Interaction Analysis of Mathematical Communication in Primary Teaching: The Epistemological Perspective[1]

Heinz Steinbring, Dortmund

Abstract: Communication between students and teacher in the mathematics classroom is a form of social interaction which focuses on a specific topic: *mathematical knowledge.* This knowledge cannot be introduced into classroom interaction "from the outside", but grows through the communicative process, in the course of interactive exchanges between the participants of discussion. Although mathematical communication must be seen and analysed in the same way as any other form of communication, the particularity of interactive constructions of mathematical knowledge and its specific *social epistemology within the context of teaching processes* has to be taken into consideration. Also, the institutional influences of school institutions and those of teaching (analysed in the frame of general socio-interactive research approaches) must be considered. An epistemology-oriented interaction research approaches the specificity of a *mathematical* classroom and communication culture in its analyses.

Kurzreferat: *Interaktionsanalyse der mathematischen Kommunikation im Grundschulunterricht. Die epistemologische Perspektive.* Die Kommunikation zwischen Schülerinnen, Schülern und Lehrerin im Mathematikunterricht ist als *soziale* Interaktion zugleich mit einem spezifischen Inhalt befaßt: dem *mathematischen* Wissen. Dieses Wissen kann nicht unveränderlich "von außen" in den Unterricht hineingegeben werden, es entsteht im Kommunikationsprozess durch den interaktiven Austausch zwischen den Beteiligten. Obwohl die mathematische Kommunikation einerseits wie eine jede andere Kommunikationsform zwischen Menschen betrachtet und analysiert werden kann, muss andererseits die Besonderheit der interaktiven Konstruktion mathematischen Wissens und seiner "Epistemologie im Kontext der Unterrichtsprozesse" berücksichtigt werden. Die epistemologisch orientierte Interaktionsforschung verfolgt in ihren Analysen das Problem der Spezifik einer *mathematischen* Unterrichts- bzw. Kommunikationskultur.

ZDM-Classification: C52, E22, F32

1. The role of mathematical knowledge in interaction research

The qualitative analysis of mathematical communication always has to start – explicitly or implicitly – from assumptions about the status of mathematical knowledge. There are different ways of coping with this requirement. There could be a general assumption, to observe and to analyse mathematics teaching in the same way as every other form of teaching, without taking into account the particularities of mathematical knowledge. Or another, typical, assumption could be that mathematical communication is, indeed, determined by the specific subject matter dealing with an "objective", correct subject matter knowledge, and therefore the analysis of mathematical communication could reach an unequivocal assessment as "true / false" or "good / bad teaching".

A fundamental assumption for different research approaches to mathematical interaction is the idea that the mathematical subject matter cannot be introduced into the teaching-learning process as a ready made curricular product, but that the subject matter knowledge can only be mutually generated during the interactive process. This assumption contains the following contradiction: Teaching is an activity oriented towards a pre given goal which requires the step by step administration of subject matter to students. In contrast to this, learning is seen as an active process of construction and development, which, through interactivities, is the basis for the emergence of new knowledge. When starting from this supposition the central objective for interpretative research is to reconstruct and understand interactive knowledge development in the mathematics classroom as an evolving autonomous process dependent on internal conditions. Ready made mathematical knowledge cannot be the measure to assess success or failure of the teaching-learning process.

In the course of interaction analysis, interpretations must be made of the verbal and non-verbal communications of interacting persons. These qualitative interpretations have to be consistent in some way; often there will not only be one "true" explanation but several, alternative, plausible interpretations of analysed communications (cf. Voigt 1994). These interpretative analyses bring to light "typical" patterns of interaction; and comparative analysis of several "similar" teaching episodes can provide a more secure data base for decisions about the chosen interpretation of observed communication (Krummheuer 1997).

Epistemology-based interaction research in mathematics education proceeds on the assumption that a specific social epistemology of mathematical knowledge is constituted in classroom interaction and this assumption influences the possibilities and the manner of how to analyse and interpret mathematical communication. This assumption includes the view on mathematics explained above: Mathematical knowledge is not conceived as a ready made product, characterised by correct notations, clear cut definitions and proven theorems. If mathematical knowledge in learning processes could be reduced to this description, the interpretation of mathematical communication would become a direct and simple concern. When observing and analysing mathematical interaction one would only have to diagnose whether a participant in the discussion has used the "correct" mathematical word, whether he or she has applied a learned rule in the appropriate way, and then has gained the correct result of calculation, etc. The epistemology-based interaction research approach understands mathematical knowledge and mathematical concepts neither as concrete, material objects, given *a priori* in the "external" reality, nor as independently existing (platonic) ideas. For the individual cognitive agent mathematical concepts are "mental objects" (Changeux & Connes 1995; Dehaene 1997); in the course of communication mathematical concepts are constituted as "social facts" (Searle 1997) or as "cultural objects" (Hersh 1997). From an evolutionary point of view, mathematical concepts develop as cognitive and social theoretical knowledge objects in confrontation with the material and social environment.

Mathematical concepts are constructed as symbolic relational structures and are coded by means of *signs and* *symbols,* that can be combined logically in mathematical operations. The intended construction of meaning for the unfamiliar, new mathematical signs, by trying to build up reasonable relations between signs and possible contexts of reference and of interpretation, is a fundamental feature of an epistemological perspective on mathematical classroom interaction. This intended process of constructing meaning for mathematical signs is an essential element of every mathematical activity whether this construction process is performed by the mathematician in a very advanced research problem, or whether it is undertaken by a young child when trying to understand elementary arithmetical symbols with the help of the position table. The focus on this construction process enables us, to see mathematics teaching and learning at different school levels as an authentic mathematical endeavour.

2. The creation of new and generalised mathematical knowledge

The particularities of an epistemology-based approach to classroom interaction research can best be described with the help of exemplary teaching episodes dealing with the construction and justification of new mathematical knowledge. The research project "Social and epistemological constraints of constructing new knowledge in the mathematics classroom" (funded by the German Research Society, DFG; see Steinbring et al. 1998; Steinbring 1999) investigates the problem of how children in primary mathematics classrooms are able to construct and explain new knowledge with their own stories and arguments interactively.

Several teachers participated in this project, performing short experimental teaching units of 4 to 5 lessons; the young students (age 8 and 9) worked on (mathematical) problems within mathematical learning environments. One type of learning environment offered a specific context for the children to elaborate meaning or offer a first justification for arithmetical patterns. Such children are not in the position to use algebraic notation or operations to "abstract" more general relationships and formulae from arithmetical learning environments. The following question arose: How does new, more general knowledge emerge? Is it a necessary precondition for the introduction of new knowledge first to supply the new terminology, notation and definitions, which only then could permit its development and description? Or does the process of constructing new knowledge take place in a completely different manner without having to rely on the notation and formal definitions?

This problem is discussed in more detail with the help of an arithmetical example from elementary mathematics teaching (grade 2 or 3). The children are asked to work on the following question. The following "rhombus numbers" on the 1×1 table are investigated (cf. Wittmann & Müller 1990).

Taking four adjacent products one can calculate a "rhombus number" in the following way: Subtract the "vertical" sum of the two products from the "horizontal" sum of the two other products (Fig. 2). Several "rhombus numbers" are calculated in the 1×1 table. All rhombus numbers are equal to 1. Why? In what way can the new knowledge, the new arithmetical relation, behind these rhombus numbers be described, developed, understood and justified? An "experienced" mathematician would perhaps at once approach this problem with "letters". "Replacing numbers by letters" could be a first solving strategy in which the concrete, specific numbers are exchanged with a new, general notation (algebraic letters). Is it possible simply to construct the new knowledge by introducing new, general descriptions and notation (Fig. 3)? Obviously we cannot do so as. It is not sufficient only to use a new notation. It is more important to take an existing relation between the given numbers into account (Fig. 4).

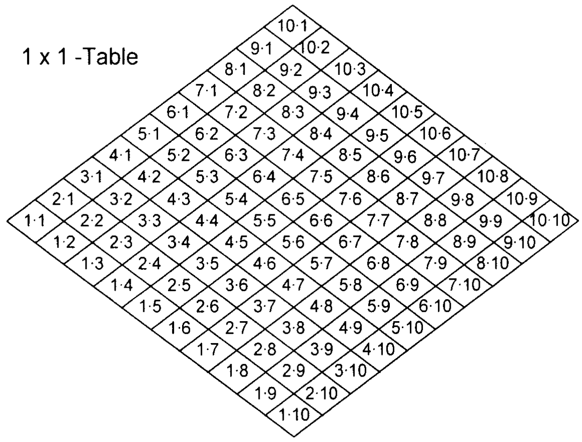
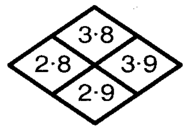
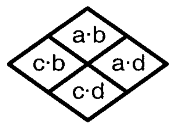


Fig. 1: The one-times-one-table



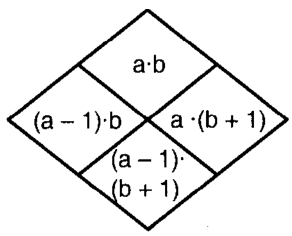
$$(2 \cdot 8 + 3 \cdot 9) - (3 \cdot 8 + 2 \cdot 9) =$$
$$(16 + 27) - (24 + 18) =$$
$$43 - 42 = 1$$

Fig. 2



$$(c \cdot b + a \cdot d) - (a \cdot b + c \cdot d) =$$
$$-(a - c) \cdot (b - d) = ???$$

Fig. 3



$$(a-1)b + a(b+1) - ab - (a-1)(b+1) =$$
$$ab - b + ab + a - ab - ab + b - a + 1 = 1$$

Fig. 4

Normally this can be done using the new terminology to represent the "difference of 1" between several numbers.

Then the rules of algebra are used and the result is determined. This indeed leads to the solution "1" in a general manner. Has the new knowledge relation been understood? Algebra automatically furnishes a result but without necessarily or substantially increasing the insight into the new relation. One aspect of this example is already important: the simple replacement of concrete numbers by algebraic signs is insufficient. New knowledge inevitably needs the identification of the arithmetical relation: Certain numbers differ by 1. Why is this important for the problem question? Is it the case across the whole arithmetical pattern?

The significant underlying mathematical relation can already be identified in the frame of the arithmetical problem situation and it is the fundamental base for the construction and justification of new knowledge. Let us compare the relationships between the numbers in this arithmetical pattern.

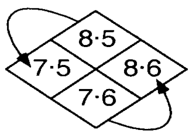


Fig. 5

We observe: The left product in the horizontal row contains 1 times 5 less than the product above it in the vertical column; the right product in the horizontal row contains 1 times 6 more than the lower product in the vertical column. Therefore the horizontal row contains 6 - 5 = 1 more than the vertical column. We have only used the relationships "$1 \cdot 5-$ less" (or "1 times the second factor less") and "$1 \cdot 6-$ more" (or "1 times the second factor more" (the two second factors just differ by 1)). Unlike the mere meaning of concrete arithmetical signs by letters, these important arithmetical relationships, now identified, form the central elements of the mathematical justification.

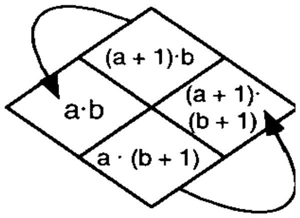


Fig. 6

These arithmetical relations are the proper meaning that could also offer deeper understanding of the algebraic procedure. The algebraic situation could be seen in an analogous way to the arithmetical interpretation: The left product in the horizontal row contains 1 times b less than the product above in the vertical column; the right product in the horizontal row contains 1 times $(b+1)$ more than the lower product in the vertical column. Therefore the difference between the horizontal sum and the vertical sum is constant: $b+1-b=1$. The choice of algebraic notation was not arbitrary but it had additional advantages with regard to the underlying important arithmetical relationship.

In our example the specific property of rhombus numbers (always having the constant value 1) can be conceived as a central property of the new knowledge. This new knowledge only emerges by the construction of essential relations, and this necessarily has to be done in the existing frame of mathematical structures and notation. The *construction of a new mathematical relation* in the course of knowledge development precedes the mere *naming of the concrete numbers by more general signs.*

How are these considerations connected with the epistemology-based approach to interaction analysis? Since the new mathematical knowledge cannot simply be constructed in new notation and ready made definitions, but relies upon identifying relevant mathematical relations within the context of existing mathematical structures and expressions, a main consequence is that primary children in learning mathematics have, and are able, to approach and bring forward their descriptions and constructions of new mathematical knowledge with their own words and ideas. Hence, when observing children's statements, it is impossible to deduce a construction or a justification of new knowledge from the use of abstract notation, universal definitions or the introduction of variables immediately. The students shall, must and can only try to make attempts to justify by using their own descriptions and covered in the expressions they have used until now.

But when the young students describe new relations and new knowledge with the old exemplary interactively, and concrete interpretations and references partially, an epistemological analysis is faced with the problem of finding out the extend to which these documented statements and contributions, with their customary, familiar descriptions, contain justifications of *new* knowledge or whether they are mere repetitions of knowledge already known mediated by the use of familiar expressions. A very careful analysis is needed to be able to recognise initial hints and potential relational structures in the children's personal remarks that permit the attribution of an intention aiming at a new mathematical relation within the existing mathematical problem field.

3. Epistemological and communicative aspects in interaction analysis

The construction of new mathematical knowledge is constituted by establishing relations between signs / symbols (diagrams, operational signs etc.) and objects or reference contexts (concrete and abstract ones). The new, or partially unknown, signs and symbol systems are more and more enriched with meaning by referring them to certain (concrete or also structural) reference contexts. In the frame of the epistemology-oriented interaction research the epistemological triangle serves as a central, theoretical instrument for describing and analysing processes of constructing new mathematical knowledge (Steinbring 1989, 1991, 1999).

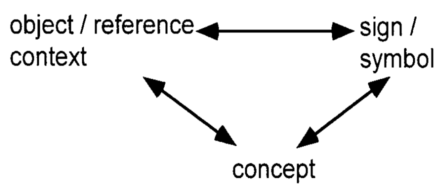


Fig. 7: The epistemological triangle

The particularity of this epistemological triangle consists in the specific reciprocal relationships between the three corners “sign / symbol”, “object / reference context” and “concept”. These relations are not explicitly regulated; rather they form a balancing system. As knowledge develops interactively, the interpretation of sign systems and of corresponding reference contexts will change and modify. It is important to notice that the object or reference context is not given *a priori* in a definite and unequivocal manner, but changes into a structural setting in the course of knowledge development. From this perspective, mathematical meaning can be seen as the interpretation of relatively new and unknown signs with reference to a more familiar (structural) reference context.

The epistemological triangle expresses the ways in which mathematical signs are endowed with meaning in mathematical interaction by referring these signs to specific contexts of reference. Mathematical interaction in the everyday classroom can be regarded as a process of communication between different participants. How could communication be characterised in a theoretical way? The German sociologist Luhmann (1997, 1996) takes communication as the fundamental and constitutive element of any social system. “Communication is the final element or the specific operation ... of social systems. It is the synthesis of three selections: (1) conveyance; (2) information; (3) understanding of the distinction between information and conveyance” (Baraldi, Corsi & Esposito 1997, p. 89). From this basic theoretical perspective Luhmann describes society and all its social subsystems. In what follows, I only refer to the central “mechanism” of communication insofar as it is relevant to the theoretical approach.

“One can speak of communication, when Alter understands, that Ego has conveyed an information; this information can then be attributed to him. The conveyance of an information (Alter says for instance “It rains”) is not itself an information. Communication only can realise this as an information, if it is understood: if the information (“It rains”) and Alter’s intention for this conveyance (Alter for example will cause Ego to take an umbrella with him) are seen as distinct selections. Without understanding there cannot be observed any communication: Alter waves to Ego, and Ego walks on quietly, because he has not understood that the waving was a greeting. The understanding realises the fundamental distinction of communication: the distinction between conveyance and information” (Baraldi, Corsi & Esposito 1997, p. 89).

When communication takes place, one can observe a sequence of conveyances and reconstruct the information that is different from these conveyances, by trying to identify the understanding in this process. Luhmann explains the difference between conveyance, information and understanding with the help of de Saussure’s distinction between signifier (signifiant), signified (signifié) and sign (signe) (cf. Fig. 8). Accordingly, in the course of interaction “signifiers” are conveyed that have to be distinguished from the intended information (the “signifieds”), and only the observation of this distinction can produce understanding with the production of a “sign”.

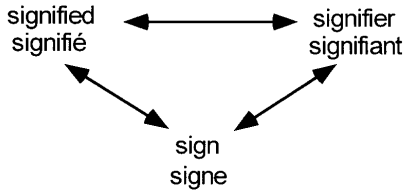


Fig. 8: The semiotic triangle according to de Saussure

The conveyor can only convey a signifier, but the signified that is intended by the conveyor, which alone could lead to an understandable sign, remains open and relatively uncertain; in principle it can only be constructed by the receiver of the conveyance, in a way that he or she himself or herself articulates a new signified.

“... we start with the situation of the receiver of the conveyance, hence the person who observes the conveyor and who ascribes to him the conveyance, but not the information. The receiver of the conveyance has to observe the conveyance as the designation of an information, hence both together as a sign ....” (Luhmann 1997, p. 210).

The receiver must not ascribe the possible signified strictly to the conveyor but he has to construct the signified himself; the signified and the sign is constituted within the process of communication.

In this way, communication can become a self reproducing, living system, by the succession of one conveyance after the other. And the central mechanism of communication consists in the following: The conveyances (signifiers) of the teacher or of different students do not yet contain the information (a signified); the distinction and consequently the reference between signifier and signified can only be established in interaction by the participants of communication, otherwise communication cannot happen. In contrast to this position, in mathematics teaching there is often the tacit assumption that mathematical signifiers possess one and only one definite signified. This assumption can lead to a destruction of authentic mathematical communication. There is the danger that the conveyance already has to be taken as the information; no distinction is made between the conveyance and the intended information in mathematical communication.

The openness and potential ambiguity of conveyed signifiers in mathematical interaction is a necessary constraint for authentic mathematical communication: First, the young students are only able to use the old, familiar frame of notations and of descriptions for expressing the intended new relations and new knowledge. The mechanism explaining the functioning of communicative systems, as elaborated in Luhmann’s theoretical position, explains and justifies this way of trying to grasp the intended new knowledge by using old and familiar descriptions in an open but changed way. That then has to be understood in interaction by seeing the distinction between the conveyed signifiers and the intended signified (the new information) and thus constructing a new sign or a new mathematical relation (conceptual aspect). Also, this theoretical approach to communication clearly states that there is absolutely no other way of constituting new knowledge in communication: there is no possibility of starting with a perfect, abstract terminology (a system of signifiers) that would correspond in an unequivocal manner to a definite and precise system of signifieds; such a fixed linkage between signifiers and signifieds would destroy the mathematical communication.

4. Children's subjective construction of new knowledge – epistemological analysis of two teaching episodes

Two teaching episodes (see Appendix) have been observed and documented in the research project: "Social and epistemological constraints of constructing new knowledge in the mathematics classroom" (funded by the German Research Society, DFG; see Steinbring 2000). One objective of this project was to question how children in primary mathematics classrooms (grades 3 and 4) describe and develop the construction and justification of new mathematical knowledge in their own words. The children worked in arithmetical learning environments. The arithmetical problem field of the first teaching episode (grade 4) centred on "number walls" (cf. Wittmann & Müller 1990). Number walls are constructed by first choosing four numbers to be placed on the four stones at the base level,and then the next level of three stones is filled by adding the two numbers immediately below. By using this rule of addition the whole wall is filled with numbers.

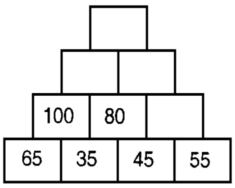


Fig. 9

In this lesson the children had started with four numbers, 35, 45, 55, and 65, and they had tried to construct different number walls by permuting these four numbers and placing them in different ways. The task was to find out different goal numbers at the top of the wall: What is the smallest and what is the biggest number that can be so "constructed"? The teacher then collected several number walls the students had constructed, and put them on the blackboard (see Appendix). The short episode addresses the question: What are the reasons for the variation of the top number when the base stones are exchanged?

The mathematical topic of the second episode (in a mixed class of grades 3 and 4) was the arithmetical environment of special "number squares", so called "crossing out number squares" (cf. Wittmann & Müller 1990). These number squares are constructed in the following way: First one adds some numbers given in the border row and border column of a table (cf. Fig. 10).

+	13	9	4
10	23	19	
12	25		
18			

Fig. 10

The squares thus created have the following property: You can choose (or circle) in a $(3 \cdot 3)$ number square three numbers arbitrarily but there must be one and only one circled number in every row and in every column. The sum of the three numbers chosen is always constant (cf. Fig. 11, 12).

+	13	9	4
10	㉓	19	14
12	25	21	⑯
18	31	㉗	22

Fig. 11

+	13	9	4
10	23	19	⑭
12	㉕	21	16
18	31	㉗	22

Fig. 12

The children discovered that the sum of three circled numbers – they always spoke of the "magical number" – is constant, here 66. In the course of this lesson the relation between the border numbers and the magical number was discussed: The sum of the six border numbers was equal to the magical number. The reason given was: Every border number is "in play" only once when it contributes, with one other border number, to the sum of three circled numbers. In this short episode, this justification is first repeated together with a new aspect of argumentation; then a student presents a completely new justification.

4.1 First episode: Relationships between the middle base numbers and the goal numbers in a four level number wall

At the blackboard the teacher displayed nine different number walls (see Appendix), which – working in pairs – the children had constructed. The systematic exchange of the four base stones led to walls with big and small top numbers.

1	174	Problem: Why does the top number remain the same when by exchanging the two middle numbers the two border numbers in the second level change?

The teacher points at the second and third stone, 45 and 35 (in the lower and in the middle number wall with top number "360": When these are exchanged then the sixth stone, 80, remains the same. But when in the lower wall the numbers 35, 65 on the third and fourth stone are exchanged, then in comparison with the middle wall there would be a change in the seventh stone. But the top number remains the same: Why?

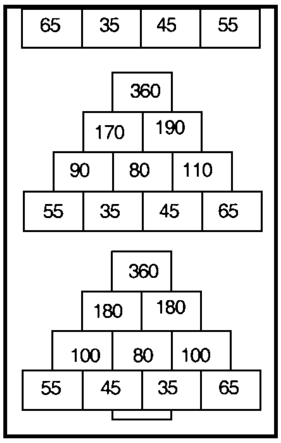


Fig. 13

2.1	175	Timo compares the corresponding border numbers of two number walls: They equalise.

Timo points at the fifth and the seventh stone in the lower number wall, both 100. Then he also points at the fifth and the seventh stone in the middle wall: 110 and 90. He states that this equalises, once ten more, and once ten less.

2.2	176–177	Nele repeats the explanation.

Nele once again explains this relation. She points at 110 on the seventh stone of the middle wall, and she says, the tens are distributed differently: Here 10 more and here (fifth stone) 10 less. In the lower wall the fifth and the seventh stone are equal.

2.3	178–185	The difference of "+10" and "−10" is noticed.

The teacher compares the "lower" fifth stone with the "middle" fifth stone and says: 10 less. She writes "−10" onto the fifth stone of the middle wall.

Then she points questioning at the seventh stone "110" of the middle wall. Children call "10 more", the teacher writes "+ 10" onto the seventh stone of the middle wall. Subsequently another student remarks similar differences of ten in the two numbers of the third level of the walls.

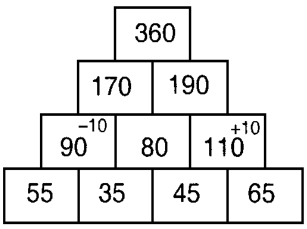


Fig. 14

Elements of an analysis of the first episode

The starting problem situation of this episode can be modelled with the help of the epistemological triangle in the following way:

See Fig. 23!

(Explanation of terminology: The circled numbers 1, 2, 3, etc. indicate the sequence of direct pointing at different stones in different number walls).

By posing the problem question and simultaneously pointing at different stones in two number walls the teacher constructs a new mathematical sign: a new arithmetical pattern in two number walls with the same top number "360". This new sign now has to be explained and justified: Why does the top number remain unchanged despite the fact that the border numbers in the second level of the wall are different?

Timo argues in the following way: At the fifth and the seventh stone in the lower number wall there is always "100", and correspondingly there is "90" and "110" at the fifth and the seventh stone in the middle number wall. These numbers compensate for each other, one being 10 more, and the other 10 less.

The construction of a mathematical sign

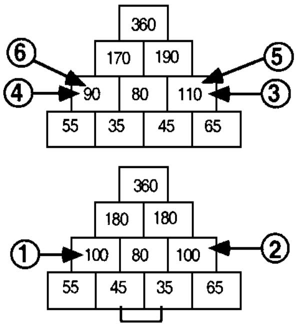


Fig. 15

Implicitly the existing numbers are compared with the number "100". The question as to what could be the reason for this "arithmetical equality" of "10 more" and "10 less", is neither posed nor do the children ask why this should lead then to the same top number (for instance because these are border numbers that are added in the construction of number walls only one time). Timo's statement contains observable arithmetical regularities, but no further justifying relation for this invariance of the top number is mentioned.

Nele explains the constancy of the top numbers in a similar way. She seems to take up the remark the "ten" from Timo's statement and she says that the tens are differently distributed. She too states that the fifth stone (middle wall) contains "10 less", and the seventh stone "10 more". The corresponding stones in the lower wall are equal, she says. This contribution is a repetition of the observation of arithmetical differences and "similarities" in the given number pattern. Here too, no "deeper" reasons are provided as to why this "arithmetical symmetry" should emerge, and why it could give rise to the constancy of the top number.

The construction of a mathematical sign

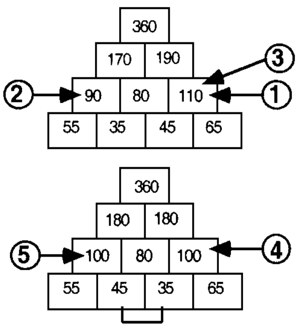


Fig. 16

The teacher emphasises the “arithmetical symmetry” as mentioned in their own words by both children; she repeats this observation and then writes “-10” and “$+10$” onto the fifth and the seventh stone. In the end another student transfers this comparison to the third level of number walls.

The construction of a mathematical sign

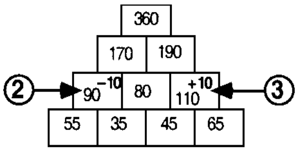


Fig. 17

4.2 Second episode: Relationships between the border numbers and the magical number in a 3×3 number square

At the end of this lesson, just as this episode (see appendix) starts, the teacher asks for repetitions of the argument as to why the magical number “66” remains constant. First Judith repeats her justification with other words. She again uses the description, that every border number is just “one time in play” in the additive construction of a circled number. Here she adds a new aspect to the argument: “... and therefore you may ... circle in a row only one number. Yes, because otherwise a number would be included twice” (396). Judith gives a kind of “indirect” reasoning: If you would have two circled numbers – which is strictly forbidden by the rule of circling numbers in the number square – then one would have a border number in two cases. This then shows why one border number is indeed only one time “in play” with the circled numbers.

+	13	9	4
10	23	19	14
12	25	21	16
18	31	27	22

Fig. 18

Later Kim presents a completely new justification not at all mentioned before.

8.4	407–411	Kim’s argument: One can break the magical number 66 into six (new) border numbers; then one can use these to construct a number square with the same magical number.

Kim advances her argument by constructing any adequate number square with the help of the magical number “66”. She introduces a partition of 66 into any six terms of a sum: “ehm, mh, you only need to break the sixty-six to pieces, into six numbers. And then you have to, when it is snipped into pieces, one would, one should then place them simply somewhere there still, and then calculate the numbers.” (408). She proposes to separate 66 into six terms of a sum (”snipped into pieces”), to order these in some way as border numbers (”place them simply somewhere”) and then calculate the numbers of the 3x3 square by addition of border numbers (”then calculate the numbers”).

Subsequently certain numbers have to be circled according to the known algorithm and their sum has to be calculated: “And, and, then, then, then one only has to do this with the number square. And then you get always sixty-six.” (408).

Kim is asked to explain her argument more concretely: She says that three numbers should be placed at the border row and the other three at the side, the border column (410). Kim again then refers to the given exemplary numbers when explaining the calculation of numbers in the square: “And then one has somehow to calculate, there for instance nine and ten makes nineteen” (410). She then remarks simply – as before: “And then, and when one would make this, one would always get sixty-six” (410). Here she does not again explain that all border numbers appear exactly one time as a term in an addition. The teacher acclaims Kim’s contribution and in this way confirms the justification.

Elements of an analysis of the second episode

As a kind of recapitulation several students again formulate the explanations for the “trick”. First Judith repeats her justification with new descriptions she had developed before.

From an epistemological perspective, Kim’s argument starts with a sort of “inversion” of the statement “The sum of the border numbers is equal to the magical number”: “A given magical number (66) can be decomposed into six (arbitrary) terms of a sum, and these six numbers can be used as border numbers for the construction of a number square.”

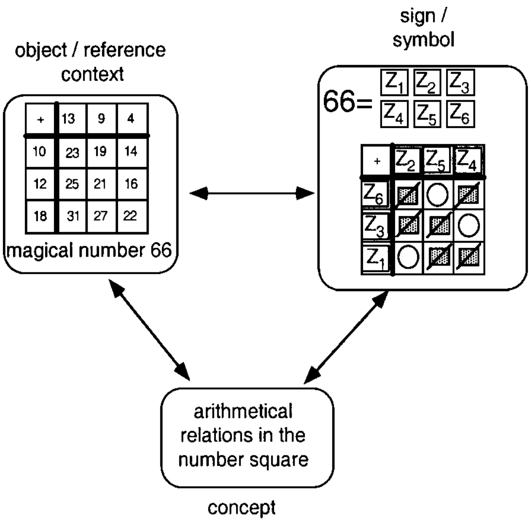


Fig. 19

In her argument Kim proposes the following construction: (1) Decompose 66 into six arbitrary numbers; (2) Place these six numbers on the border; (3) “Calculate” the number square; (4) Perform the crossing out algorithm. Kim does not again explicitly mention the relationship between two border numbers and one circled number; she states (5) The starting number 66 is the magical number. This completes her argument. The vagueness of the first version of her argument (408) is partially concretised and confirmed in the repetition of the argument (410) with reference to examples. This allows the display, in the epistemological triangle, of the construction of the mathematical object "general (magical) number square built up from a pre given magical number".

The presentation of the new mathematical "sign / symbol" as given in the epistemological triangle in the form of an abstract diagram has not been constructed by Kim in this way; she has developed a verbal description of this new symbol. The relation between "object / reference context" and "sign / symbol" is also exchangeable in this example. New structures are imposed into the given number square, and conversely, concrete numbers and arithmetical problems from the given reference context are used to explain new and more general relations.

5. Concluding remarks: Comparing the analysis of the two episodes

During the first episode the teacher formulates an open – partially inapplicable – problem question; nevertheless the children react in the expected manner. Using an arithmetical regularity, Timo constructs a possible justifying relationship: He compares the fifth and the seventh stone and he states that these two stones "compensate", once ten more and once ten less. He says "That's why, ... ", meaning that therefore the top stone remains unchanged. Also Nele uses this relation to justify the invariance of the top number: On both stones in question the tens are differently distributed, and therefore they (together) equal, and that is the reason for the top number remaining unchanged. The teacher confirms these proposals writing "-10" and "$+10$" on the fifth and the seventh stone.

The interactively constituted argument can be summarised in the following way: The top stone remains unchanged because – despite their change – the border stones in the second level of the wall "compensate". But this remark is only a first possible aspect of a complete argument. The students have observed and stated the "compensation" of two numbers. But what are the reasons for this compensation, what kind of justification could be given? And further: Why does this "compensation" lead to the invariance of the top number? Obviously the lower number wall at the right (with top number "380", see Appendix) possesses in the fifth and the seventh stone the same kind of "compensation" as the number wall discussed in the episode, however it has a different top number. There must be additional conditions. Or, is it also possible to justify the invariance of the top number on the basis of a compensation between the fifth and the sixth stone?

The arguments developed in this interaction refer to the arithmetical surface structure of the numbers in this number wall; an arithmetical compensation is used to conclude the invariance of the top number. But only the help of deeper relations responsible for the construction of the number walls would permit the argument to be completed. During the second episode Kim constructs a new relationship between border numbers, magical square and magical number. Whereas up to now the direction of argumentation started from the given 3×3 number square looking at the magical number to be calculated as the sum of three circled numbers and then at the border numbers generating the number square – the sum of these numbers giving the magical number – Kim now proceeds "conversely": Starting from a magical number and its decomposition into six border numbers, a magical number square with just this (starting) number can be constructed. On this basis the new knowledge relation can be summarised in the following way.

- The chain of argumentation as interactively developed in this class can be characterised – in this way: In certain 3×3 number squares – that are constructed by the addition of border numbers – magical numbers can be calculated from three circled numbers, and these magical numbers are constant because the sum of the six border numbers is equal to the magical number and because each border number appears exactly one time in the addition tasks for the three circled numbers.
- The argumentation changes with Kim's justification in the following manner: A number is chosen (66), and this number is decomposed into six border numbers, that allow by addition the construction of a 3×3 magical square. Each border number appears in one circled number once in the addition task. The chosen number is, by construction, equal to the sum of border numbers, and hence it is also equal to the sum of three circled numbers: the magical number.

A first comparison of these two episodes shows that from a communicative perspective the type of interaction in both situations has a similar, "open" character. In the first episode, the two students Timo and Nele do not use the numbers in the number walls simply in a definite, direct manner. They try to elaborate *relations* between different numbers for justifying the invariance of the top number. The two numbers "90" and "110" compensate and hence they argue that this represents the same constancy as the two numbers "100" and "100". In this way, an arithmetical relation is established. The signifier "90 and 110 compensate" has to be interpreted by the other participants of communication; this signifier has not yet a definite signified, but it has to be constructed in communication. In this way, one can observe a "true", open mathematical interaction in this episode, in the way as it is described in the theoretical approach of Luhmann (see part 3). In the second episode Kim conveys different, open signifiers that have to be understood by constructing new signifieds in communication. One signifier to be interpreted is: "... you only need to break the sixty-six to pieces, into six numbers. And then you have to, when it is snipped into pieces, one would, one should then place them simply somewhere there..." This signifier needs a new interpretation for the way the magical number "66" is used now, i.e. intentionally as the generating number for six border numbers. This example also demonstrates the "open" use of signifiers in communication, as required in Luhmann's theoretical description. In this first comparison one can identify similar uses of signifiers in communication; the students do not take the numbers directly for calculating arithmetical results, etc. But they intend to construct relations between numbers that could lead to a justification and a possible generalisation. In this way, a relation between border numbers (in the second level) and the top number is constructed in the first episode; in the second episode, the magical number is interpreted in a new relational way as the partition of six border numbers.

A fundamental problem of the epistemological interaction analysis is the adequate interpretation of mathematical statements in communication, in which with their own words, in the frame of existing mathematical knowledge and familiar descriptions, new knowledge has to be constructed and to be justified interactively by children. In both episodes one can observe such mathematical statements with the intended construction of new knowledge; from a general, communicative perspective both communication processes are open, in a way that communicated signifiers have to be interpreted. They do not have a given, definite signified beforehand. And also the arguments developed in these two situations are not yet "complete", and they are presented in a situated context, not as "strictly logical" justifications; further one can state that the use of available, familiar, partially concrete means of description in the actual frame of means of mathematical description makes the analysis difficult: is there indeed an intended construction of new knowledge or is some already known knowledge simply repeated?

But from an epistemological perspective, there is an important difference in the interactive construction of new knowledge in these two episodes. The relation constructed by Kim "The magical square is the result of a chosen magical number" goes beyond the perception of patterns and structures on the arithmetical surface on number squares and takes into account underlying, deeper structural relations. Kim's argument cannot be plausibly interpreted immediately from her perception of observable arithmetical regularities; it can only be reconstructed as a partial argument in the frame of invisible, deeper mathematical relations. The argument developed in the first episode remains on the surface of observable arithmetical regularities. A relation is constructed from observing the difference between "90" and "110" as a balance to "100", and this regularity is taken directly as the basis for the justification of the invariance of the top number. In short the argument developed here is: "Because the two numbers 90 and 110 balance to 100, and hence there is some sort of equivalence compared with the two border numbers of 100 in the other number wall, therefore the top numbers also do not differ, they remain the same". But this argument remains on the surface, it does not take into account the deeper, relational structure, i.e. the construction rules for number walls.

The critical evaluation of the arguments given in the first episode as well as the assessment of the arguments presented in the course of the second episode require an analysis from an epistemological perspective. This analysis focuses upon the possible functional relations in the mathematical "object" being communicated here and their reconstruction as "symbolic relational structures" in line with the assumptions about the nature of mathematical knowledge developed above. In the context of the interpretations being interactively constituted, the observer has to develop a potential spectrum of knowledge relations and networks, which support his / her capacity to recognise and help to develop those arguments and justifications produced and adequately to assess them and be able to reflect on them from an epistemological perspective.

6. Acknowledgement

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Author

Steinbring, Heinz, Prof. Dr., Institut für Entwicklung und Erforschung des Mathematikunterrichts (IEEM), FB 1, Universität Dortmund, Vogelpothsweg 87, D-44221 Dortmund.
E-mail: heinz.steinbring@math.uni-dortmund.de

Appendix: Transcripts of two teaching episodes

1. Episode:
Relationships between central base numbers and the goal number in a number wall of four levels

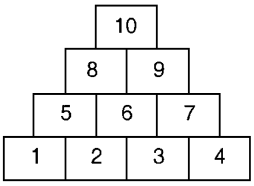

Fig. 20

[To describe the position of numbers on the number wall the stones are numbered consecutively from bottom to top and from left to right in the transcripts.]

The following number walls are displayed on the blackboard.

See Fig. 24!

174 T # Mh, I repeat the question again, Sascha, because it is really difficult now. ... Well, when we change in the middle, [*points at the second and third stone "45" and "35" and then at the sixth stone "80" of the lower number wall with goal number "360"*] then here [*points at the second and third stone "35" and "45" of the number wall in the middle with goal number "360", subsequently she points at the sixth stone "80"*] we get always the same, sure. Yes, you have found this correctly. But, when we change here [*points at the third and fourth stone "35" and "65" of the lower number wall with goal number "360"*] this stone changes [*points at the seventh stone "100" of the lower number wall and at the seventh stone "110" of the middle number wall with goal number "360"*] Nevertheless there is the same goal number. [8 sec pause]

174b S Because eh... That is always the same.

174c T Timo!

175 Ti That's why, [*goes to the blackboard*] well, yes, ehm, here, here is hundred, hundred [*points at the fifth and seventh stone "100" of the lower number wall with goal number "360"*] and here is hundred-ten and ninety. [*points at the seventh and then at the fifth stone in the middle number wall with goal number "360"*] That compensates again. There is ten more and there is ten less. Then it's equal again. ... [*he points at the seventh stone "110" and at the fifth stone "90" in the middle number wall with goal number "360"*]

176 T Who of you could explain this again? Then I could write it down, Nele!

177 N Well this is always. [*goes to the blackboard*] Here, the tenths are differently distributed, [*points at the seventh stone "110" of the middle number wall with goal number "360"*] then here ten less [*points at the fifth stone "90"*] and there ten more. [*points at the seventh stone "110"*] And here it is then equal. [*points alternately at the seventh and the at the fifth stone "100" in the middle number wall with goal number "360"*]

178 T [points first at the fifth stone "100" of the lower number wall and subsequently at the fifth stone "90" in the middle number wall with goal number "360"] Ten less. [writes "-10" into the fifth stone of the middle number wall with goal number "360"] Can you see this?

179 S # The difference is equal. #

180 T # A bit small, yes? And here accordingly then? [*points at the seventh stone "110" in the middle number wall with goal number "360"*] ...

181 S Ten more

182 S Ten more

183 T Yes. [*writes "$+10$" into the seventh stone*]

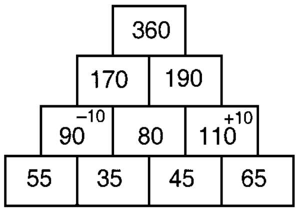

Fig. 21

Then, another student remarks on similar differences of ten in the two numbers of the third level of the walls.

2. Episode:
Relationships between border numbers and the magical number in a 3×3 number square

+	13	9	4
10	23	19	14
12	25	21	16
18	31	27	22

Fig. 22

At the end of this lesson the teacher asks for the justifications why the magical number of this number square is always 66 to be repeated. First, Judith repeats her own justification in other words. Later, Kim presents a completely new justification.

396 Ju Well. Yes, well, when you there calculate eighteen plus twelve plus ten plus thirteen plus nine plus four then it is always sixty-six. Yes, ehm, now when you add all these many numbers. We have the thirty-one that, that is the same as the eighteen and the thirteen. Well, ehm, and, and from the plus numbers, ehm from the border numbers that, yes, that, that now, every border number comes now one time into play, when you cross out something there, or so. This, this, yes ... and therefore you may ... ehm, then, yes, then, circle in a row only one number. Yes, because otherwise a number would be included twice.

397 S Ahh!

...

...

407 T Mhm. Kim.

408 Ki ehm, mh, you only need to break the sixty-six to pieces, into six numbers. And then you have to, when it is snipped into pieces, one would, one should then place them simply somewhere there still, and then calculate the numbers. And, and, then, then, then one only has to do this with the number square. And then you get always sixty-six.

409 T Super, really super! Please show where you should place the six numbers which you have snipped out.

410 Ki [goes to the blackboard] Yes, simply there above, three pieces and there, and then there a plus sign, [points at the according places in the border of the square]. And then one has somehow to calculate, there for instance nine and ten makes nineteen. And then, and when one would make this, one would always get sixty-six.

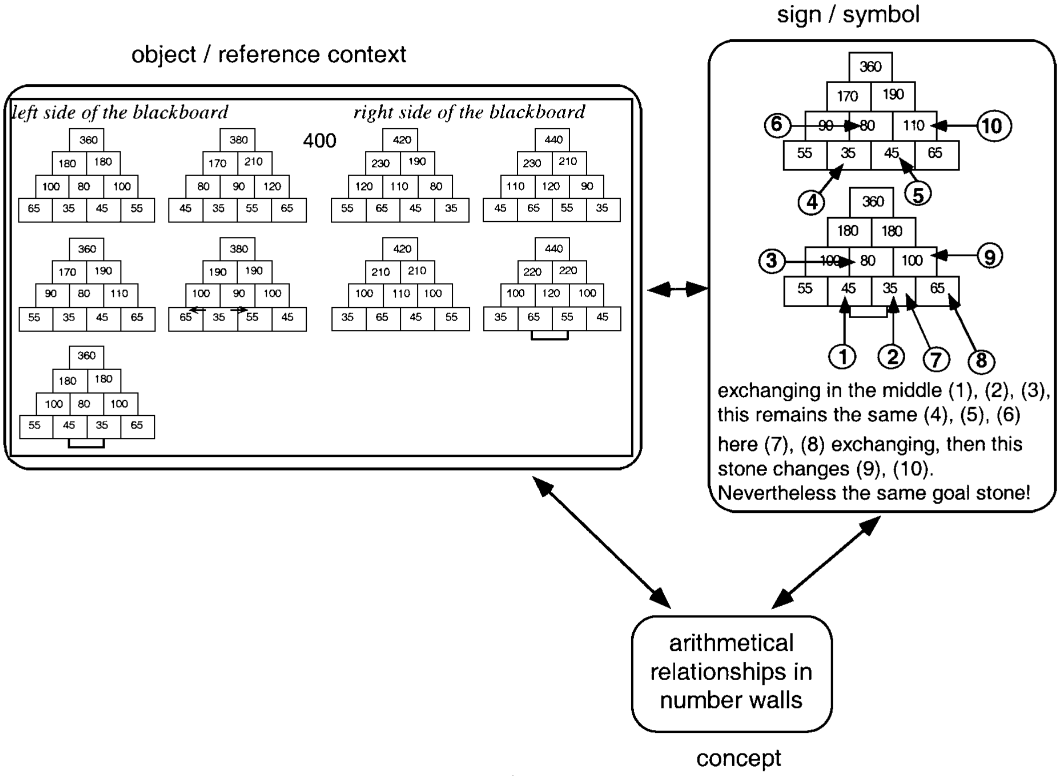

Fig. 23

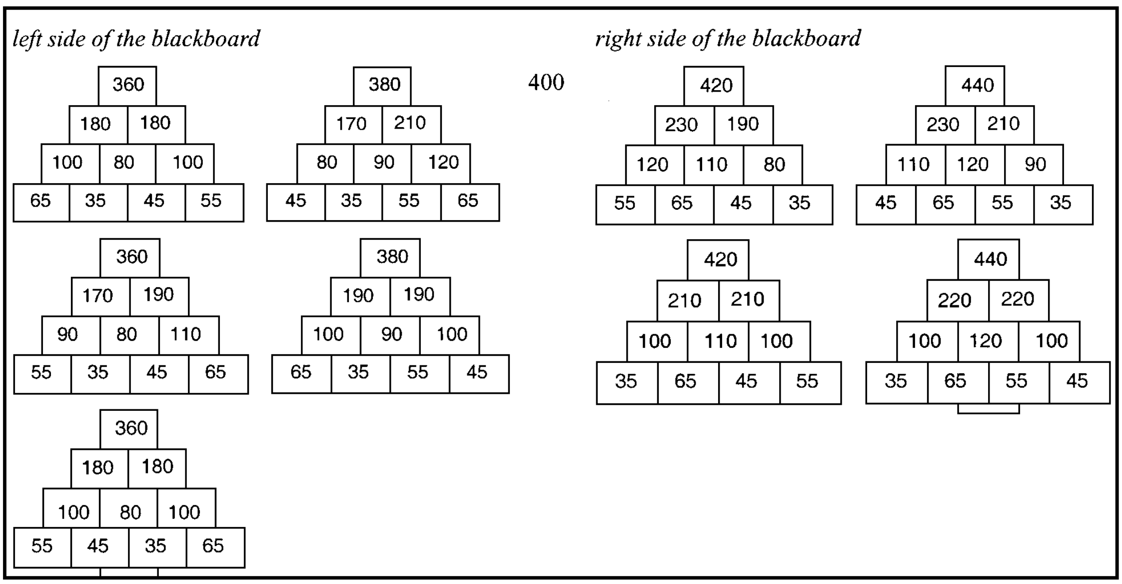

Fig. 24