

# An Integral Operator Representation of Classical Periodic Pseudodifferential Operators

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**Abstract.** In this note we prove that every classical 1-periodic pseudodifferential operator  $A$  of order  $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$  can be represented in the form

$$(Au)(t) = \int_0^1 \left[ \kappa_\alpha^+(t-s)a_+(t,s) + \kappa_\alpha^-(t-s)a_-(t,s) + a(t,s) \right] u(s) ds$$

where  $\alpha_\pm$  and  $a$  are  $C^\infty$ -smooth 1-periodic functions and  $\kappa_\alpha^\pm$  are 1-periodic functions or distributions with Fourier coefficients  $\hat{\kappa}_\alpha^+(n) = |n|^\alpha$  and  $\hat{\kappa}_\alpha^-(n) = |n|^\alpha \text{sign}(n)$  ( $0 \neq n \in \mathbb{Z}$ ) with respect to the trigonometric orthonormal basis  $\{e^{in2\pi t}\}_{n \in \mathbb{Z}}$  of  $L^2(0,1)$ . Some explicit formulae for  $\kappa_\alpha^\pm$  are given. The case of operators of order  $\alpha \in \mathbb{N}_0$  is discussed, too.

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## 1. Periodic pseudodifferential operators

By  $H^\lambda$  ( $\lambda \in \mathbb{R}$ ) we denote the Sobolev space of 1-periodic functions or distributions  $u$  having a finite norm

$$\|u\|_\lambda = \left( \sum_{n \in \mathbb{Z}} \underline{n}^{2\lambda} |\hat{u}(n)|^2 \right)^{\frac{1}{2}}$$

where

$$\hat{u}(n) = \int_0^1 u(t) e^{-in2\pi t} dt = \langle u, e^{-in2\pi t} \rangle$$

are the Fourier coefficients of  $u$  and  $\underline{n} = \max\{1, |n|\}$ . As usual,  $\mathcal{L}(H^\lambda, H^\mu)$  denotes the space of linear bounded operators from  $H^\lambda$  into  $H^\mu$ . Every operator  $A \in \mathcal{L}(H^\lambda, H^\mu)$  is of the form

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$$u(t) = \sum_{n \in \mathbb{Z}} \hat{u}(n)e^{in2\pi t} \mapsto (Au)(t) = \sum_{n \in \mathbb{Z}} \sigma(t, n)\hat{u}(n)e^{in2\pi t} \tag{1}$$

(and one writes  $A = Op \sigma$ ) where

$$\sigma(t, n) = e^{-in2\pi t} A e^{in2\pi t}$$

is called the *symbol* of  $A$ . Indeed, the Fourier series

$$u(t) = \sum_{n \in \mathbb{Z}} \hat{u}(n)e^{in2\pi t}$$

of  $u \in H^\lambda$  converges in  $H^\lambda$ , therefore the series

$$(Au)(t) = \sum_{n \in \mathbb{Z}} \hat{u}(n) A e^{in2\pi t} = \sum_{n \in \mathbb{Z}} \hat{u}(n)\sigma(t, n)e^{in2\pi t}$$

converges in  $H^\mu$ . Clearly,  $\sigma(t, n)$  is 1-periodic in  $t$ .

A complex-valued function

$$\sigma = \sigma(t, n) \quad (t \in \mathbb{R}, n \in \mathbb{Z})$$

is called a *periodic symbol of degree  $\alpha$*  ( $\alpha \in \mathbb{R}$ ), denoted  $\sigma \in \Sigma^\alpha$ , if it is  $C^\infty$ -smooth and 1-periodic in  $t$  and satisfies the inequalities

$$\left| \left( \frac{\partial}{\partial t} \right)^j \Delta_n^k \sigma(t, n) \right| \leq c_{jk} n^{\alpha-k} \quad (j, k \in \mathbb{N}_0, t \in \mathbb{R}, n \in \mathbb{Z}). \tag{2}$$

Here  $\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}$ , and  $\Delta$  is the (forward) difference operator:

$$(\Delta\psi)(n) = \psi(n + 1) - \psi(n)$$

for  $\psi : \mathbb{Z} \rightarrow \mathbb{C}$ . An operator  $A = Op \sigma$  of form (1) with  $\sigma \in \Sigma^\alpha$  is called a *periodic pseudodifferential operator of order  $\alpha$* , denoted  $A \in Op \Sigma^\alpha$ . This definition originates from [1, 2]. Equivalent definitions can be found in [2 - 4, 12]. It occurs (see, e.g., [12]) that  $A \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$  for any  $\lambda \in \mathbb{R}$  if  $A \in Op \Sigma^\alpha$ .

Introduce a  $C^\infty$ -smooth function  $h : \mathbb{R} \rightarrow \mathbb{R}$  satisfying

$$h(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 0 \neq k \in \mathbb{Z} \end{cases}$$

$$h \in S(\mathbb{R}), \quad \text{i.e.} \quad \sup_{\xi \in \mathbb{R}} |\xi^j h^{(k)}(\xi)| < \infty \quad (j, k \in \mathbb{N}_0)$$

$$\forall k \in \mathbb{N} \exists h_k \in S(\mathbb{R}) \quad \text{such that} \quad h^{(k)}(\xi) = (\Delta^k h_k)(\xi) \quad (\xi \in \mathbb{R})$$

(see [12] for a construction of  $h$ ). The formula

$$\sigma(t, \xi) = \sum_{n \in \mathbb{Z}} \sigma(t, n)h(\xi - n) \quad (\xi \in \mathbb{R})$$

defines a prolongation  $\sigma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$  of  $\sigma \in \Sigma^\alpha$ . It occurs that (2) implies the inequalities

$$\left| \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial \xi} \right)^k \sigma(t, \xi) \right| \leq c'_{jk} (1 + |\xi|)^{\alpha-k} \quad (j, k \in \mathbb{N}_0; t, \xi \in \mathbb{R}). \tag{3}$$

Indeed,

$$\Delta_\xi h(\xi - n) = h(\xi + 1 - n) - h(\xi - n) = -\overline{\Delta}_n h(\xi - n)$$

where  $\overline{\Delta}$  is the backward difference operator, thus

$$\begin{aligned} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial \xi} \right)^k \sigma(t, \xi) &= \sum_{n \in \mathbb{Z}} \left( \frac{\partial}{\partial t} \right)^j \sigma(t, n) h^{(k)}(\xi - n) \\ &= \sum_{n \in \mathbb{Z}} \left( \frac{\partial}{\partial t} \right)^j \sigma(t, n) \Delta_\xi^k h_k(\xi - n) \\ &= (-1)^k \sum_{n \in \mathbb{Z}} \left( \frac{\partial}{\partial t} \right)^j \sigma(t, n) \overline{\Delta}_n^k h_k(\xi - n) \\ &= \sum_{n \in \mathbb{Z}} h_k(\xi - n) \left( \frac{\partial}{\partial t} \right)^j \Delta_n^j \sigma(t, n) \end{aligned}$$

(summation by parts on the last step). Since  $h_k \in \mathcal{S}(\mathbb{R})$  we have  $|h_k(\xi - n)| \leq c_r (1 + |\xi - n|)^{-r}$  with any  $r > 0$ ; we take  $r > |\alpha - k| + 1$ . Due to (2), we obtain (3):

$$\left| \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial \xi} \right)^k \sigma(t, \xi) \right| \leq c_r c_{jk} \sum_{n \in \mathbb{Z}} (1 + |\xi - n|)^{-r} \underline{n}^{\alpha-k} \leq c'_{jk} (1 + |\xi|)^{\alpha-k}.$$

The converse is also true: if  $\sigma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$  satisfies (3), then its restriction to  $\mathbb{R} \times \mathbb{Z}$  satisfies (2). Thus we may assume that the symbol  $\sigma \in \Sigma^\alpha$  is defined and  $C^\infty$ -smooth on  $\mathbb{R} \times \mathbb{R}$ , 1-periodic in  $t$  and satisfies (3). Nevertheless, only the values on  $\mathbb{R} \times \mathbb{Z}$  of  $\sigma$  are used to define  $A = Op \sigma \in Op \Sigma^\alpha$ .

A symbol  $\sigma \in \Sigma^\alpha$  is called *classical* or *polyhomogeneous*, denoted  $\sigma \in \Sigma_{cl}^\alpha$ , if it admits an asymptotic expansion

$$\sigma(t, \xi) \sim \sum_{j=0}^{\infty} \sigma_j(t, \xi), \quad \text{i.e. } \sigma - \sum_{j=0}^{N-1} \sigma_j \in \Sigma^{\alpha-N} \quad (N \in \mathbb{N}) \tag{4}$$

where  $\sigma_j \in \Sigma^{\alpha-j}$  are positively homogeneous of degree  $\alpha - j$  in  $\xi$  for  $|\xi| \geq 1$ :

$$\sigma_j(t, \tau\xi) = \tau^{\alpha-j} \sigma_j(t, \xi) \quad (|\xi| \geq 1, \tau \geq 1).$$

Clearly,

$$\sigma_j(t, \xi) = \begin{cases} \sigma_j(t, 1)\xi^{\alpha-j} =: a_j^+(t)\xi^{\alpha-j} & \text{for } \xi \geq 1 \\ \sigma_j(t, -1)|\xi|^{\alpha-j} =: a_j^-(t)|\xi|^{\alpha-j} & \text{for } \xi \leq -1 \end{cases} \tag{5}$$

with  $a_j^\pm \in C_1^\infty(\mathbb{R})$  where  $C_1^\infty(\mathbb{R})$  denotes the set of 1-periodic  $C^\infty$ -smooth functions on  $\mathbb{R}$ . The corresponding  $A = Op\sigma$  is called a *classical periodic pseudodifferential operator* of order  $\alpha$ , denoted  $A \in Op\Sigma_{cl}^\alpha$ . It follows from (4) and (5) that

$$A \sim \sum_{j=0}^\infty A_j, \quad \text{i.e. } A - \sum_{j=0}^{N-1} A_j \in Op\Sigma^{\alpha-N} \quad (N \in \mathbb{N})$$

where

$$\begin{aligned} A_j &= [a_j^+(t)P^+ + a_j^-(t)P^-]L^{\alpha-j} \sim Op\sigma_j \\ P^+u &= \sum_{n \geq 0} \hat{u}(n)e^{in2\pi t} \\ P^-u &= \sum_{n < 0} \hat{u}(n)e^{in2\pi t} \\ L^\lambda u &= \sum_{n \in \mathbb{Z}} \underline{n}^\lambda \hat{u}(n)e^{in2\pi t} \quad (\lambda \in \mathbb{R}). \end{aligned}$$

Let us comment on the polyhomogeneity of a symbol. It occurs that a symbol  $\sigma \in \Sigma^\alpha$  belongs to  $\Sigma_{cl}^\alpha$  if and only if  $|\xi|^{-\alpha}\sigma(t, \xi)$  behaves in a regular manner as  $\xi \rightarrow \pm\infty$ , or equivalently,  $\sigma_*(t, \eta) = |\eta|^\alpha\sigma(t, \frac{1}{\eta})$  with  $\eta = \frac{1}{\xi}$  behaves in a regular manner as  $\eta \rightarrow \pm 0$ . Namely, if  $\sigma_*$  has  $C^\infty$ -smooth continuations to  $\eta = +0$  and  $\eta = -0$ , then the Taylor expansions

$$\begin{aligned} \sigma_*(t, \eta) &= \sum_{j=0}^{N-1} a_j^+(t)\eta^j + \mathcal{O}(\eta^N) \quad (\eta \rightarrow +0, a_j^+(t) = \frac{1}{j!}(\frac{\partial}{\partial \eta})^j \sigma_*(t, \eta)|_{\eta=+0}) \\ \sigma_*(t, \eta) &= \sum_{j=0}^{N-1} a_j^-(t)\eta^j + \mathcal{O}(\eta^N) \quad (\eta \rightarrow -0, a_j^-(t) = \frac{1}{j!}(\frac{\partial}{\partial \eta})^j \sigma_*(t, \eta)|_{\eta=-0}) \end{aligned}$$

hold true for all  $N \in \mathbb{N}$ . Returning to  $\xi = \frac{1}{\eta}$  and  $\sigma(t, \xi) = |\xi|^\alpha\sigma_*(t, \xi^{-1})$  we have

$$\begin{aligned} \sigma(t, \xi) &= \sum_{j=0}^{N-1} a_j^+(t)|\xi|^{\alpha-j} + \mathcal{O}(\xi^{\alpha-N}) \quad (\xi \rightarrow +\infty) \\ \sigma(t, \xi) &= \sum_{j=0}^{N-1} a_j^-(t)|\xi|^{\alpha-j} + \mathcal{O}(\xi^{\alpha-N}) \quad (\xi \rightarrow -\infty) \end{aligned}$$

and it can be checked that by those  $a_\pm$  the asymptotic expansion (4) - (5) is defined.

## 2. Integral operator representation of periodic pseudodifferential operators

Here we follow some ideas from [3, 5, 8 - 10]. For  $A \in Op\Sigma^\alpha$  ( $\alpha < -1$ ) and  $u \in H^0 = L^2(0, 1)$  we have

$$\begin{aligned} (Au)(t) &= \sum_{n \in \mathbb{Z}} \sigma(t, n) \hat{u}(n) e^{in2\pi t} \\ &= \sum_{n \in \mathbb{Z}} \sigma(t, n) e^{in2\pi t} \int_0^1 u(s) e^{-in2\pi s} ds \\ &= \int_0^1 \sum_{n \in \mathbb{Z}} \sigma(t, n) e^{in2\pi(t-s)} u(s) ds \\ &= \int_0^1 \mathcal{K}(t, t-s) u(s) ds \end{aligned}$$

where the series

$$\mathcal{K}(t, s) = \sum_{n \in \mathbb{Z}} \sigma(t, n) e^{in2\pi s}$$

converges uniformly in  $t, s \in \mathbb{R}$  due to the estimate  $|\sigma(t, n)| \leq c_{00} \underline{n}^\alpha$  (see (2)). Thus  $\mathcal{K}(t, s)$  is continuous on  $\mathbb{R} \times \mathbb{R}$ . Moreover,  $\mathcal{K}(t, s)$  is  $C^\infty$ -smooth for  $s \in \mathbb{R} \setminus \mathbb{Z}$  (and then  $\mathcal{K}(t, t-s)$  is  $C^\infty$ -smooth for  $t-s \notin \mathbb{Z}$ ). Indeed, consider the product

$$\begin{aligned} (e^{-i2\pi s} - 1)\mathcal{K}(t, s) &= \sum_{n \in \mathbb{Z}} \sigma(t, n) (e^{i(n-1)2\pi s} - e^{in2\pi s}) \\ &= \sum_{n \in \mathbb{Z}} [\sigma(t, n+1) - \sigma(t, n)] e^{in2\pi s} \\ &= \sum_{n \in \mathbb{Z}} [\Delta\sigma(t, n)] e^{in2\pi s}. \end{aligned}$$

Repeating the multiplications by  $(e^{-i2\pi s} - 1)$  we obtain

$$(e^{-i2\pi s} - 1)^l \mathcal{K}(t, s) = \sum_{n \in \mathbb{Z}} [\Delta^l \sigma(t, n)] e^{in2\pi s} \quad (l \in \mathbb{N}).$$

Now estimate (2) yields that  $(e^{-i2\pi s} - 1)^l \mathcal{K}(t, s)$  is  $l$ -times continuously differentiable on  $\mathbb{R} \times \mathbb{R}$ . Since  $l$  is arbitrary,  $\mathcal{K}(t, s)$  is infinitely smooth for  $(t, s)$  satisfying  $e^{-i2\pi s} - 1 \neq 0$ , i.e. for  $s \in \mathbb{R} \setminus \mathbb{Z}$ . Also the case  $\alpha \in [-1, 0)$  can be treated, but then  $\mathcal{K}(t, t-s)$  is weakly singular for  $t = s$ .

For  $u \in H^l$  ( $l \in \mathbb{N}_0$ ) integration by parts yields

$$\hat{u}(n) = \int_0^1 u(s) e^{-in2\pi s} ds = \frac{1}{(2\pi in)^l} \int_0^1 u^{(l)}(s) e^{-in2\pi s} ds$$

and

$$(Au)(t) = \int_0^1 \mathcal{K}_l(t, t-s)u^{(l)}(s) ds, \quad \mathcal{K}_l(t, s) = \sum_{N \in \mathbb{Z}} \frac{\sigma(t, n)}{(2\pi in)^l} e^{in2\pi s}.$$

Now already for  $\sigma \in \Sigma^\alpha$  with  $\alpha < l - 1$ , the series converges uniformly and defines a continuous kernel  $\mathcal{K}_l$  on  $\mathbb{R} \times \mathbb{R}$ ; for  $s \in \mathbb{R} \setminus \mathbb{Z}$ ,  $\mathcal{K}_l(t, s)$  is again  $C^\infty$ -smooth.

One can try to represent  $\mathcal{K}$  and  $\mathcal{K}_l$  in the form of products

$$\mathcal{K}(t, t-s) = a(t, s)\kappa(t-s) \quad \text{and} \quad \mathcal{K}_l(t, t-s) = a_l(t, s)\kappa_l(t-s),$$

respectively, where  $a$  and  $a_l$  are  $C^\infty$ -smooth on the whole  $\mathbb{R} \times \mathbb{R}$  whereas  $\kappa$  and  $\kappa_l$  are  $C^\infty$ -smooth on  $\mathbb{R} \setminus \mathbb{Z}$ . With some specifications we shall succeed in the case of classical periodic pseudodifferential operators. For a general (non-classical) periodic pseudodifferential operator a similar representations does not exist.

We point out also the following inverse result from [11].

**Theorem 1.** *An integral operator defined by*

$$(Au)(t) = \int_0^1 \kappa(t-s)a(t,s)u(s) ds$$

with a  $C^\infty$ -smooth 1-biperiodic function  $a$  and 1-periodic function or distribution  $\kappa$  belongs to  $Op\Sigma^\alpha$  if  $\kappa$  satisfies

$$|\Delta^k \hat{\kappa}(n)| \leq c_k \underline{n}^{\alpha-k} \quad (k \in \mathbb{N}_0, n \in \mathbb{Z})$$

or, equivalently, if the extended function  $\hat{\kappa} : \mathbb{R} \rightarrow \mathbb{C}$  (defined by  $\hat{\kappa}(\xi) = \sum_{n \in \mathbb{Z}} \hat{\kappa}(n)h(\xi - n)$ ) or in some other way) satisfies

$$\left| \left( \frac{d}{d\xi} \right)^k \hat{\kappa}(\xi) \right| \leq c'_k (1 + |\xi|)^{\alpha-k} \quad (k \in \mathbb{N}_0, \xi \in \mathbb{R}).$$

Thereby  $A$  has asymptotic expansions  $A \sim \sum_{j=0}^\infty A_j$  with

$$(A_j u)(t) = a_j(t) \int_0^1 \kappa_j(t-s)u(s) ds = a_j(t) \sum_{n \in \mathbb{Z}} \hat{\kappa}_j(n) \hat{u}(n) e^{in2\pi t}$$

where

$$\left. \begin{aligned} \hat{\kappa}_j(n) &= \frac{1}{j!} \Delta^j \hat{\kappa}(n) \quad (n \in \mathbb{Z}) \\ a_j(t) &= \left( \frac{1}{2\pi i} \frac{\partial}{\partial s} - (j-1) \right) \left( \frac{1}{2\pi i} \frac{\partial}{\partial s} - (j-2) \right) \dots \left( \frac{1}{2\pi i} \frac{\partial}{\partial s} - 1 \right) \frac{1}{2\pi i} \frac{\partial}{\partial s} a(t, s) \Big|_{s=t} \end{aligned} \right\}$$

respectively

$$\left. \begin{aligned} \hat{\kappa}_j(n) &= \frac{1}{j!} \left( \frac{d}{d\xi} \right)^j \hat{\kappa}(\xi) \Big|_{\xi=n} \quad (n \in \mathbb{Z}) \\ a_j(t) &= \left( \frac{1}{2\pi i} \frac{\partial}{\partial s} \right)^j a(t, s) \Big|_{s=t}. \end{aligned} \right\}$$

### 3. Integral operator representation of classical periodic pseudodifferential operators

Here we first formulate and at the end prove the main results of the paper.

**Theorem 2.** *Every operator  $A \in Op\Sigma_{cl}^\alpha$  with  $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$  can be represented in the form*

$$(Au)(t) = \int_0^1 \left[ \kappa_\alpha^+(t-s)a_+(t,s) + \kappa_\alpha^-(t-s)a_-(t,s) + a(t,s) \right] u(s) ds \tag{6}$$

where  $a_\pm, a \in C_1^\infty(\mathbb{R} \times \mathbb{R})$ , i.e.  $a_\pm, a$  are  $C^\infty$ -smooth and 1-periodic with respect to both arguments and  $\kappa_\alpha^\pm$  are 1-periodic functions or distributions defined by their Fourier coefficients

$$\left. \begin{aligned} \hat{\kappa}_\alpha^+(n) &= |n|^\alpha \\ \hat{\kappa}_\alpha^-(n) &= |n|^\alpha \text{sign}(n) \end{aligned} \right\} \quad (0 \neq n \in \mathbb{Z}). \tag{7}$$

Conversely, every integral operator of form (6) – (7) with  $a_\pm, a \in C_1^\infty(\mathbb{R} \times \mathbb{R})$  belongs to  $Op\Sigma_{cl}^\alpha$ .

Note that (7) define  $\kappa_\alpha^\pm$  uniquely up to a constant addend  $\hat{\kappa}_\alpha^\pm(0)$ . Changing  $\hat{\kappa}_\alpha^\pm(0)$ , only the coefficient  $a \in C_1^\infty(\mathbb{R} \times \mathbb{R})$  changes in (6). In (6), the integral means the usual Lebesgue integral for  $\alpha < 0$  and  $u \in H^0 = L^2(0,1)$ . For  $\alpha \geq 0$  and  $u \in H^\mu$  ( $\mu > \alpha + \frac{1}{2}$ ) the integral can be understood as the dual product between  $H^\mu$  and  $H^{-\mu}$ , since  $\kappa_\alpha^\pm \in H^{-\mu}$ . The case of  $u \in H^\lambda$  with an arbitrary  $\lambda \in \mathbb{R}$  can be understood through the approximation of  $u$  by smooth functions, e.g.

$$Au = \lim_{N \rightarrow \infty} AP_Nu, \quad P_Nu = \sum_{|n| \leq N} \hat{u}(n)e^{in2\pi t}.$$

Here  $P_Nu \rightarrow u$  in  $H^\lambda$  and  $AP_Nu \rightarrow Au$  in  $H^{\lambda-\alpha}$  for  $u \in H^\lambda$  (recall that  $A \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$  for any  $\lambda \in \mathbb{R}$  if  $A \in Op\Sigma^\alpha$ ).

Now consider the case  $\alpha \in \mathbb{N}_0$  excluded from Theorem 2.

**Theorem 3.** *Every operator  $A \in Op\Sigma_{cl}^\alpha$  with  $\alpha = m \in \mathbb{N}_0$  has the representation*

$$\begin{aligned} (Au)(t) &= \sum_{j=0}^m \left[ c_j^+(t)u^{(m-j)}(t) + c_j^-(t)(H_0u^{(m-j)})(t) \right] \\ &+ \int_0^1 \left[ \kappa_{-1}^+(t-s)a_+(t,s) + \kappa_{-1}^-(t-s)a_-(t,s) + a(t,s) \right] u(s) ds \end{aligned} \tag{8}$$

where  $c_j^\pm \in C_1^\infty(\mathbb{R})$  and  $a_\pm, a \in C_1^\infty(\mathbb{R} \times \mathbb{R})$ ,  $H_0$  is the Hilbert transformation

$$(H_0u)(t) = \frac{1}{i} \text{p.v.} \int_0^1 \cot \pi(s-t)u(s) ds = \sum_{n \geq 1} \hat{u}(n)e^{in2\pi t} - \sum_{n \leq -1} \hat{u}(n)e^{in2\pi t},$$

and

$$\kappa_{-1}^+(t) = -2 \log |\sin \pi t| \tag{9}$$

$$\kappa_{-1}^- \text{ is the 1-periodic extension of } t \mapsto -2\pi it \text{ from } [0, 1) \text{ to } \mathbb{R} \tag{10}$$

(these functions satisfy (7) with  $\alpha = -1$ ).

**Remark 1.** Using the periodic Dirac delta function and its derivatives one can also (8) represent as an integral operator.

**Remark 2.** Clearly,  $\Sigma_{cl}^\alpha \subset \Sigma_{cl}^{\alpha+1}$ , therefore we actually have different possible integral operator representations of an operator  $A \in Op \Sigma_{cl}^\alpha$ . For instance,  $A \in Op \Sigma_{cl}^{-m}$  with an  $m \in \mathbb{N}$  can be represented in the form (6) with  $\alpha = -1$  and  $\kappa_{-1}^\pm$  defined in (9) - (10); the order  $-m$  of the operator can be discovered by properties of the coefficients  $a_\pm$ :

$$\left(\frac{\partial}{\partial s}\right)^j a_\pm(t, s) \Big|_{s=t} = 0 \quad (t \in \mathbb{R}; j = 0, \dots, m - 2).$$

Operators of type (6) have been examined in [6, 12]. They often appear solving boundary integral equations on closed curves (see, e.g., [5 - 7, 12]).

**Proof of Theorem 2.** Let  $A \in Op \Sigma_{cl}^\alpha$ , i.e. its symbol  $\sigma(t, \xi)$  has the asymptotic expansion (4) with  $\sigma_j$  of form (5). We regularize the functions  $|\xi|^\beta$  in the neighbourhood of  $\xi = 0$  putting

$$\left. \begin{aligned} \phi_\beta(0) &= 0 \\ \phi_\beta(\xi) &= \phi_0(\xi)|\xi|^\beta \quad (\xi \in \mathbb{R} \setminus \{0\}) \end{aligned} \right\}$$

where  $\phi_0 \in C^\infty(\mathbb{R})$  satisfies

$$\phi_0(\xi) = \begin{cases} 1 & \text{for } |\xi| \geq 1 \\ 0 & \text{for } |\xi| \leq \frac{1}{2}. \end{cases}$$

Thus we have

$$\sigma(t, \xi) \sim \sum_{j=0}^\infty \left[ b_j^+(t) \phi_{\alpha-j}(\xi) + b_j^-(t) \phi_{\alpha-j}(\xi) \text{sign}(\xi) \right] \tag{11}$$

where

$$b_j^+(t) = \frac{1}{2}[a_j^+(t) + a_j^-(t)] \quad \text{and} \quad b_j^-(t) = \frac{1}{2}[a_j^+(t) - a_j^-(t)].$$

On the other hand, by Theorem 1 the integral operator defined in (6) is a (classical) periodic pseudodifferential operator with the symbol  $\tilde{\sigma}$  having the asymptotical expansion

$$\begin{aligned} \tilde{\sigma}(t, \xi) \sim & \sum_{j=0}^\infty \frac{1}{j!} \left(\frac{\partial}{\partial \xi}\right)^j \phi_\alpha(\xi) \left(\frac{1}{2\pi i} \frac{\partial}{\partial s}\right)^j a_+(t, s) \Big|_{s=t} \\ & + \sum_{j=0}^\infty \frac{1}{j!} \left(\frac{\partial}{\partial \xi}\right)^j \phi_\alpha(\xi) \text{sign}(\xi) \left(\frac{1}{2\pi i} \frac{\partial}{\partial s}\right)^j a_-(t, s) \Big|_{s=t}. \end{aligned}$$



Representation (6) of  $A \in Op \Sigma_{cl}^\alpha$  takes place if  $\sigma \sim \tilde{\sigma}$ , i.e.  $\sigma - \tilde{\sigma} \in \Sigma^{-\infty}$ . For  $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$  this means that  $a_\pm \in C_1^\infty(\mathbb{R} \times \mathbb{R})$  satisfy

$$\frac{\alpha(\alpha - 1) \cdots (\alpha - j + 1)}{j!} \left( \frac{1}{2\pi i} \frac{\partial}{\partial s} \right)^j a_\pm(t, s) \Big|_{s=t} = b_j^\pm(t) \quad (t \in \mathbb{R}, j \in \mathbb{N}_0).$$

Thus, to prove Theorem 2, we simply have to solve the following elementary problem: given  $b_j \in C_1^\infty(\mathbb{R})$  ( $j \in \mathbb{N}_0$ ), construct  $a \in C_1^\infty(\mathbb{R} \times \mathbb{R})$  such that

$$\left( \frac{\partial}{\partial s} \right)^j a(t, s) \Big|_{s=t} = b_j(t) \quad (t \in \mathbb{R}, j \in \mathbb{N}_0).$$

A solution may be given by a regularization and periodization of the Taylor series:

$$a(t, s) = \sum_{l=0}^\infty \frac{b_l(t)}{l!} [\chi(s - t)]^l \psi_{N_l}(s - t).$$

Here  $\chi \in C_1^\infty(\mathbb{R})$  satisfies  $\chi(s) = s$  for  $|s| \leq \frac{1}{4}$ , and  $\psi_N \in C_1^\infty(\mathbb{R})$  ( $N \in \mathbb{N}$ ) satisfies

$$\psi_N(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{1}{8N} \\ 0 & \text{for } \frac{1}{4N} \leq |t| \leq \frac{1}{2}. \end{cases}$$

More concretely, we define

$$\psi_N(t) = \sum_{j \in \mathbb{Z}} \psi(Nt + j) \quad \text{where } \psi \in C^\infty(\mathbb{R}) \quad \text{with } \psi(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{1}{8} \\ 0 & \text{for } |t| \geq \frac{1}{4}. \end{cases}$$

The numbers  $N_l \geq 1$  should be chosen so that the series itself and the series after applying  $\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial s}\right)^k$  ( $j, k \in \mathbb{N}_0$ ) will converge uniformly for  $t, s \in \mathbb{R}$ . A sufficient condition is given by  $N_l \geq d_l$  where

$$d_l = \max_{0 \leq n \leq l} \max_{0 \leq t \leq 1} |b_l^{(n)}(t)|.$$

Indeed, applying  $\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial s}\right)^k$  we obtain a finite number of series of the type

$$\sum_{l=p}^\infty \frac{b_l^{(n)}(t)}{(l-p)!} [\chi(s-t)]^{l-p} \psi_{N_l}^{(q)}(s-t) \quad (n \leq j; p, q \leq j+k)$$

(notice that  $\chi(s) = s$  for  $s \in \text{supp } \psi_{N_l} \cap [-\frac{1}{2}, \frac{1}{2}]$ ). For  $l \geq n$  the members of the last series can be estimated by

$$\frac{d_l}{(l-p)!} (4N_l)^{-(l-p)} c_q N_l^q \leq \frac{c_q}{(l-p)!} 4^{-(l-p)} N_l^{-l+p+q+1}$$

guaranteeing uniform convergence of the series ■

**Proof of Theorem 3.** For  $\alpha = m \in \mathbb{N}_0$ , we present (11) as the sum

$$\sigma = \sigma_m + \sigma^{[m]}$$

with

$$\sigma_m(t, \xi) = \sum_{j=0}^m \left[ b_j^+(t) \phi_{m-j}(\xi) + b_j^-(t) \phi_{m-j}(\xi) \text{sign}(\xi) \right]$$

and

$$\begin{aligned} \sigma^{[m]}(t, \xi) &= \sum_{j=m+1}^{\infty} \left[ b_j^+(t) \phi_{m-j}(\xi) + b_j^-(t) \phi_{m-j}(\xi) \text{sign}(\xi) \right] \\ &= \sum_{j=0}^{\infty} \left[ b_{j+m+1}^+(t) \phi_{-1-j}(\xi) + b_{j+m+1}^-(t) \phi_{-1-j}(\xi) \text{sign}(\xi) \right] \end{aligned}$$

and the representation (8) - (10) for  $A \in Op \Sigma_{cl}^m$  follows immediately from Theorem 2. Thereby,

$$c_j^\pm(t) = (2\pi i)^{j-m} b_j^\pm(t) \quad (0 \leq j \leq m)$$

and the theorem is proved ■

**Remark 3.** As it can be seen from the proof, also some non-classical periodic pseudodifferential operators have an integral operator representation similar to (6). Namely, if  $\sigma \in \Sigma^\alpha$  has an asymptotic expansion

$$\sigma(t, \xi) \sim \sum_{j=0}^{\infty} \left[ a_j^+(t) \gamma^{(j)}(\xi) + a_j^-(t) \gamma^{(j)}(\xi) \text{sign}(\xi) \right]$$

where

$$\left. \begin{aligned} a_j^\pm &\in C_1^\infty(\mathbb{R}) \\ \gamma &\in C^\infty(\mathbb{R}) \quad \text{with} \quad |\gamma^{(j)}(\xi)| \leq c_j (1 + |\xi|)^{\alpha-j} \quad (\xi \in \mathbb{R}, j \in \mathbb{N}_0) \end{aligned} \right\},$$

then  $A = Op \sigma$  can be represented in the form

$$(Au)(t) = \int_0^1 \left[ \kappa_+(t-s) a_+(t, s) + \kappa_-(t-s) a_-(t, s) + a(t, s) \right] u(s) ds$$

where

$$a_\pm, a \in C_1^\infty(\mathbb{R} \times \mathbb{R}) \quad \text{and} \quad \begin{cases} \hat{\kappa}_+(n) = \gamma(n) \\ \hat{\kappa}_-(n) = \gamma(n) \text{sign}(n) \end{cases} \quad (0 \neq n \in \mathbb{Z}).$$

### 4. Functions $\kappa_\alpha^\pm$

Here we present some formulae of functions  $\kappa_\alpha^\pm$  satisfying (7). For  $\alpha = -1$  these formulae are well-known (see (9) - (10)). Consider the case  $-1 < \alpha < 0$ . Introduce the function

$$\kappa_\alpha(t) = t^{|\alpha|-1} + \sum_{j=1}^{\infty} [(t+j)^{|\alpha|-1} - \gamma_j] \quad (0 < t \leq 1, -1 < \alpha < 0) \tag{12}$$

where

$$\gamma_j = \int_0^1 (t+j)^{|\alpha|-1} dt = \frac{1}{|\alpha|} [(j+1)^{|\alpha|} - j^{|\alpha|}].$$

Note that the series in (12) converges uniformly in  $t \in [0, 1]$ , since  $\gamma_j$  as the mean value of  $(t+j)^{|\alpha|-1}$  in  $[0, 1]$  has a representation  $\gamma_j = (t_j+j)^{|\alpha|-1}$  with a  $t_j \in (0, 1)$ , and

$$\left. \begin{aligned} (t+j)^{|\alpha|-1} - \gamma_j &= (t+j)^{|\alpha|-1} - (t_j+j)^{|\alpha|-1} = (|\alpha|-1)(t'_j+j)^{|\alpha|-2}(t-t_j) \\ |(t+j)^{|\alpha|-1} - \gamma_j| &\leq (1-|\alpha|)j^{|\alpha|-2} \end{aligned} \right\}$$

where  $t'_j \in (t, t_j) \subset (0, 1)$ . Clearly, also the series obtained after differentiations converge uniformly. Thus,  $\kappa_\alpha \in C^\infty(0, 1]$ . Moreover,  $\kappa_\alpha$  is decreasing and  $0 < \kappa_\alpha(t) < t^{|\alpha|-1}$  ( $0 < t \leq 1$ ).

Define

$$\kappa_\alpha^\pm(t) = \gamma_\alpha^\pm[\kappa_\alpha(t) \pm \kappa_\alpha(1-t)] \quad (0 < t < 1) \tag{13}$$

where

$$\left. \begin{aligned} \gamma_\alpha^\pm &= \frac{1}{c_\alpha^\pm} \\ c_\alpha^+ &= 2(2\pi)^\alpha \Gamma(|\alpha|) \cos \frac{|\alpha|\pi}{2} \\ c_\alpha^- &= -2i(2\pi)^\alpha \Gamma(|\alpha|) \sin \frac{|\alpha|\pi}{2} \end{aligned} \right\}$$

and

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt \quad (0 < \beta < 1)$$

is the Euler function. We preserve the designations  $\kappa_\alpha^\pm$  also for the 1-periodic extensions of those functions and assert that

$$\left. \begin{aligned} \hat{\kappa}_\alpha^+(n) &= |n|^\alpha \\ \hat{\kappa}_\alpha^-(n) &= |n|^\alpha \text{sign}(n) \end{aligned} \right\} \quad (0 \neq n \in \mathbb{Z}, -1 < \alpha < 0). \tag{14}$$

To prove this we first find the Fourier coefficients of  $\kappa_\alpha(t)$  ( $0 < t \leq 1$ ) which we also regard as extended up to a 1-periodic function. Clearly,

$$\left. \begin{aligned} \hat{\kappa}_\alpha(0) &= \int_0^1 t^{|\alpha|-1} dt = \frac{1}{|\alpha|} \\ \hat{\kappa}_\alpha(n) &= \sum_{j=0}^{\infty} \int_0^1 (t+j)^{|\alpha|-1} e^{-in2\pi t} dt \quad (0 \neq n \in \mathbb{Z}) \end{aligned} \right\}.$$

With the changes of variables

$$\left. \begin{aligned} t + j &= s \\ 2\pi|n|s &= \tau \end{aligned} \right\}$$

we have

$$\begin{aligned} \hat{\kappa}_\alpha(n) &= \sum_{j=0}^{\infty} \int_j^{j+1} s^{|\alpha|-1} e^{-in2\pi s} ds \\ &= \int_0^{\infty} s^{|\alpha|-1} e^{-in2\pi s} ds \\ &= (2\pi|n|)^\alpha \int_0^{\infty} \tau^{|\alpha|-1} e^{-\text{sign}(n)i\tau} d\tau. \end{aligned}$$

It is known that

$$\int_0^{\infty} \tau^{\beta-1} e^{\pm i\tau} d\tau = e^{\pm i\frac{\pi}{2}\beta} \Gamma(\beta) \quad (0 < \beta < 1)$$

(see, e.g., [13: p. 69] for a proof). Thus

$$\hat{\kappa}_\alpha(n) = (2\pi)^\alpha \Gamma(|\alpha|) e^{-i\text{sign}(n)\frac{\pi}{2}|\alpha|} |n|^\alpha \quad (0 \neq n \in \mathbb{Z}).$$

The Fourier coefficient of functions  $v$  and  $w$  such that  $w(t) = v(1 - t)$  are related by  $\hat{w}(n) = \hat{v}(-n)$ . Therefore, the Fourier coefficients of  $\kappa_\alpha^\pm$  defined by (13) are as follows:

$$\hat{\kappa}_\alpha^\pm(n) = \gamma_\alpha^\pm (2\pi)^\alpha \Gamma(|\alpha|) \left( e^{-i\text{sign}(n)\frac{\pi}{2}|\alpha|} \pm e^{i\text{sign}(n)\frac{\pi}{2}|\alpha|} \right) |n|^\alpha \quad (0 \neq n \in \mathbb{Z}).$$

This results to (14).

Now we have formulae of  $\kappa_\alpha^\pm$  satisfying (7) for  $-1 \leq \alpha < 0$ . The following obvious remark makes possible to extend the result for other  $\alpha \in \mathbb{R}$ .

**Remark 4.** The formulae

$$\left. \begin{aligned} \kappa_{\alpha-1}^+(t) &= 2\pi i \int_0^t [\kappa_\alpha^-(s) - \hat{\kappa}_\alpha^-(0)] ds \\ \kappa_{\alpha-1}^-(t) &= 2\pi i \int_0^t [\kappa_\alpha^+(s) - \hat{\kappa}_\alpha^+(0)] ds \end{aligned} \right\} \quad (\alpha < 0)$$

and

$$\left. \begin{aligned} \kappa_{\alpha+1}^+(t) &= \frac{1}{2\pi i} \frac{d}{dt} \kappa_\alpha^-(t) \\ \kappa_{\alpha+1}^-(t) &= \frac{1}{2\pi i} \frac{d}{dt} \kappa_\alpha^+(t) \end{aligned} \right\} \quad (\alpha \in \mathbb{R})$$

hold where  $\frac{d}{dt}$  means the periodic distribution derivative.

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