Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 1, 269–278

On Infinite-Horizon Optimal Control Problems

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Abstract. In this paper, we consider infinite-horizon optimal control problems. First, by a suitable change of variable, we transform the problem to a finite-horizon nonlinear optimal control problem. Then the problem is modified into one consisting of the minimization of a linear functional over a set of positive Radon measure. The optimal measure is approximated by a finite combination of atomic measures and the approximate solution of the fist problem is find by the optimal solution of a finite-dimensional linear programming problem. The solution of this problem is used to find a piecewise constant control for the original one, and finally by using the approximate control signals we obtain the approximate trajectories.

Keywords: *Infinite-horizon problems, measure theory, optimal control, linear programming*

AMS subject classification: 49 J 15

1. Introduction

A powerful method has recently been used to solve optimal control problems, replacing the classical problem by problems in measure spaces (see, for example, Wilson and Rubio [18], Rubio [13 - 14], Kamyad et al. [10], Farahi et al. [8], and Effati [6]). Smirnov [16] presents necessary conditions of optimality for an infinite-horizon optimal control problem (see also [8]). The maximum principle for this problem without transversality conditions at infinity appeared in [4, 12]. Transversality conditions were derived by Aubin and Clarke [1], Michel [11] (a non-smooth version of this result appeared in [7, 19] for some dynamical optimization problems arising from mathematical economics).

In this paper, we transform the infinite-horizon problem to a finite-horizon problem, that is, the interval $[0,\infty)$ to $[0,1)$. First we construct a sequence of compact intervals $\{[0,1-\frac{1}{n}]\}_{n\geq 1}$ which approach $[0,1)$ when $n\to\infty$. Hence by constructing a sequence, we reduce the above problem to the known cases, that is to a problem on compact sets. By choosing a big positive number $n=n_0$ we obtain an approximate solution of the problem in the closed interval $[0,1-\frac{1}{n_0}]$. Of course, if we choose n_0 be very large, we could get a better approximation for the original problem. The aim of this paper is to derive optimal control $\hat{u}$ and the corresponding trajectory $\hat{x}$ for infinite-horizon optimal control problem by using measure theory. We now consider infinite-horizon optimal control problems with fixed and point $x(0)=x^0$ and $\lim_{t\to\infty} x(t) = x^1 = 0$.

Let us consider the problem

$$\inf \int_0^\infty \varphi(t,x(t),u(t))\,dt \tag{1}$$

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ISSN 0232-2064 / $ 2.50

subject to

$$x'(t) = f(t, x(t), u(t)) \tag{2}$$
$$u(t) \in U \subseteq \mathbb{R}^k, U \text{ compact} \tag{3}$$
$$x(0) = x^0, \lim_{t\to\infty} x(t) = x^1 = 0 \tag{4}$$
$$x(t) \in A \subseteq \mathbb{R}^n, \ A \text{ compact}$$

where

$$f : [0,\infty) \times \mathbb{R}^n \times U \to \mathbb{R}^n$$
$$\varphi : [0,\infty) \times \mathbb{R}^n \times U \to \mathbb{R}$$

are Lebesgue integrable functions so that they are continuously differentiable in x, and $\varphi(t) > 0$ for all $t \in [0,\infty)$. A pair

$$w = [x, u] \qquad \text{with} \quad \begin{cases} x : [0,\infty) \to \mathbb{R}^n \\ u : [0,\infty) \to \mathbb{R}^k \end{cases}$$

is said to be *admissible* if u is measurable and bounded, the trajectory function x is absolutely continuous, and conditions (2) - (4) are satisfied. We say that an admissible pair $\hat{w} = [\hat{x}, \hat{u}]$ is an *optimal solution* of problem (1) - (4) if

$$\int_0^\infty \varphi(t, x(t), u(t))\, dt \geq \int_0^\infty \varphi(t, \hat{x}(t), \hat{u}(t))\, dt.$$

for any admissible pair $w = [x, u]$. Also, we assume that the set Υ of all admissible pairs is non-empty.

2. Transformation of infinite- to finite-horizon problems

In this section, by a change of variable, we transform the interval $[0,\infty)$ to $[0,1)$, and then obtain optimal control and the corresponding trajectory in this interval. The change of variable is

$$\theta = \tfrac{2}{\pi}\tan^{-1}(t) \qquad \text{or} \qquad t = \tan\left(\tfrac{\pi}{2}\theta\right). \tag{5}$$

Then the above problem is transformed into the variational nonlinear optimal control problem

$$\inf \int_{[0,1)} \tfrac{\pi}{2}\varphi\Big(\tan\left(\tfrac{\pi}{2}\theta\right), x\big(\tan\left(\tfrac{\pi}{2}\theta\right)\big), u\big(\tan\left(\tfrac{\pi}{2}\theta\right)\big)\Big)\sec^2\left(\tfrac{\pi}{2}\theta\right) d\theta$$

subject to

$$\left.\begin{array}{l} x'\big(\tan\left(\tfrac{\pi}{2}\theta\right)\big) = f\Big(\tan\left(\tfrac{\pi}{2}\theta\right), x\big(\tan\left(\tfrac{\pi}{2}\theta\right)\big), u\big(\tan\left(\tfrac{\pi}{2}\theta\right)\big)\Big) \\ u\big(\tan\left(\tfrac{\pi}{2}\theta\right)\big) \in U \subseteq \mathbb{R}^k \\ x(0) = x^0, \ \lim_{\theta\to 1^-} x\big(\tan\left(\tfrac{\pi}{2}\theta\right)\big) = x^1 = 0 \\ x\big(\tan\left(\tfrac{\pi}{2}\theta\right)\big) \in A \subseteq \mathbb{R}^n. \end{array}\right\}$$

Assume

$$\left.\begin{aligned} y(\theta) &= x\big(\tan\left(\tfrac{\pi}{2}\theta\right)\big) \\ v(\theta) &= u\big(\tan\left(\tfrac{\pi}{2}\theta\right)\big). \end{aligned}\right\}$$

Then we get the variational problem

$$\inf \int_{[0,1)} \varphi\big(\tan\left(\tfrac{\pi}{2}\theta\right), y(\theta), v(\theta)\big)\tfrac{\pi}{2}\sec^2\left(\tfrac{\pi}{2}\theta\right)d\theta \tag{6}$$

subject to

$$y'(\theta) = \tfrac{\pi}{2} f\big(\tan\left(\tfrac{\pi}{2}\theta\right), y(\theta), v(\theta)\big)\sec^2\left(\tfrac{\pi}{2}\theta\right) \tag{7}$$

$$v(\theta) \in U \subseteq \mathbb{R}^k \tag{8}$$

$$y(0) = y^0 = x^0, \lim_{\theta\to 1^-} y(\theta) = y^1 = 0 \tag{9}$$

$$y(\theta) \in A \subseteq \mathbb{R}^n.$$

3. Classical control problems

We may transform the above control problems to an infinite-dimensional linear programming problem. Let us consider $\Omega = J \times A \times U$, where $J = [0,1)$. Assume $J_\varepsilon = [0, 1-\varepsilon]$ and $\Omega_\varepsilon = J_\varepsilon \times A \times U$. Since J_ε, U and A are compact subsets of $\mathbb{R}$, $\mathbb{R}^k$ and $\mathbb{R}^n$, respectively, then Ω_ε is a compact subset of $\mathbb{R}^{1+k+n}$, and Ω_ε approaches to Ω as $\varepsilon \to 0$. Let $w = [y, v]$ be an admissible pair for the variational problem and B an open ball in $\mathbb{R}^{n+1}$ containing $J \times A$. Let $C'(B)$ be the space of all bounded real-valued continuously differentiable functions on B such that the first derivative is also bounded.

Let $\phi \in C'(B)$ and define the function ϕ^g by

$$\phi^g(\theta, y(\theta), v(\theta)) = \phi_y(\theta, y(\theta))\, g(\theta, y(\theta), v(\theta)) + \phi_\theta(\theta, y(\theta)) \tag{10}$$

with $(\theta, y(\theta), v(\theta)) \in \Omega$ for all $\theta \in [0,1)$, where the function $g : \Omega \to \mathbb{R}^n$ is defined by

$$g(\theta, y(\theta), v(\theta)) = f\big(\tan\left(\tfrac{\pi}{2}\theta\right), y(\theta), v(\theta)\big)\tfrac{\pi}{2}\sec^2\left(\tfrac{\pi}{2}\theta\right).$$

Further, define the function $f_0 : \Omega \to \mathbb{R}$ by

$$f_0(\theta, y(\theta), v(\theta)) = \varphi\big(\tan\left(\tfrac{\pi}{2}\theta\right), y(\theta), v(\theta)\big)\tfrac{\pi}{2}\sec^2\left(\tfrac{\pi}{2}\theta\right).$$

By these definitions, the problem in Section 2 is transformed to the problem to find the infimum of the functional

$$I[y, v] = \int_{[0,1)} f_0(\theta, y(\theta), v(\theta))\, d\theta \tag{11}$$

subject to

$$y'(\theta) = g(\theta, y(\theta), v(\theta)) \tag{12}$$
$$v(\theta) \in U \subseteq \mathbb{R}^k \tag{13}$$
$$y(0) = y^0 = x^0, \ \lim_{\theta \to 1^-} y(\theta) = y^1 = 0 \tag{14}$$
$$y(\theta) \in A \subseteq \mathbb{R}^n.$$

Since $w = [y, v]$ is an admissible pair, we have

$$\begin{aligned}\int_{[0,1)} \phi^g(\theta, y(\theta), v(\theta))\, d\theta &= \int_{[0,1)} \big(\phi_y(\theta, y(\theta))y'(\theta) + \phi_\theta(\theta, y(\theta))\big)\, d\theta \\ &= \int_{[0,1)} \phi'(\theta, y(\theta))\, d\theta \\ &= \phi(1, y^1) - \phi(0, y^0) \\ &= \Delta\phi \end{aligned} \tag{15}$$

for all $\phi \in C'(B)$. Let $D(J^\circ)$ be the space of infinitely differentiable real-valued functions with compact support in J° (see [17]), where $J^\circ = (0, 1)$. Define

$$\psi_j(\theta, y(\theta), v(\theta)) = y_j(\theta)\psi'(\theta) + g_j(\theta, y(\theta), v(\theta))\psi(\theta) \tag{16}$$

for $j = 1, 2, ..., n_1$ and all $\psi \in D(J^\circ)$. Then, if $w = [y, v]$ is an admissible pair, we have for $j = 1, 2, ..., n_1$ and $\psi \in D(J^\circ)$

$$\begin{aligned}\int_{[0,1)} \psi_j(\theta, y(\theta), v(\theta))\, d\theta &= \int_{[0,1)} y_j(\theta)\psi'(\theta)\, d\theta + \int_{[0,1)} g_j(\theta, y(\theta), v(\theta))\psi(\theta)\, d\theta \\ &= y_j(\theta)\psi(\theta)\big|_J - \int_{[0,1)} \big(y_j'(\theta) - g_j(\theta, y(\theta), v(\theta))\big)\psi(\theta)\, d\theta \\ &= 0\end{aligned}$$

since the trajectory and control functions in an admissible pair satisfy (12) on J°, and since the function ψ has compact support in J^o, $\psi(0) = \psi(1) = 0$. And also, by choosing a functions which depend only to the variable θ, we have

$$\int_{[0,1)} f(\theta, y(\theta), v(\theta))\, d\theta = a_f \qquad (f \in C_1(\Omega))$$

where $C_1(\Omega)$ is a subspace of the space $C(\Omega)$ of all bounded continuous functions on Ω depending only on the variable θ.

The mapping

$$\Lambda_w : F \to \int_J F(\theta, y(\theta), v(\theta))\, d\theta \qquad (F \in C_{00}(\Omega))$$

defines a positive linear functional on $C_{00}(\Omega)$, the space of all bounded continuous functions with compact support. By the Riesz representation theorem (see [15: p. 40/Theorem 2.14]) there exists a unique positive Radon measure μ on Ω such that

$$\Lambda_w(F) = \int_J F(\theta, y(\theta), v(\theta))\, d\theta = \int_\Omega F\, d\mu \equiv \mu(F) \qquad (F \in C_{00}(\Omega)).$$

Thus, the minimization of the functional (11) over Υ is equivalent to the minimization of

$$I(\mu) = \lim_{\varepsilon \to 0^+} \int_{\Omega_\varepsilon} f_0\, d\mu = \int_\Omega f_0 d\mu \equiv \mu(f_0) \in \mathbb{R} \tag{17}$$

over the set of all measures μ corresponding to admissible pairs w, which satisfy

$$\left.\begin{array}{l} \mu(\phi^g) = \Delta\phi \quad (\phi \in C'(B) \\ \mu(\psi_j) = 0 \quad (j = 1, 2, ..., n_1;\ \psi \in D(J^\circ)) \\ \mu(f) = a_f \quad (f \in C_1(\Omega)). \end{array}\right\} \tag{18}$$

We shall consider the minimization of (17) over the set Q of all positive Radon measures on Ω satisfying (18). This is an infinite-dimensional linear programming problem, and all the functions in (17) - (18) are linear with respect to the measure μ. Furthermore, the measure μ is required to be positive.

Let $M^+(\Omega)$ be the set of all positive Radon measures on Ω. The functional $I : Q \to \mathbb{R}$ defined by

$$I(\mu) = \int_\Omega f_0\, d\mu \equiv \mu(f_0) \in \mathbb{R} \qquad (\mu \in Q)$$

is a linear continuous functional on the set Q with weak*-topology. Let $\varepsilon = \frac{1}{n}$ and $\Omega_{1/n} = [0, 1 - \frac{1}{n}] \times A \times U$ for $n \in \mathbb{N}$. Denote now

$$Q_n = \{\mu \in M^+(\Omega_{1/n}) : \mu \text{ satisfies } (18)\}.$$

Lemma 3.1. *Suppose $m, n \in \mathbb{N}$ and $n > m$. Then*

$$Q_m \subset Q_n \subset \ldots \subset Q. \tag{19}$$

Proof. Since $M^+(\Omega_{1/m}) \subset M^+(\Omega_{1/n}) \subset \ldots \subset M^+(\Omega)$, this implies (19) ∎

Lemma 3.2. *If $I(\mu) = \int_\Omega f_0 d\mu$, $I^* = \inf_Q I(\mu)$ and $I_n^* = \inf_{Q_n} I(\mu)$, then $\lim_{n\to\infty} I_n^* = I^*$.*

Proof. From Lemma 3.1, for $n > m$ we have

$$-\infty < \inf_Q I(\mu) = I^* \leq \inf_{Q_n} I(\mu) = I_n^* \leq \inf_{Q_m} I(\mu) = I_m^*.$$

Therefore the sequence $\{I_n^*\}$ is non-increasing and bounded below; it converges, to a number $c \geq \inf_Q I(\mu)$. Suppose that $c > \inf_Q I(\mu)$. Then there is a $v \in Q$ such that

$$c > I(v) \geq \inf_Q I(\mu). \tag{20}$$

By Lemma 3.1 there is an $h \in \mathbb{N}$ such that $v \in Q_h$. We have $I(v) \geq \inf_{Q_h} I(\mu) = I_h^*$. Then $I(v) \geq c$ and this is a contradiction to (20). Thus $c = \inf_Q I(\mu)$ ∎

Lemma 3.3. *Let $m, n \in \mathbb{N}$ with $m > n$, $I_n(\mu) = \int_{\Omega_{1/n}} f_0 d\mu$ and $I(\mu) = \int_\Omega f_0 d\mu$. Then $\lim_{n\to\infty} \inf_Q I_n(\mu) = \inf_Q I(\mu)$.*

Proof. Since f_0 is non-negative on $[0, 1)$ and

$$I_n(\mu) < I_m(\mu) < \ldots < I(\mu) = \int_\Omega f_0 d\mu,$$

it follows that

$$\inf_Q I_n(\mu) \le \inf_Q I_m(\mu) \le \ldots \le \inf_Q I(\mu). \tag{21}$$

Thus, the sequence $\{\inf_Q I_n(\mu)\}$ is increasing and bounded from above; so

$$\lim_{n\to\infty} \inf_Q I_n(\mu) = \sup_n \Big(\inf_Q I_n(\mu)\Big) = \inf_Q I(\mu)$$

(for the second equality see (21)) ∎

Theorem 3.1. *If $I_n(\mu) = \int_{\Omega_{1/n}} f_0 d\mu$ and $I(\mu) = \int_\Omega f_0 d\mu$, then*

$$\lim_{n\to\infty} \inf_{Q_n} I_n(\mu) = \inf_Q I(\mu).$$

Proof. From Lemmas 3.2 and 3.3 we have

$$\lim_{n\to\infty} \inf_{Q_n} I(\mu) = \lim_{n\to\infty} \inf_Q I_n(\mu) = \inf_Q I(\mu)$$

and since

$$\inf_Q I_n(\mu) \le \inf_{Q_n} I_n(\mu) \le \inf_{Q_n} I(\mu).$$

Thus $\lim_{n\to\infty} \inf_{Q_n} I_n(\mu) = \inf_Q I(\mu)$ and the statement is proved ∎

Remark 3.1. Note that Ω here is not a compact space. For the compact case Rubio in [13] has shown that Q is weak*-compact. But in our situation, by using the Banach-Alaoglu Theorem, one can show that Q actually is weak*-compact.

4. Approximation

We now estimate the optimal control by a nearly-optimal piecewise constant control. We first assume n be a large number $n = n_0$. Then we minimize the functional

$$I_{n_0}(\mu) = \int_{\Omega_{1/n_0}} f_0 d\mu$$

over a subset of $M^+(\Omega_{1/n_0})$ which is defined by requiring only a finite number of the constraints in (18) to be satidfied (still infinite-dimensional). This will be achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate spaces, and then selecting a finite number of them. In the first step, we obtain an approximation to the optimal measure μ^* by a finite combination of atomic measures, that is from [13: Theorem A.5] μ^* has the form

$$\mu^* = \sum_{i=1}^{N} \alpha_i^* \delta_{z_i^*} \qquad (\alpha_i \geq 0, z_i^* \in \Omega_{\frac{1}{n_0}}).$$

Here δ_z is the unitary atomic measure characterized by $\delta_z(F) = F(z)$ where $F \in C_{00}(\Omega_{1/n_0})$ and $z \in \Omega_{1/n_0}$. Then we construct a piecewise constant control function corresponding to the finite-dimensional problem. Therefore in the infinte-dimensional linear programming problem (17) with restriction defined by (18) we shall consider only a finite number M_1 of the functions ϕ of the type

$$\phi_1 = y_1, \phi_2 = y_2, ..., \phi_n = y_n$$
$$\phi_{n+1} = y_1^2, \phi_{n+2} = y_2^2, ...$$

and, also, only a finite number of functions ψ^k $(k = 1, 2, ..., M_2)$ defined in (16), when the functions ψ in (16) are of the form $\sin(2\pi r\theta)$, $1 - \cos(2\pi r\theta)$ $(r \in \mathbb{N})$ are considered. Also, only a finite number L of functions f of the type

$$f_s(\theta) = \begin{cases} 1 & \text{if } \theta \in J_s \\ 0 & \text{otherwise} \end{cases} \qquad \text{with } J_s = \left(\frac{s-1}{L}, \frac{s}{L}\right) \quad (s = 1, \ldots, L)$$

will be considered. The set $\Omega_{1/n_0} = J_{1/n_0} \times A \times U$ will be covered with a grid, where the grid will be defined by taking all points in Ω_{1/n_0} as

$$Z_j = \left(\theta_j, y_{1_j}, y_{2_j}, \ldots, y_{n_j}, v_{1_j}, v_{2_j}, \ldots, v_{k_j}\right);$$

the points in the grid will be numbered sequentially from 1 to N. Of course, we only need to construct the control function, since the trajectory is then simply the corresponding solution of the differential equation (12), which can be estimated numerically. Thus, instead of the infinite-dimensional linear programming problem (17) with restriction defined by (18) can be approximated by the following linear programming problem which Z_i for $i = 1, \ldots, N$ belongs to a dense subset Ω_{1/n_0}.

The linear programming problem consists of minimizing the linear form

$$\sum_{j=1}^{N} \alpha_j f_0(Z_j)$$

over the set $\alpha_j \geq 0$, subject to

$$\left.\begin{aligned} &\sum_{j=1}^{N} \alpha_j \phi_i^g(Z_j) = \Delta\phi_i \quad (i = 1, ..., M_1) \\ &\sum_{j=1}^{N} \alpha_j \psi^k(Z_j) = 0 \quad (k = 1, ..., M_2) \\ &\sum_{j=1}^{N} \alpha_j f_s(t_j) = a_f \quad (s = 1, ..., L). \end{aligned}\right\}$$

Therefore, by solving this problem, we get the coefficients α_j $(j = 1, ..., N)$, and then we construct a piecewise constant control function; that is, from analysis Rubio (see [13]) we obtain the piecewise constant control function v, and then from (12) obtains the trajectory function y.

5. Numerical example

Consider the problem

$$\inf \tfrac{1}{2} \int_0^\infty (x^2(t) + \alpha u^2(t))\, dt,$$

α constant, subject to

$$x''(t) = -x'(t) + u(t)$$

where $x(0) = x'(0) = 0.1$ and $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$. Suppose $x_1(t) = x(t)$, $x_2(t) = x'(t)$, and $\alpha = 4$. Then the above problem transform into the form

$$\inf \tfrac{1}{2} \int_0^\infty (x_1^2(t) + 4u^2(t))\, dt$$

subject to

$$\left.\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -x_2(t) + u(t) \end{aligned}\right\}$$

where $x_1(0) = x_2(0) = 0.1$ and $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$. Using the Euler-Lagrange equations (see [17]), we obtain the optimal paths as

$$\left.\begin{aligned} x_1(t) &= \left[0.1 + \left(0.1 + \frac{0.1}{\sqrt{2}}\right) t\right] \exp\left(\frac{-t}{\sqrt{2}}\right) \\ x_2(t) &= \left[0.1 - \left(0.1 + \frac{0.1}{\sqrt{2}}\right) \frac{t}{\sqrt{2}}\right] \exp\left(\frac{-t}{\sqrt{2}}\right). \end{aligned}\right\}$$

Now by a suitable change of variable in (5), the problem is transformed into the variational problem

$$\inf \int_{[0,1)} \tfrac{\pi}{4}(y_1^2(\theta) + 4v^2(\theta)) \sec^2\left(\tfrac{\pi}{2}\theta\right) d\theta$$

subject to

$$\left.\begin{aligned} y_1'(\theta) &= \tfrac{\pi}{2} y_2(\theta) \sec^2\left(\tfrac{\pi}{2}\theta\right) \\ y_2'(\theta) &= \tfrac{\pi}{2}(-y_2(\theta) + v(\theta)) \sec^2\left(\tfrac{\pi}{2}\theta\right) \end{aligned}\right\}$$

where $y_1(0) = y_2(0) = 0.1$ and $\lim_{\theta\to 1^-} y_1(\theta) = \lim_{\theta\to 1^-} y_2(\theta) = 0$.

Let $\theta \in J_{1/n_0} = [0, 1 - \frac{1}{n_0}]$, $\frac{1}{n_0} = 0.01$, and $y(\theta) = [y_1(\theta), y_2(\theta)] \in A = A_1 \times A_2$, where $A_1 = [0, 0.1]$ and $A_2 = [-0.08, 0.1]$. And let the set $J_{1/n_0} = [0, 1 - \frac{1}{n_0}]$ be divided into 15 equal subintervals, the sets A_1, A_2 and $U = [-1, 1]$ be devided into 10 equal subintervals, so that $\Omega_{1/n_0} = J_{1/n_0} \times A \times U$ is devided into 15000 equal subsets. We assumed

$$Z_p = (\theta_p, y_{1_p}, y_{2_p}, v_p) \qquad (p = 1, \ldots, 15000)$$

where

$$p = i + 10(j-1) + 100(k-1) + 1000(l-1) \qquad \left(\begin{matrix} i,j,k = 1,\dots,10 \\ l = 1,\dots,15 \end{matrix}\right),$$

$$y_{1_p} = \frac{1}{100}(i), \qquad y_{2_p} = -\frac{1}{10} + \frac{2}{100}(j), \qquad v_p = -1 + \frac{2}{100}(k)$$

$$\theta_p = \begin{cases} \frac{0.8}{10}(l) & \text{for } l = 1,\dots,10 \\ 0.9 + 0.02(l-10) & \text{for } l = 11,\dots,14 \\ \frac{0.99}{15}(l) & \text{for } l = 15, \end{cases}$$

$M_1 = 2$, $M_2 = 8$ and $L = 15$. In this example, the optimal value of the cost function is 0.1102. Graphs of the piecewise constant control function and the trajectory functions can be seen in Figure 1 - 3.

Figure 1: Piecewise constant optimal control

Figure 2: Solid line precise solution Figure 3: Solid line precise solution

(ooooo represents approximate solution)

Note. One should note that we have not discussed the rate of accuracy near the end point: that is when the index is making larger, we get a better accuracy, but what is the rate of this accuracy?

This problem needs more work and we appreciate the referees for evaluate suggestions and especially for this note which brings to our attention.

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Received 30.12.1998; in revised form 05.10.1999