The Geometry of M. C. Escher's Circle-Limit-Woodcuts[1]

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Abstract: Maurits Cornelis Escher has been impelled by the idea of visualizing infinity within the finite region offered by the frame of a normal picture. After some interesting attempts, which did not at all satisfy him, he received inspiration from a printed figure given in a paper on symmetry by the outstanding geometer Harold Scott MacDonald (called Donald) Coxeter. This figure served to illustrate a tessellation in the hyperbolic plane represented by the interior of a circle according to a geometric model proposed by Jules Henri Poincaré. The points of the limiting circle (called *Circle Limit* in the theory of automorphic functions) have infinite hyperbolic distance from the interior points of the hyperbolic plane. As a consequence the tiles of the tessellation appear to decrease infinitely when passing closer and closer to the border. This was exactly the idea Escher needed in order to make infinity visible. So in the years 1958 to 1960 he produced four woodcuts based on the idea of hyperbolic tilings and patterns, though he claimed not to understand the mathematical background of his pictures. The lecture presents a mathematical analysis joined by a computer-reconstruction of Escher's *Circle-Limit*-Woodcuts. The mystery of these pictures remains untouched and the ingenuity of their invention and construction emerges in even greater lucidness.

Kurzreferat: *Die Geometrie von M. C. Eschers Kreislimit-Holzschnitten.* Maurits Cornelis Escher suchte intensiv nach Möglichkeiten, mit graphisch-malerischen Mitteln unendliche Wiederkehr in endlichen Figuren darzustellen. Nach einigen interessanten Versuchen, die ihn jedoch nicht völlig befriedigten, stieß er in einer Abhandlung über Symmetrie des bedeutenden Geometers Harold Scott MacDonald (gen. Donald) Coxeter auf eine Abbildung, die er für sein Vorhaben als Offenbarung empfand. Die Abbildung zeigt eine Parkettierung der hyperbolischen Ebene, die man sich – einer Idee von Jules Henri Poincaré folgend – als das Innere eines Kreises vorstellen kann. Die Punkte des Randes (in der Theorie der automorphen Funktionen als Grenzkreis bezeichnet) haben in dieser Geometrie unendlichen Abstand von den inneren Punkten der hyperbolischen Ebene. Infolgedessen scheinen die Pflastersteine des Parketts zum Kreisrand hin unaufhaltsam zu schrumpfen, obwohl sie in der hyperbolischen Geometrie kongruent sind. Escher erkannte hier ein Prinzip, das er für seine Vorstellungen von der Visualisierung des Unendlichen in einer endlichen Figur für hervorragend geeignet hielt. So entstanden in den Jahren 1958 bis 1960 die vier sogenannten Kreislimit-Holzschnitte, in denen er hyperbolische Parkettierungen realisierte, wobei er stets betonte, den mathematischen Hintergrund seiner Bilder nicht zu verstehen.

Im folgenden wird eine mathematische Analyse der Kreislimit-Holzschnitte gegeben und mit dem Versuch einer Computer-Rekonstruktion verknüpft. Das Geheimnis dieser Bilder wird dadurch nicht angetastet. Vielmehr tritt die konstruktive Phantasie Eschers mit umso größerer Deutlichkeit hervor.

ZDM-Classification: G90, M80

1. Introduction

The Dutch artist M.C. Escher was fascinated by esthetic objects giving an idea of *infinity* (Locher 1984; Ernst 1978; Escher 1971). Plane Euclidean ornaments for instance represent such objects. In these ornaments the perception has to be strongly supported by imagination, because infinity is suggested by the repetition of a motif. Physically this repetition can be done only a finite number of times. It therefore has to be performed in mind thus initiating a meditative process. Escher intended to improve the evidence in visualisations of infinity. So he started several attempts to show infinity in a finite region. He achieved his final success after he had seen a diagram which Donald Coxeter, the outstanding British-Canadian geometer, had used in a mathematical paper on *hyperbolic geometry.* Here I have reconstructed this diagram (Fig. 1) by methods of fractal geometry (Herfort 1993, Herfort/Klotz 1997).

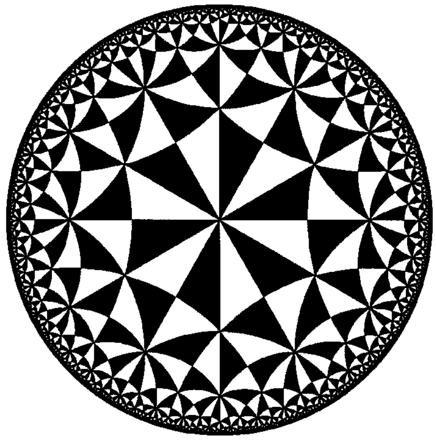

Fig. 1: Coxeter's Diagram (reconstruction) which inspired Escher

Stimulated by this mathematical diagram (Fig. 1) Escher produced four very exciting woodcuts between 1958 and 1960 (Fig.2).

2. Features of hyperbolic geometry

Analysing these pictures leads on one hand to possibilities of reconstruction, on the other hand to new constructions of similar pictures according to the same principle. In the classroom this fact could be used as a stimulation and motivation to look at the geometry behind these woodcuts. So the aim of this paper is to show along which lines the argumentation might go, if pupils are to get some insight into non-Euclidean geometry. Of course there is no need to develop a complete course on hyperbolic geometry. But the concept of *reflection in a circle* is important.

2.1 The concept of reflection in a circle

As Coxeter has exemplarily shown (Coxeter 1963) this matter can be treated by *elementary construction.* Even global properties as the *conservation of angle* and *of circles* can be derived without *algebraic tools* as for instance complex-number-calculations. Besides these conservation laws students should know some analogies between reflection in a circle and reflection in a line:

- Reflecting a reflected point in the same reflecting manifold (line or circle) results in the original point. (In other words: Reflections are involutions.)
- The reflecting manifold (line or circle) consists of fixed points.

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[1]Paper presented to the 8th International Conference on Geometry, Nahsholim (Israel), March 7–14, 1999

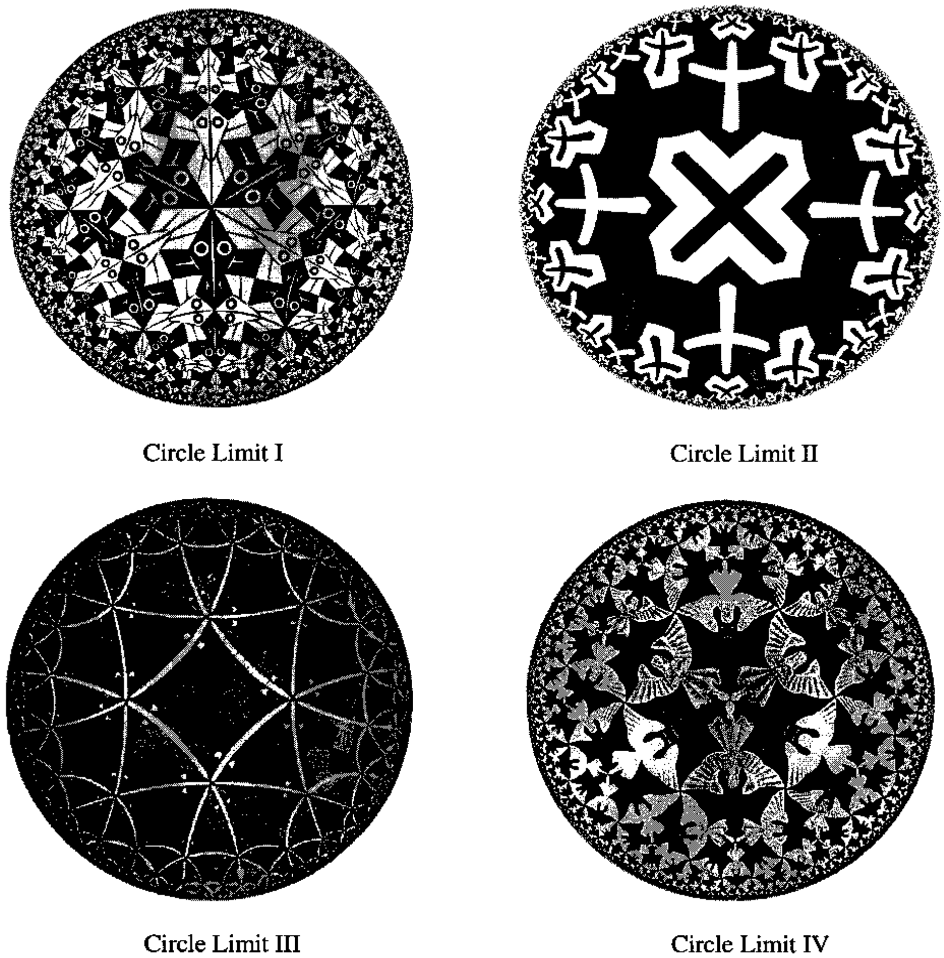

Fig. 2: M. C. Escher, Circle Limit I–IV, Woodcuts 1958–1960

- The orientation of a closed curve is changed by reflection.
- Reflections are homeomorphisms. (Even more they are isometries, if the metric is properly defined. As we do not propose to introduce a hyperbolic metric in the classroom-situation, we confine ourselves to the properties of a topological mapping. We can do so, as only topological properties are needed for the games we want to introduce later.)

The Coxeter-diagram suggests that triangles will be the matters of our main interest. But the concept of triangle has to be extended. Now the sides of a triangle are circular arcs. There is however a small exception: triangles having the center of the surrounding circle as one of its corners are bordered by two ordinary sides and only one circular arc. All the triangle-sides are part of a circle, which intersects the surrounding circle under an angle of 90°. We may regard the straight lines through the center as circles with center at infinity. For a better distinction from ordinary triangles we shall use the notion of *hyperbolic triangle* for these triangles within the surrounding circle.

Having the Coxeter-diagram in mind we are looking at a so called *tessellation of the circular region* (bounded by the surrounding circle) *with hyperbolic triangles.* That means: Every point of the circular region belongs to at least one triangular region (including the boundary) and the triangles are not overlapping. That means: If a point belongs to more than one triangular region it lies on the boundaries of these regions.

What happens to a hyperbolic triangle from the tessellation when it is reflected in one of its circular sides? From the knowledge listed above about circular reflection students will be able to conclude that the image of the triangle is again a hyperbolic triangle which is *equiangular* with the original one. They will notice that this reflected triangle belongs to the tessellation. Thus it can be conjectured, that the whole tessellation can be generated by continued reflection beginning with one particular triangle.

2.2 Tiling of a circular region by hyperbolic triangles

The particular starting-triangle may be ABC. Reflection in AC will be symbolized by 1; reflection in AB by 2 and finally reflection in BC by 3.

The whole tessellation now can be generated by continued application of these 3 reflections composing them in all possible orders. The symbol 3212132 for instance may be interpreted as:

Apply 2, then apply 3, then apply 1, ... , then apply 3

It should be read from the right to the left.

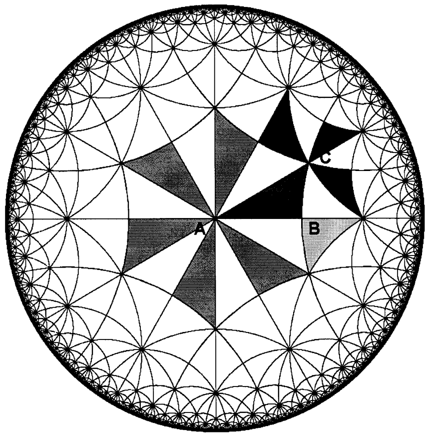

Fig. 3: Basic hyperbolic rotations, centers A, B and C

In order to achieve good practice in recognition of the generating process and to become familiar with the coding, I propose two games A and B:

(A) Write down an arbitrary sequence of the figures $1, 2, 3$ and find out the hyperbolic triangle into which the triangle ABC is transported when the application meant by the given sequence is performed.

(B) Pick up an arbitrary triangle from the tessellation and find out which application maps the triangle ABC into the given one. There are many ways to do this. Find a good – that means a "short" – way.

Some results of these games should be summarized as shortening rules:

(i) A sequence containing 11 or 22 or 33 can be shortened by simply discarding these figures; due to the involutary character of reflections.

(ii) Obviously the map 12 symbolizes an *anticlockwise rotation with center A*. The angle of rotation is twice the angle occuring in the tringle at point A. Performing the rotation 12 six times 121212121212 brings the triangle ABC back to itself. So if such a piece occurs in a sequence it can be discarded.

(iii) A similar phenomenon emerges in the points B and C. But here the composition 31 (resp. 23) is not a rotation in the Euclidean sense. It has nevertheless all important properties of a rotation. So we shall call it here an *anticlockwise hyperbolic rotation with center* C (resp. B). The angle of rotation is twice the angle at point C (resp. point B). We find:

(a) If a sequence contains a piece 31313131 it can be canceled.

(b) If a sequence contains a piece 2323 it can be canceled.

We call A, B and C *centers of rotation.* The minimal number of repetition necessary to "come back" to the original triangle ABC we shall call the order of the center of rotation. So in this case

A has order 6,
B has order 2,
C has order 4.

By the order of the points A, B, C the angles at A, B and C are uniquely determined, and so is the structure of the *set of mappings generated by the three hyperbolic reflections* 1, 2, 3, *where the basic hyperbolic rotations* 12, 31, 23 *are subject to the relations*

$(12)^6 = id,$
$(31)^4 = id,$
$(23)^2 = id.$

This set is called the *hyperbolic reflection group* $T^*(6, 2, 4)$. If we only regard the applications which are generated by the basic rotations 12, 31, 23 and which are represented by all sequences with an even number of digits, we denote this set as the *hyperbolic rotation-generated subgroup* $T(6, 2, 4)$.

As became clear the group $T^*(6, 2, 4)$ generates the Coxeter-tessellation with hyperbolic triangles, which are all images of the *fundamental region,* the triangle ABC. Also the subgroup $T(6, 2, 4)$ generates a tessellation of the circular region. In this case however the fundamental region has *double size.* A union of ABC with one of the neighbouring hyperbolic triangles can be regarded as fundamental region if the subgroup $T(6, 2, 4)$ shall generate a tessellation.

If k, l, m are integers satisfying the so-called *condition of hyperbolicity* $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$ there can be constructed a triangle ABC within the circular region, which can act as a fundamental region for a triangular tessellation generated by the reflection group $T^*(k, l, m)$. Again the union of ABC with an adjacent triangle can act as a fundamental region for the corresponding rotation-generated subgroup $T(k, l, m)$. As a union of two adjacent triangles this fundamental region in general is a quadrangle and so in general we get a tessellation with quadrangles.

There exists an infinite manifold of hyperbolic tessellations from which M.C. Escher could have started. Let us now look from which groups he started producing the 4 woodcuts. Knowing the group and the design of the single motif filling the fundamental region the hyperbolic Escher-ornament can be reconstructed fairly well.

3. Analysis of the Circle-Limit-Woodcuts

3.1 Circle Limit I

In the case of *Circle Limit I* Escher refers to the Coxeter-tessellation. Therefore we superpose the picture (Fig. 4) with the net of the Coxeter-tessellation. Now we see, that the picture may be built up beginning with the weakly

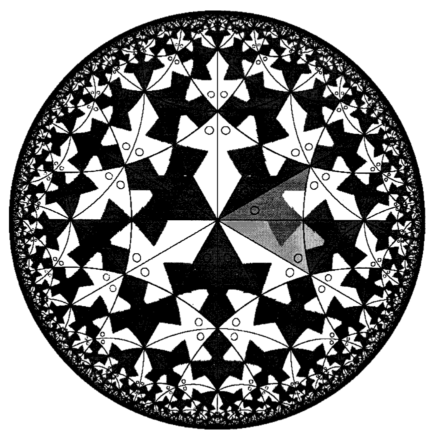

Fig. 4: Reconstruction of Circle Limit I, triangular tessellation generated by T*(6,2,4), fundamental region of T(6,2,4) highlighted

shaded part of the central area, which can be regarded as a fundamental region of the group $T(6,2,4)$. But neither the group $T(6,2,4)$ is "acting" here, nor the reflection group $T^*(6,2,4)$. Rather a subgroup of $T^*(6,2,4)$ generated by the rotation 1212, the reflection 1 and the rotation 23 builds up the whole picture from the coloured motif.

It should be annotated, that the fundamental region of this generating subgroup used by Escher emerges from a very inventive deformation of its fundamental region.

3.2 Circle Limit II

With some practice in looking at those hyperbolic ornaments students should notice that *Circle Limit II* has a center of rotation at A of order 4 and two centers of rotation of order 3. So it seems to be self-evident, that we superimpose the net of the triangular tessellation generated by $T^*(4,3,3)$ (Fig. 5).

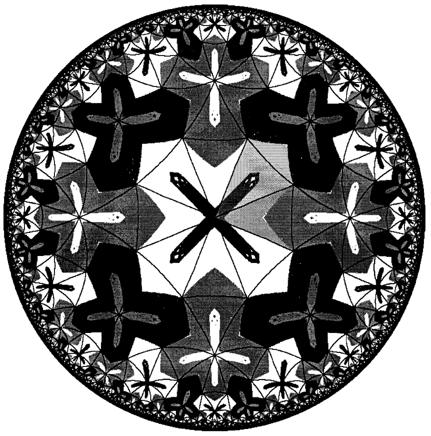

Fig. 5: Reconstruction of Circle Limit II, triangular tessellation generated by $T^*(4,3,3)$, fundamental region of $T(4,3,3)$ highlighted

Those parts of the central area which are weakly shaded may be regarded as the motif of the hyperbolic ornament. The rotation-generated group $T(4,3,3)$ produces the ornament when applied to this highlighted region. Again it is obvious that a quadrangular fundamental region of the group $T(4,3,3)$ has to be deformed in a sophisticated manner in order to gain this motif.

There occur three kinds of crosses distinguished by their colouring. Crosses of the same type can be generated by applying a particular subgroup of $T(4,3,3)$ to a special one of those crosses, while crosses of the two other types are gained by applying the two different cosets to the same special cross (Herfort/Klotz 1997).

Remark: As in *Circle Limit IV* also a slightly different "larger" group can be seen acting here. Using the obvious symmetry in the central cross we can produce this hyperbolic ornament by application of a subgroup of $T^*(8,2,3)$ to a fundamental region gained by cutting off half of the highlighted region. The correct subgroup is then generated by 2, 1212 and 31.

3.3 Circle Limit III

As in the case of *Circle Limit II* the trained student will notice a center of rotation of order 4 in the center of the circular region (Fig. 6). Besides he will discover two centers of order 3. So the tessellation he will superimpose ought to be again $T^*(4,3,3)$.

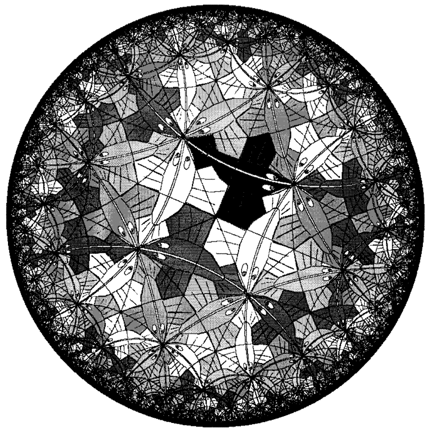

Fig. 6: Reconstruction of Circle Limit III, triangular tessellation generated by $T^*(4,3,3)$, fundamental region of $T(4,3,3)$ highlighted

Undoubtedly Escher appreciated this picture in a particular manner. He presented one of the imprints to Coxeter proudly emphasizing that he himself had invented its basic symmety system. Indeed it seems to be the most interesting one among the four *Circle-Limit*-woodcuts. As in the case of *Circle Limit II* its colouring presents an additional mathematical structure. A particular subgroup and its cosets may again be regarded as responsible for the four different colours (Herfort/Klotz 1997). But there is a mystery: The white fishbones forming a circular line along which the equicoloured fishes are swimming cause irritation and mislead the viewer. Escher believed that these lines intersect the surrounding circle under an angle of $90°$ (Locher 1984). He obviously thought, that they belong to the net of the tessellation. But surprisingly they do *not* as can be seen from the given reconstruction. Coxeter (1996) disproved Escher's assumption.

Again the motif is highlighted by changing here the strength of its shading. The dark fish can be regarded as a fundamental region of the group $T(4,3,3)$ for it emerges by a very clever deformation from two adjacent triangles belonging to the tessellation related to $T^*(4,3,3)$. Indeed the total ornament is generated by application of the rotation-generated group $T(4,3,3)$.

3.4 Circle Limit IV

Now in his last woodcut *Circle Limit IV* Escher returns to a unicoloured design (Fig. 7). This does not happen by accident, I believe. On first glance the ornament seems to be generated by the group $T(3,4,4)$. So we shall superimpose the net of the triangular tessellation generated by $T^*(3,4,4)$.

Here the union of an angle with a devil can be regarded as a fundamental region of the tessellation, and it is obvious that this figure again arises from two adjacent triangles by a clever deformation. Indeed the group $T(3,4,4)$ generates the hyperbolic ornament when applied to this figure.

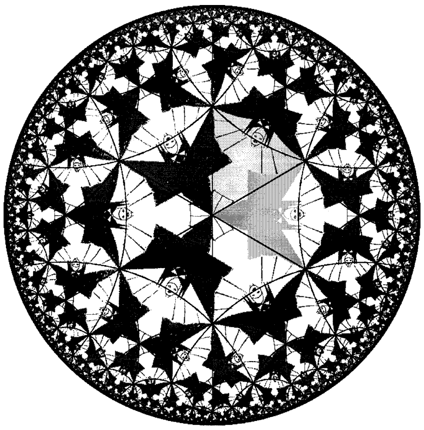

Fig. 7: Reconstruction of Circle Limit IV, triangular tessellation generated by $T^*(3,4,4)$, fundamental region of $T(3,4,4)$ highlighted

But critically revising our result we notice that there is still a symmetry in the figure which has not been used in the production of this ornament. So a "larger" group could be more adapted for its description. In fact we may describe it as generated by $T^*(6,2,4)$. But then we have to choose the subgroup generated by 1212, 2, 13. In this case the fundamental region would shrink to half an angle plus half a devil. It is more likely that Escher based his construction on this group because he could use the same principle of construction as in *Circle Limit I*. Maybe that in both cases he did not find a sufficiently interesting subgroup for colouring.

4. Final remark

The method by which these hyperbolic Escher-ornaments – and completely new hyperbolic ornaments as well – can be generated originates from the theory of *Iterated Function Systems* (IFS), a particular part of *fractal geometry*. The IFS-method is famous as well for its esthetical output as for its practical use in the field of image-compression. The motif of the hyperbolic ornament – that means its restriction to the fundamental region of its symmetry group – is designed as the attractor of an IFS. This motif-generating IFS has to be hierarchized by the symmetry-group. A set of generators of this group is treated as an IFS though being a set of "hyperbolic isometries" and not a set of *contracting mappings* (For details of the method cf. Herfort/Klotz 1997).

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