial^{n-j}\Phi}{\partial z^{n-k}\partial \overline{z}^{k-j}} &= 2^{j-n}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)^{n-k}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)^{k-j}\Phi \\ &= 2^{j-n}\sum_{q=0}^{n-k}\sum_{\nu=0}^{k-j}(-1)^{q}i^{q+\nu}\binom{n-k}{q}\binom{k-j}{\nu}\frac{\partial^{n-j}\Phi}{\partial x^{n-q-\nu-j}\partial y^{q+\nu}} \\ &= 2^{j-n}\sum_{q=0}^{n-k}\sum_{\nu=0}^{k-j}(-1)^{q}i^{q+\nu}\binom{n-k}{q}\binom{k-j}{\nu}i^{q+\nu}z^{1-n}G_{q+\nu+j}(z) \\ &= 2^{j-n}z^{1-n}\sum_{q=0}^{n-k}\binom{n-k}{q}\sum_{\nu=0}^{k-j}(-1)^{\nu}\binom{k-j}{\nu}G_{q+\nu+j}(z)\end{aligned} \tag{17}$$

for $n \geq k \geq j$.

3. The holomorphic functions φ_ρ can be expressed by the derivatives of the poly-analytic function Φ with respect to z and $\overline{z}$.

Thus

$$\frac{\partial^{n-1}\Phi}{\partial\overline{z}^{n-1}} = \frac{\partial^{n-1}}{\partial\overline{z}^{n-1}} \sum_{\rho=0}^{n-1} \overline{z}^\rho \varphi_\rho(z) = (n-1)!\varphi_{n-1}(z).$$

On the other hand, it follows from (17), with $k = n$ and $j = 1$, that

$$\frac{\partial^{n-1}\Phi}{\partial\overline{z}^{n-1}} = (2z)^{1-n} \sum_{\nu=0}^{n-1} (-1)^\nu \binom{n-1}{\nu} G_{\nu+1}(z) \qquad \text{on } \partial D.$$

Hence we may conclude that

$$\varphi_{n-1}(z) = \frac{1}{(n-1)!}(2z)^{1-n} \sum_{\nu=0}^{n-1} (-1)^\nu \binom{n-1}{\nu} G_{\nu+1}(z) \qquad \text{on } \partial D.$$

Next we have, on the one hand,

$$\begin{aligned}\frac{\partial^{n-2}\Phi}{\partial\overline{z}^{n-2}} &= \frac{\partial^{n-2}}{\partial\overline{z}^{n-2}}\Big(\overline{z}^{n-2}\varphi_{n-2}(z) + \overline{z}^{n-1}\varphi_{n-1}(z)\Big) \\ &= (n-2)!\,\varphi_{n-2}(z) + (n-1)!\overline{z}\,\varphi_{n-1}(z).\end{aligned}$$

On the other hand, we deduce from (17), with $k = n$ and $j = 2$, that

$$\frac{\partial^{n-2}\Phi}{\partial\overline{z}^{n-2}} = 2^{2-n} z^{1-n} \sum_{\nu=0}^{n-2} (-1)^\nu \binom{n-2}{\nu} G_{\nu+2}(z) \qquad \text{on } \partial D.$$

So we can obtain for φ_{n-2} the representation

$$\varphi_{n-2}(z) = \frac{1}{(n-2)!}\left[2^{2-n} z^{1-n} \sum_{\nu=0}^{n-2} (-1)^\nu \binom{n-2}{\nu} G_{\nu+2}(z) - (n-1)!\,\overline{z}\,\varphi_{n-1}(z)\right]$$

on ∂D.

Similarly we compute $\varphi_{n-3}, \dots, \varphi_1, \varphi_0$. Suppose we have computed $\varphi_{n-1}, \varphi_{n-2}, \dots, \varphi_{n-j+1}$. Then we compute φ_{n-j} as

$$\frac{\partial^{n-j}\Phi}{\partial\overline{z}^{n-j}} = \frac{\partial^{n-j}}{\partial\overline{z}^{n-j}} \sum_{\rho=n-j}^{n-1} \overline{z}^\rho \varphi_\rho(z) = (n-j)!\varphi_{n-j}(z) + \sum_{\rho=n-j+1}^{n-1} \varphi_\rho(z) \frac{\partial^{n-j}\overline{z}^\rho}{\partial\overline{z}^{n-j}}.$$

On the other hand, for $k = n$ formula (17) yields

$$\frac{\partial^{n-j}\Phi}{\partial\overline{z}^{n-j}} = 2^{j-n} z^{1-n} \sum_{\nu=0}^{n-j} (-1)^\nu \binom{n-j}{\nu} G_{\nu+j}(z).$$

We thus arrive at the general representation

$$\varphi_{n-j}(z) = \frac{1}{(n-j)!}\left[2^{j-n} z^{1-n} \sum_{\nu=0}^{n-j} (-1)^{\nu} \binom{n-j}{\nu} G_{\nu+j}(z) - \sum_{\rho=n-j+1}^{n-1} \varphi_{\rho}(z) \frac{\partial^{n-j}\overline{z}^{\rho}}{\partial \overline{z}^{n-j}}\right]$$

for φ_{n-j} $(j = 1, \ldots, n)$. Hence all n holomorphic functions φ_j $(j = 0, \ldots, n-1)$ are uniquely determinable, and with them the polyanalytic function Φ as well. Furthermore, since $a_k, b_k, c_k \in W_{1-\frac{1}{p},p}(\partial D)$ $(2 < p < \infty)$, we conclude that $z^{1-n} G_k(z) \in W_{1,p}(D)$ (cf. [1, 10, 12, 14, 15, 19, 20]). It thus follows from (12) that

$$t^{1-n} G_1(t) = \sum_{\alpha=0}^{n-1} \binom{n-1}{\alpha} \sum_{j=\alpha}^{n-1} \frac{j!}{(j-\alpha)!} t^{\alpha-j} \frac{d^{n-\alpha-1}}{dt^{n-\alpha-1}} \varphi_j(t) \in W_{1-\frac{1}{p},p}(\partial D)$$

and, in particular,

$$\frac{d^{n-\alpha-1}}{dt^{n-\alpha-1}} \varphi_j(t) \in W_{1-\frac{1}{p},p}(\partial D) \qquad (j, \alpha = 0, \ldots, n-1).$$

Hence

$$\frac{d^{n-1}}{dt^{n-1}} \varphi_j \in W_{1-\frac{1}{p},p}(\partial D) \qquad (j = 0, 1, \ldots, n-1).$$

It now follows from the properties of traces of functions that

$$\varphi_j \in W_{n-\frac{1}{p},p}(\partial D) \quad \text{and} \quad \varphi_j,\ \Phi \in W_{n,p}(D) \qquad (j = 1, \ldots, n)$$

and the estimates

$$\left.\begin{aligned} \|\Phi\|_{p,D} &\le C_1(p, D) \max_k \|c_k\|_{p,\partial D} \\ \|\Phi\|_{j,p,D} &\le C_2(p, D) \max_k \|c_k\|_{1-\frac{1}{p},p,\partial D} \end{aligned}\right\} \qquad (j, k = 1, \ldots, n; 2 < p < \infty) \tag{18}$$

hold (cf. [1, 10, 12, 14, 15, 19, 20]).

4. The generalized Riemann-Hilbert problem for equation (1)

We now take up the following boundary value problem for the function w:

$$\frac{\partial^n w}{\partial \overline{z}^n} = F\left(z, w, \left\{\frac{\partial^{m+k} w}{\partial z^m \partial \overline{z}^k}\right\}\right) \quad \text{on} \quad D$$

$$\operatorname{Re}\left[(a_k + i b_k) \frac{\partial^{n-1} w}{\partial x^{n-k} \partial y^{k-1}}\right](t) = c_k(t) \quad \text{on} \quad \partial D \qquad (k = 1, \ldots, n) \tag{19}$$

$$n \ge m, k \in \mathbb{N}_0, m + k \le n, (0,0) \ne (m,k) \ne (0,n), n \in \mathbb{N}$$

where $a_k, b_k, c_k \in W_{1-\frac{1}{p},p}(\partial D)$ $(2 < p < \infty)$ are prescribed real-valued functions on ∂D with $(a_k + ib_k)(t) \neq 0$ for all $t \in \partial D$.

It was shown earlier that for every polyanalytic function $\Phi \in W_{n,p}(D)$ $(2 < p < \infty)$ there exists a unique solution $w \in W_{n,p}(D)$ to the partial differential equation (1). This solution is represented by (2). We shall now exploit the arbitrariness of the polyanalytic function Φ to construct the solution of the boundary value problem (1), (19). For this purpose we shall write Φ as

$$\Phi = \Phi_c + \Phi_{(w,h)}$$

where Φ_c, $\Phi_{(w,h)}$ are solutions of the boundary value problems

$$\begin{aligned}
&\operatorname{Re}\left[(a_j + ib_j)\,\frac{\partial^{n-1}\Phi_c}{\partial x^{n-j}\partial y^{j-1}}\right](t) = c_j(t) \quad \text{on } \partial D \\
&\operatorname{Re}\left[(a_j + ib_j)\,\frac{\partial^{n-1}}{\partial x^{n-j}\partial y^{j-1}}\,\Phi_{(w,h)}\right](t) \\
&\qquad = -\operatorname{Re}\left[(a_j + ib_j)\,\frac{\partial^{n-1}}{\partial x^{n-j}\partial y^{j-1}} T_{0,n,D}F(\cdot, w, \{h_{m,k}\})\right](t) \\
&\qquad := g_{w,h),j}(t) \quad \text{on } \partial D
\end{aligned} \tag{20}$$

for $j = 1, \ldots, n$. Since $F(z, w, \{h_{m,k}\}) \in L_p(D)$ $(2 < p < \infty)$, then $T_{0,n,D}F \in W_{n,p}(D)$ (cf. [5, 6]). Moreover,

$$\begin{aligned}
&\frac{\partial^{n-1}}{\partial x^{n-k}\partial y^{k-1}}\, T_{0,n,D}F(z) \\
&= i^{k-1}\sum_{\alpha=0}^{n-k}\binom{n-k}{\alpha}\sum_{\beta=0}^{k-1}(-1)^{\beta}\binom{k-1}{\beta}\frac{\partial^{n-1}}{\partial z^{n-\alpha-\beta-1}\partial \overline{z}^{\alpha+\beta}}\, T_{0,n,D}F(z) \\
&= i^{k-1}\sum_{\alpha=0}^{n-k}\binom{n-k}{\alpha}\sum_{\beta=0}^{k-1}(-1)^{\beta}\binom{k-1}{\beta} T_{\alpha+\beta+1-n,n-\alpha-\beta,D}F(z) \\
&\in W_{1,p}(D),
\end{aligned}$$

i.e. $g_{(w,h),j} \in W_{1-\frac{1}{p},p}(\partial D)$ $(2 < p < \infty; j = 1, \ldots, n)$ (cf. [1, 10, 11, 12, 19, 20]).

Polyanalytic functions which satisfy boundary conditions of the form (20) have been constructed earlier, and we deduce from there that Φ_c, $\Phi_{(w,h)} \in W_{n,p}(D)$ $(2 < p < \infty)$ and, in particular, the estimates

$$\begin{aligned}
\|\Phi_{(w,h)}\|_{p,D} &\le C(p, D)\,\|g_{(w,h)j}\|_{p,\partial D} \le C_1(p, D)\,\|T_{0,n,D}F\|_{1,p,D} \\
\|\Phi_{(w,h)}\|_{k,p,D} &\le C_2(p, D)\,\|T_{0,n,D}F\|_{n+1-k,p,D}
\end{aligned} \tag{21}$$

hold for $k, j = 1, \ldots, n$ and $2 < p < \infty$.

We now define a mapping $\mathbb{Q}$ in the Banach space $\mathcal{L}_p(D)$ $(2 < p < \infty)$. For any tuple $(w, \{h_{m,k}\}) \in \mathcal{L}_p(D)$ we set

$$(W, \{H_{m,k}\}) = \mathbb{Q}(w, \{h_{m,k}\})$$

where

$$W(z) = \Phi_c(z) + \Phi_{(w,h)}(z) + T_{0,n,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z)$$

$$H_{m,k}(z) = \frac{\partial^{m+k}}{\partial z^m \partial \overline{z}^k}\left(\Phi_c(z) + \Phi_{(w,h)}(z)\right) + T_{-m,n-k,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z)$$

$$n \geq m, k \in \mathbb{N}_0, m+k \leq n, (0,0) \neq (m,k) \neq (0,n), n \in \mathbb{N}.$$

The operator $\mathbb{Q}$ is uniquely defined, and it maps the Banach space $\mathcal{L}_p(D)$ $(2 < p < \infty)$ into itself. Moreover, the following result holds.

Theorem 3. *If $(w, \{h_{m,k}\})$ is a fixed point of the operator $\mathbb{Q}$, then w is the solution of the given differential equation (1) which also satisfies the boundary conditions (19).*

We next derive the conditions to be imposed in order that $\mathbb{Q}$ has a fixed point. Suppose $(W, \{H_{m,k}\}), (\widetilde{W}, \{\widetilde{H}_{m,k}\})$ are the respective images of $(w, \{h_{m,k}\})$, $(\widetilde{w}, \{\widetilde{h}_{m,k}\}) \in \mathcal{L}_p(D)$ $(2 < p < \infty)$. If we set

$$\varphi = \Phi_{(w,h)} - \Phi_{(\widetilde{w},\widetilde{h})} \qquad \text{and} \qquad f = F\big(z, w, \{h_{m,k}\}\big) - F\big(z, \widetilde{w}, \{\widetilde{h}_{m,k}\}\big),$$

then

$$W - \widetilde{W} = \varphi + T_{0,n,D}f, \qquad H_{m,k} - \widetilde{H}_{m,k} = \frac{\partial^{m+k}\varphi}{\partial z^m \partial \overline{z}^k} + T_{-m,n-k,D}F$$

and

$$\begin{aligned}
&\gamma\, \|W - \widetilde{W}\|_{p,D} \\
&\quad \leq \gamma\Big(C_1(p,D)\, \|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p\Big)\|f\|_{p,D} \\
&\quad \leq \bigg(L_1 \max\Big\{\max_{m+k<n} \|h_{m,k} - \widetilde{h}_{m,k}\|_{p,D}, \|w - \widetilde{w}\|_{p,D}\Big\} \\
&\qquad + L_2 \max_{m+k=n} \|h_{m,k} - \widetilde{h}_{m,k}\|_{p,D}\bigg)\gamma\Big(C_1(p,D)\, \|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p\Big) \\
&\quad \leq \Big(C_1(p,D)\, \|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p\Big)(L_1 + \gamma L_2)\big\|(w, \{h_{m,k}\}) - (\widetilde{w}, \{\widetilde{h}_{m,k}\})\big\|.
\end{aligned}$$

Similary we arrive at

$$\begin{aligned}
\gamma\, \|H_{m,k} - \widetilde{H}_{m,k}\|_{p,D} &\leq \Big(C_2(p,D)\, \|T_{0,n,D}\|_{n-m-k,p} + \|T_{-m,n-k,D}\|_p\Big) \\
&\qquad \times (L_1 + \gamma L_2)\, \big\|(w, \{h_{m,k}\}) - (\widetilde{w}, \{\widetilde{h}_{m,k}\})\big\|
\end{aligned}$$

and

$$\begin{aligned}
\|H_{\alpha,\beta} - \widetilde{H}_{\alpha,\beta}\|_{p,D} &\leq \Big(C_3(p,D)\, \|T_{0,n,D}\|_{1,p} + \|T_{-\alpha,n-\beta,D}\|_p\Big) \\
&\qquad \times \Big(\frac{1}{\gamma}L_1 + L_2\Big)\big\|(w, \{h_{m,k}\}) - (\widetilde{w}, \{\widetilde{h}_{m,k}\})\big\|
\end{aligned}$$

for $0 < m+k < n, \alpha + \beta = n$ and $(\alpha, \beta) \neq (0, n)$. Consequently, on account of relations (8), we arrive at the estimate

$$\|(W, \{H_{m,k}\}) - (\widetilde{W}, \{\widetilde{H}_{m,k}\})\|_p \leq \kappa\, \|(w, \{h_{m,k}\}) - (\widetilde{w}, \{\widetilde{h}_{m,k}\})\| \tag{22}$$

where $\left(\frac{1}{\gamma}L_1 + L_2\right)^{-1}\kappa$ is the maximum of the three quantities

$$\gamma\left(C_1(p,D)\,\|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p\right)$$
$$\gamma \max_{m+k<n}\left\{C_2(p,D)\,\|T_{0,n,D}\|_{n-m-k+1,p} + \|T_{-m,n-k,D}\|_p\right\}$$
$$C_3(p,D)\,\|T_{0,n,D}\|_{1,p} + \|\Pi_D\|_p$$

If $\kappa < 1$, then the mapping $\mathbb{Q}$ is contractive in $\mathcal{L}_p(D)$ $(2 < p < \infty)$ and it has therefore exactly one fixed element $(w, \{h_{m,k}\}) \in \mathcal{L}_p(D)$, by the Banach fixed point theorem.

The contractiveness of $\mathbb{Q}$ imposes certain restrictions on the constants L_1, L_2, γ and the size of the domain D. Going through an argument similar to the one presented earlier for the case of the existence of a general solution, we can secure the contractiveness of $\mathbb{Q}$, and hence the existence of a solution $w \in W_{n,p}(D)$ $(2 < p < \infty)$ of the boundary value problem posed. It is easy to establish that the solution is unique.

Theorem 4. *Under the assumptions* (A1) - (A3), (15) *and* $\kappa < 1$ *the generalized Riemann-Hilbert boundary value problem* (1), (19) *admits a unique solution* $w \in W_{n,p}(D)$ $(2 < p < \infty)$.

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