Felix Klein Meets Napoléon[1]

Dedicated to Prof. R. Artzy

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Abstract: In this article we will show how, by teaching our students geometric transformations, we can help them see the way to the solution when faced with a challenging geometric problem. In this way they will understand that they should not let themselves be restricted by the tools they use.

Kurzreferat: *Felix Klein trifft Napoleon.* Im Rahmen der Abbildungsgeometrie zeigen wir, wie wir unseren Schülern helfen können, den Lösungsweg zu sehen, wenn sie mit einer herausfordernden Aufgabe konfrontiert sind. Auf diese Weise werden sie verstehen, dass sie sich nicht selbst durch die Werkzeuge einschränken, die sie benutzen.

ZDM-Classification: G40, G50

1. Introduction

Transformation geometry is aimed at liberating the teaching of geometry from domination by Euclid. The Euclidean geometry style in teaching is distinguished by emphasis on rigid figures: rigid congruences of triangles are considered to be the main method of proof in elementary geometry.

The notion of transformation gives a central concept to modern geometry teaching, and instead of the static treatment of Euclid we have a dynamic geometry.

Students should be taught not only transformation geometry but also shown how a geometric transformation, or a series of geometric transformations, can help the problem solver see the way to the solution, and not let himself be restricted by the tools he uses.

We would like to cite a phrase from **[1]**: " ... algebraic proofs are superior to geometric proofs which may contain errors based on mere geometric intuition". Indeed, the geometric proofs appear to show their weaknesses when required to handle "non-standard" problems.

In this paper we first lay out and construct our tools in the *Basics.* Then we apply them by taking a common matriculation problem, given in Israel at the 5 point mathematics examination, and we solve the problem in a dynamic way. By comparison, one can easily solve the same problem in a rigid way, as it is being done in high school.

Finally, in *Old problems new solutions* we present well known problems **[2]–[7]** and some problems from mathematical olympiads, and we solve them with the tools we have constructed in *Basics.*

Since 1983 I have been teaching at the University of Haifa, Israel. In the classroom, in different courses, my students have had to cope with problems similar to the matriculation problem mentioned above. When given a generalized problem (a regular n-polygon with $n > 4$) they often complained, saying the solution is a nightmare. We felt much better when a new method was presented to solve these problems, using a powerful tool – the dynamic conception. As a result the obstacles were reduced so that the solutions become shorter and easier.

The reader can follow the challenge and hopefully may be stimulated to continue using the central ideas which have been presented in this paper.

2. Basics

a. If $P = (x, y)$ is a point in $\mathbb{R}^2$ than the image of P under a rotation about the origin with an angle α, in a direction opposite to the classic watch, is given by:

$$\rho_{0,\alpha}(P) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (1)$$

and the image of a point P under a rotation around a point $M = (e, f)$ with an angle α is:

$$\rho_{M,\alpha}(P) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x-e \\ y-f \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}. \qquad (2)$$

b. If $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a linear transformation given by multiplication with a matrix $D_T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $\det D_T = \Delta \neq 0$, then the image of a line $Ax + By + C = 0$, $|A| + |B| \neq 0$ under T, is a line of equation:

$$x(Ad - Bc) + y(-Ab + Ba) + C\Delta = 0. \qquad (3)$$

From (3) we obtain that the equation of a line ℓ after a rotation by an angle α, about a point $M = (e, f)$ is:

$$\rho(\ell)\colon (x-e)(A\cos\alpha - B\sin\alpha) + (y-f)(A\sin\alpha + \\ +B\cos\alpha) = 0 \text{ if } M \in \ell \qquad (4)$$

and

$$\rho(\ell)\colon (x-e)(A\cos\alpha - B\sin\alpha) + (y-f)\cdot \\ \cdot(A\sin\alpha + B\cos\alpha) + Ae + Bf + C = 0 \text{ if } M \notin \ell \qquad (5)$$

c. As in [1] we denote by A^2 an affine plane. From **[1]** page 12, Theorem 1.4.4, we know that: For every two given triangles $X_1X_2X_3$ and $Y_1Y_2Y_3$ in A^2, there exists an affinity in A^2 which maps X_j on Y_j ($j = 1, 2, 3$). We will use this result in order to find the matrix D_T of the linear transformation $T\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$, such that given two triangles OX_1X_2 and OY_1Y_2, $T(X_1) = Y_1$, $T(X_2) = Y_2$.

We denote $D_T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $X_1 = (x_1, y_1)$; $X_2 = (x_2, y_2)$, $Y_1 = (a_1, b_1)$; $Y_2 = (a_2, b_2)$.

By solving a linear system in unknowns a, b, c, d, we obtain:

$$D_T = \frac{1}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} \begin{bmatrix} \begin{vmatrix} a_1 & y_1 \\ a_2 & y_2 \end{vmatrix} & \begin{vmatrix} x_1 & a_1 \\ x_2 & a_2 \end{vmatrix} \\ \begin{vmatrix} b_1 & y_1 \\ b_2 & y_2 \end{vmatrix} & \begin{vmatrix} x_1 & b_1 \\ x_2 & b_2 \end{vmatrix} \end{bmatrix}. \qquad (6)$$

d. Let $B = \{\vec{v}_1, \vec{v}_2\}$ and $B' = \{\vec{u}_1, \vec{u}_2\}$ be two bases of $\mathbb{R}^2$.

If

$$\vec{u}_1 = \alpha_{11}\vec{v}_1 + \alpha_{21}\vec{v}_2,$$
$$\vec{u}_2 = \alpha_{12}\vec{v}_1 + \alpha_{22}\vec{v}_2,$$

we denote by N the matrix: $N = \left[\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{smallmatrix}\right]$.

[1]Paper presented to the 8th International Conference on Geometry, Nahsholim (Israel), March 7–14, 1999

Theorem 1: Let $[\vec{v}]_B$ be the coordinates of a vector $\vec{v}$ within respect to the basis B and $[\vec{v}]_{B'}$ be the coordinates of a vector $\vec{v}$ with respect to the basis B'. Then, for every $\vec{v} \in \mathbb{R}^2$,

$$[\vec{v}]_B = N[\vec{v}]_{B'}. \tag{7}$$

3. A matriculation problem

The point $M = (5, 2)$ is the point of intersection of the diagonals in a square. One of the vertices is the point $A = (2, -4)$. Find the equations of the sides.

Observation: The above problem can be generalized in the following manner:

Let $M = (a, b)$ be the center of a regular n-polygon. One of the vertices is the point $A = (c, d)$. Find the equation of the sides.

No such problem was asked, according to my best knowledge, in any matriculation examination for $n > 4$, for the obvious reason: the treatment of the problem with rigid geometry Euclid+Descartes results in great difficulties, as an understatement, while, with dynamic geometry the above generalized problem can be easily solved.

Solution – the dynamic method

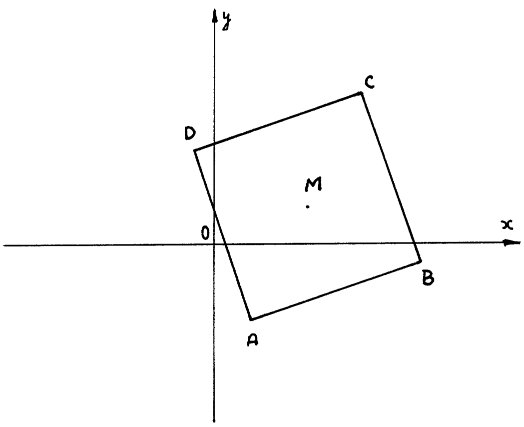

Fig. 1

According to equation (2):

$$\begin{bmatrix} x_B \\ y_B \end{bmatrix} = \rho_{M,90^\circ}(A) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -6 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

$$(AB)\colon x - 3y - 14 = 0.$$

BC is obtained from AB by rotation of AB about the point M with an angle of 90°.

We use (5) and obtain:

$$(BC)\colon y = -3x + 32.$$

The equations of the other sides can be obtained in the same way.

Observation: The same treatment for the generalized case, with $\alpha = 360^\circ/n$, can be applied if M is the center of a regular n-polygon and A is one of its vertices.

4. Old problems, new solutions

Some geometrical problems would be easier to solve if the figures involved would be in canonical position relatively to the Cartesian system.

For example an equilateral triangle is in ideal position if two of its vertices are on the x-axis and y-axis is its symmetry axis.

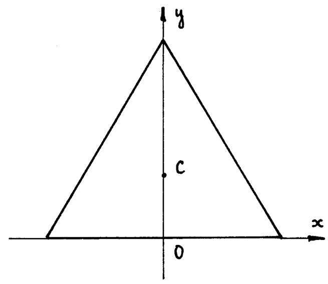

Fig. 2

Problem 1 (Problem 22 (a), Napoléon's theorem **[7]**):

Construct equilateral triangles on the sides of an arbitrary triangle ABC, exterior to it. Prove that the centers O_1, O_2, O_3 of these triangles themselves form the vertices of an equilateral triangle.

Does the assertion of this exercise remain correct if the equilateral triangles are not constructed exterior to triangle ABC, but on the same side of its sides as the triangle itself?

Problem 2 (Theorem 3.38, **[2]**):

The outer and inner Napoléon triangles of any triangle ABC differ in area by the area of ABC.

Problem 3 (Problem 4, [2], page 65):

The outer and inner Napoléon triangles have the same centers.

Solution: We choose the ox axis to be on the side BC of the triangle, the oy axis to be on the altitude AD and the origin $O = AD \cap BC$.

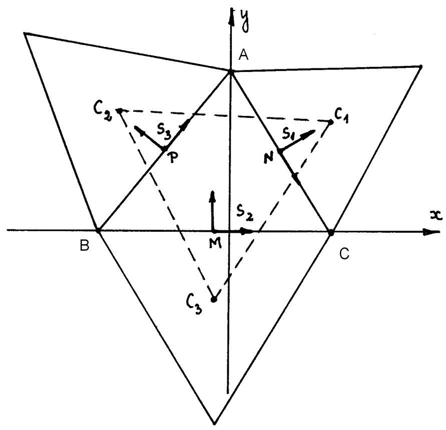

Fig. 3

We choose the coordinates $A = (0, 6a)$, $B = (-6b, 0)$, $C = (6c, 0)$ for the points A, B and C, and denote by M, N, P the middle points of the triangle and choose three orthonormal (cartesian) coordinate frames S_1, S_2, S_3 on its sides, as shown in the figure.

Let C_1, C_2, C_3 be, respectively, the centers of the three equilateral triangles constructed on the sides of the triangle ABC exterior to it and C_1', C_2', C_3' the centers of the equilateral triangles constructed on the same side of its sides as the triangle itself.

Then from (7)

$$\begin{bmatrix} x_{c_1} \\ y_{c_1} \end{bmatrix} = \begin{bmatrix} \frac{c}{\sqrt{a^2+c^2}} & \frac{a}{\sqrt{a^2+c^2}} \\ \frac{-a}{\sqrt{a^2+c^2}} & \frac{c}{\sqrt{a^2+c^2}} \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{a^2+c^2}\cdot\sqrt{3} \end{bmatrix} + \\ + \begin{bmatrix} 3c \\ 3a \end{bmatrix} = \begin{bmatrix} a\sqrt{3}+3c \\ c\sqrt{3}+3a \end{bmatrix}$$

$$\begin{bmatrix} x'_{c_1} \\ y'_{c_1} \end{bmatrix} = \begin{bmatrix} \frac{c}{\sqrt{a^2+c^2}} & \frac{a}{\sqrt{a^2+c^2}} \\ \frac{-a}{\sqrt{a^2+c^2}} & \frac{c}{\sqrt{a^2+c^2}} \end{bmatrix} \begin{bmatrix} 0 \\ -\sqrt{a^2+c^2}\cdot\sqrt{3} \end{bmatrix} + \\ + \begin{bmatrix} 3c \\ 3a \end{bmatrix} = \begin{bmatrix} -a\sqrt{3}+3c \\ -c\sqrt{3}+3a \end{bmatrix}$$

$$\begin{bmatrix} x_{c_2} \\ y_{c_2} \end{bmatrix} = \begin{bmatrix} \frac{b}{\sqrt{a^2+b^2}} & \frac{-a}{\sqrt{a^2+b^2}} \\ \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{a^2+b^2}\cdot\sqrt{3} \end{bmatrix} + \\ + \begin{bmatrix} -3b \\ 3a \end{bmatrix} = \begin{bmatrix} -a\sqrt{3}-3b \\ b\sqrt{3}+3a \end{bmatrix}$$

and

$$\begin{bmatrix} x'_{c_2} \\ y'_{c_2} \end{bmatrix} = \begin{bmatrix} a\sqrt{3}-3b \\ -b\sqrt{3}+3a \end{bmatrix}, \quad \begin{bmatrix} x_{c_3} \\ y_{c_3} \end{bmatrix} = \begin{bmatrix} 3c-3b \\ -(c+b)\sqrt{3} \end{bmatrix},$$

$$\begin{bmatrix} x'_{c_3} \\ y'_{c_3} \end{bmatrix} = \begin{bmatrix} 3c-3b \\ (c+b)\sqrt{3} \end{bmatrix}.$$

We will show now that the triangle C_1, C_2, C_3 is equilateral.

Let τ be the translation taking C_3 to $(0,0)$. We obtain:

$$\tau(C_1) = (a\sqrt{3}+3b, 2c\sqrt{3}+b\sqrt{3}+3a),$$
$$\tau(C_2) = (-a\sqrt{3}-3c, 2b\sqrt{3}+c\sqrt{3}+3a), \quad \tau(C_3) = 0.$$

We use (6) and compute the matrix D_T of a linear transformation T such that:

$$T\big(\tau(C_1)\big) = X_1 \text{ where } X_1 = (1, \sqrt{3}),$$
$$T\big(\tau(C_2)\big) = X_2 \text{ where } X_2 = (-1, \sqrt{3}),$$

and obtain:

$$D_T = \frac{1}{2\sqrt{3}} \begin{bmatrix} 6a+3b\sqrt{3}+3c\sqrt{3} & 3b-3c \\ 3c-3b & 6a+3b\sqrt{3}+3c\sqrt{3} \end{bmatrix},$$

which is a similarity.

We denote by ONT the outer Napoléon triangle, by INT the inner Napoléon triangle and by S_{ABC} the area of triangle ABC.

Solution of Problem 2

$$(C_1C_2)^2 = 12a^2 + 12b^2 + 12c^2 + \\ + 12ab\sqrt{3} + 12ac\sqrt{3} + 12bc,$$
$$(C'_1C'_2)^2 = 12a^2 + 12b^2 + 12c^2 - 12ab\sqrt{3} - \\ - 12ac\sqrt{3} + 12bc,$$
$$S_{ONT} - S_{INT} = (C_1C_2)^2 \cdot \frac{\sqrt{3}}{4} - (C'_1C'_2)^2 \cdot \frac{\sqrt{3}}{4} = \\ = \frac{\sqrt{3}}{4}(24)(ab+ac)\sqrt{3} = 18(ab+ac) = S_{\triangle ABC}.$$

Solution of Problem 3

As we have seen the point G of ONT is given by:

$$X_G = 2c - 2b, \quad Y_G = 2a.$$

The vertices of INT are:

$$C_1' = (a\sqrt{3}+3c, -c\sqrt{3}+3a),$$
$$C_2' = (a\sqrt{3}-3b, -b\sqrt{3}+3a),$$
$$C_3' = (3c-3b, (c+b)\sqrt{3}).$$

Then

$$X_G' = 2c - 2b = x_G, \quad Y_G' = 2a = y_G.$$

Problem 4 (Midpoint symmetric triangles areas **[5]**):
A triangle is drawn and the midpoints of its sides are found. Pairs of points are placed on each side of the triangle (P_i, Q_i), symmetric about the midpoint of the side (i.e. three free points and three dependent points). Corresponding points are joined to form two triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ as in Fig. 4. Investigate the relationship between their areas.

Solution:

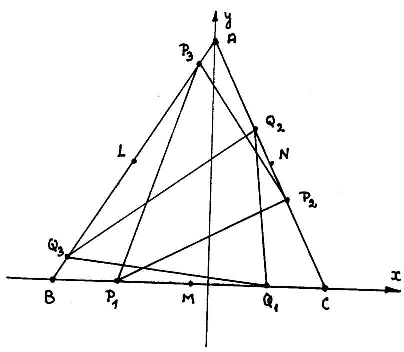

Fig. 4

We choose for the points A, B and C the coordinates $A = (0, 2a)$, $B = (-2b, 0)$, $C = (2c, 0)$ and denote by L, M, N the middle points of the sides AB, BC, CA respectively, and $P_2N = t$, $P_1M = \ell$, $LP_3 = m$. We choose the same three systems of coordinates S_1, S_2, S_3 as in the solution of Problem 1. Then the coordinates of P_1, Q_1 with respect to S_1 are $P_1 = (-\ell, 0)$, $Q_1 = (\ell, 0)$, the coordinates of P_2, Q_2 with respect to S_2 are $P_2 = (-t, 0)$, $Q_2 = (t, 0)$, and the coordinates of P_3, Q_3 with respect to S_3 are $P_3 = (m, 0)$, $Q_3 = (-m, 0)$.

We use Theorem 1 in order to compute the coordinates of P_i and Q_i $(i = 1, 3)$ with respect to XOY system of coordinates. Then:

$$P_1 = (c - b - \ell, 0); \quad Q_1 = (c - b + \ell, 0),$$

$$\begin{bmatrix} x_{P_2} \\ y_{P_2} \end{bmatrix} = \begin{bmatrix} \frac{c}{\sqrt{a^2+c^2}} & \frac{a}{\sqrt{a^2+c^2}} \\ \frac{-a}{\sqrt{a^2+c^2}} & \frac{c}{\sqrt{a^2+c^2}} \end{bmatrix} \begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ a \end{bmatrix} = \\ = \begin{bmatrix} \frac{tc}{\sqrt{a^2+c^2}} \\ \frac{-at}{\sqrt{a^2+c^2}}+a \end{bmatrix}, \quad \begin{bmatrix} x_{Q_2} \\ y_{Q_2} \end{bmatrix} = \begin{bmatrix} \frac{-tc}{\sqrt{a^2+c^2}}+c \\ \frac{at}{\sqrt{a^2+c^2}}+a \end{bmatrix},$$

$$\begin{bmatrix} x_{P_3} \\ y_{P_3} \end{bmatrix} = \begin{bmatrix} \frac{b}{\sqrt{a^2+b^2}} & \frac{-a}{\sqrt{a^2+b^2}} \\ \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \end{bmatrix} \begin{bmatrix} m \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ a \end{bmatrix} = \\ = \begin{bmatrix} \frac{mb}{\sqrt{a^2+b^2}}-b \\ \frac{ma}{\sqrt{a^2+b^2}}+a \end{bmatrix}, \quad \begin{bmatrix} x_{Q_3} \\ y_{Q_3} \end{bmatrix} = \begin{bmatrix} \frac{-mb}{\sqrt{a^2+b^2}}-b \\ \frac{-ma}{\sqrt{a^2+b^2}}+a \end{bmatrix}.$$

Then, denoting $d=\sqrt{a^2+b^2}$, $e=\sqrt{a^2+c^2}$:

$$\text{Area } Q_1Q_2Q_3 = \frac{1}{2}\left|\det\begin{bmatrix} c-b+\ell & 0 & 1 \\ \frac{-tc}{e}+c & \frac{at}{e}+a & 1 \\ \frac{-mb}{d}-b & \frac{-ma}{d}+a & 1 \end{bmatrix}\right| =$$
$$= \frac{1}{2}\left|a\,\frac{\ell td+\ell me+ctm+ed+btm+bed}{ed}\right|,$$

$$\text{Area } P_1P_2P_3 = \frac{1}{2}\left|\det\begin{bmatrix} c-b-\ell & 0 & 1 \\ \frac{tc}{e}+c & \frac{-at}{e}+a & 1 \\ \frac{mb}{d}-b & \frac{ma}{d}+a & 1 \end{bmatrix}\right| =$$
$$= \frac{1}{2}\left|a\frac{\ell td+\ell me+ctm+ed+btm+bed}{ed}\right|,$$

Therefore Area $P_1P_2P_3=$ Area $Q_1Q_2Q_3$.

Problem 5 (Based on a question from the 1975 Mathematical Olympiad):
ABC is an arbitrary triangle and points P, Q, R are such as to yield the angles shown in the figure below. Show that $\angle QPR = 90°$ and $QR = RP$.

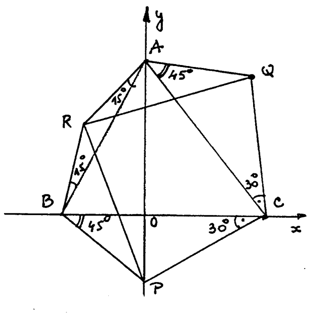

Fig. 5

Solution
First we compute the coordinates of the points P, Q and R using the techniques that we described in *Basics,* and then we will show that $\rho_{R,90°}(P)=Q$ which implies $RP=RQ$ and $RP\perp RQ$.

The equation of the line AC is: $ax+cy=ac$. AQ is obtained from AC by a rotation of an angle 45° about C, therefore with (4) we obtain:

$$(AQ)\colon x(a-c)+y(a+c)=a(a+c).$$

The equation of QC is obtained in a similar way.

The coordinates of Q are the solution of the system:

$$\begin{cases} x(a\sqrt{3}+c)+y(-a+c\sqrt{3})=c(a\sqrt{3}+c) \\ x(a-c)+y(a+c)=a(a+c) \end{cases} \Rightarrow$$
$$\Rightarrow \begin{cases} x_Q=\dfrac{a+c}{1+\sqrt{3}}, \\ y_Q=\dfrac{a\sqrt{3}+c}{1+\sqrt{3}}. \end{cases}$$

The coordinates of P are the solution of the system:

$$\begin{cases} (PC)\colon y=(\tan 30°)(x-c) \\ (PB)\colon y=-(x+b) \end{cases} \Rightarrow \begin{cases} x_P=\dfrac{c-\sqrt{3}b}{1+\sqrt{3}}, \\ y_P=\dfrac{-(b+c)}{1+\sqrt{3}}. \end{cases}$$

In order to find the coordinates of R, which is obtained by constructing an isosceles triangle, the best technique is to use (7)

$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \begin{bmatrix} \frac{b}{\sqrt{a^2+b^2}} & \frac{-a}{\sqrt{a^2+b^2}} \\ \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \end{bmatrix}\begin{bmatrix} 0 \\ \sqrt{a^2+b^2}\cdot\frac{\tan 15°}{2} \end{bmatrix} +$$
$$+\begin{bmatrix} -\frac{b}{2} \\ \frac{a}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{-a(\sqrt{3}-1)}{\sqrt{3}+1}-b \\ \frac{b(\sqrt{3}-1)}{\sqrt{3}+1}+a \end{bmatrix},$$

$$\rho_{90°,R}(P) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_P-x_R \\ y_P-y_R \end{bmatrix} + \begin{bmatrix} x_R \\ y_R \end{bmatrix} =$$
$$= \begin{bmatrix} -y_P+y_R+x_R \\ x_P-x_R+y_R \end{bmatrix} = \begin{bmatrix} x_Q \\ y_Q \end{bmatrix}.$$

From the solution of this problem we can learn that (4) and (5) can be successfully used when we construct triangles on the sides of a polygon, and (7) works well when isosceles triangles are constructed.

Problem 6 ([6]):
The midpoints and vertices of a square are joined as indicated in Fig. 6 to make a series of triangles which enclose an octagon. Is it equilateral? Is it regular?

Solution

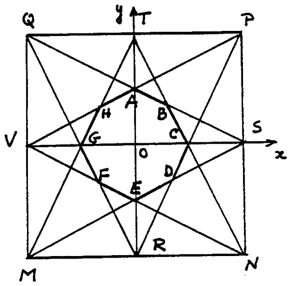

Fig. 6: Moorish Design from Toledo

Let $MNPQ$ be the square of the problem, with center at $O=(0,0)$ and sides parallel to the axes. It is clear that the octagon $ABCDEFGH$ is symmetric to the axes, and therefore is regular if

$$\rho_{0,-45°}(A)=B, \quad \rho_{0,-45°}(B)=C.$$

We compute the coordinates of A, B and C.
$A := VP \cap Oy$

$$(VP)\colon \begin{vmatrix} x-a & y-a \\ 2a & a \end{vmatrix} = 0 \Rightarrow (VP)\colon x-2y+a=0,$$
$$A=\left(0,\frac{a}{2}\right).$$

$B := QS \cap TN$

$$\left.\begin{matrix} (QS)\colon x+2y-a=0 \\ (TN)\colon 2x+y-a=0. \end{matrix}\right\} \Rightarrow B=\left(\frac{a}{3},\frac{a}{3}\right),$$

$C := PR \cap Ox$

$$(PR)\colon x+2y-a=0 \Rightarrow C=\left(\frac{a}{2},0\right).$$

Now

$$\rho_{0,-45°}(A) = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{a}{2} \end{bmatrix} \neq B.$$

It follows that the octagon $ABCDEFGH$ is not regular. It is equilateral, i.e. $AB = BC = \ldots = AH$.

Problem 7: (The XIX International Mathematical Olympiad, July 4,5, 1977):

Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments AK, BK, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon.

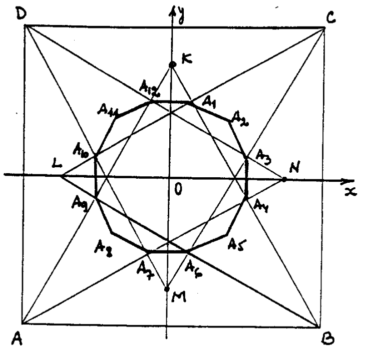

Fig. 7

Solution
Let $ABCD$ be the square of the problem, with center $(0,0)$ and sides parallel to the axes. It is clear that the dodecagon is symmetric with respect to the axes, therefore $A_1A_2A_3\ldots A_{12}$ is regular if

$$\begin{aligned}\rho_{-30^\circ,0}(A_{12}) = A_1, \quad \rho_{-30^\circ,0}(A_1) = A_2,\\ \rho_{-30^\circ,0}(A_2) = A_3, \quad \rho_{-30^\circ,0}(A_3) = A_4.\end{aligned} \qquad (*)$$

The coordinates of the points K, L, M, N are:

$$\begin{aligned}K &= \big(0, a(\sqrt{3}-1)\big), \quad L = \big(-a(\sqrt{3}-1), 0\big),\\ M &= \big(0, -a(\sqrt{3}-1)\big), \quad N = \big(a(\sqrt{3}-1), 0\big).\end{aligned}$$

Therefore the coordinates of $A_1, A_2, A_3, A_4, A_{12}$ are

$$A_1 = \left(\frac{a(2-\sqrt{3})}{2}, \frac{a}{2}\right),$$
$$A_2 = \left(\frac{a(\sqrt{3}-1)}{2}, \frac{a(\sqrt{3}-1)}{2}\right),$$
$$A_3 = \left(\frac{a}{2}, \frac{a(2-\sqrt{3})}{2}\right), \quad A_4 = \left(\frac{a}{2}, \frac{-a(2-\sqrt{3})}{2}\right),$$
$$A_{12} = \left(\frac{-a(2-\sqrt{3})}{2}, \frac{a}{2}\right).$$

A simple computation shows that indeed the relations $(*)$ take place. But then $A_1A_2A_3\ldots A_{12}$ is regular.

Problem 8:
We know from Problem 24 **[7]** that if on the sides of an arbitrary parallelogram $ABCD$ squares are constructed, lying exterior to it, their centers M_1, M_2, M_3, M_4 are themselves the vertices of a square.

The new problem: If with the notation from Fig. 8 we build new squares $A_1A_8B_8B_1$, $A_2A_3B_3B_2$, $A_4B_4B_5A_5$, $A_7A_6B_7B_6$, their centers C_1, C_2, C_3, C_4 form a parallelogram whose center is the same as the center of the parallellogram $ABCD$ and the square M_1, M_2, M_3, M_4.

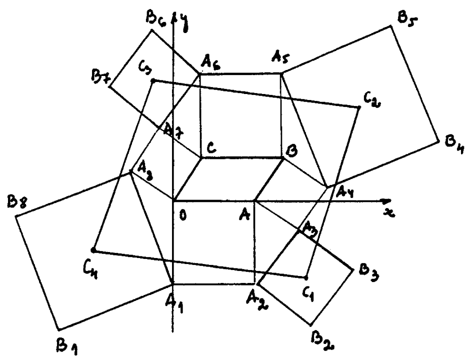

Fig. 8

Proof:
Let the coordinates of O, A, B, C be:

$$\begin{aligned}O &= (0,0), \quad A = (2a, 0),\\ B &= (2b+2a, 2c), \quad C = (2b, 2c).\end{aligned}$$

Then the coordinates of $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$ are:

$$\begin{aligned}A_1 &= (0, -2a), \quad A_2 = (2a, -2a),\\ A_5 &= (2b+2a, 2c+2a), \quad A_7 = (2b, 2c+2a),\end{aligned}$$

$$\begin{bmatrix} x_{A_3} \\ y_{A_3} \end{bmatrix} = \rho_{-90^\circ,A}(B) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2b \\ 2c \end{bmatrix} + \begin{bmatrix} 2a \\ 0 \end{bmatrix} = \begin{bmatrix} 2c+2a \\ -2b \end{bmatrix},$$

$$\begin{bmatrix} x_{A_4} \\ y_{A_4} \end{bmatrix} = \rho_{90^\circ,B}(A) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2b \\ -2c \end{bmatrix} + \begin{bmatrix} 2b+2a \\ 2c \end{bmatrix} = \begin{bmatrix} 2c+2b+2a \\ -2b+2c \end{bmatrix},$$

$$\begin{bmatrix} x_{A_7} \\ y_{A_7} \end{bmatrix} = \rho_{-90^\circ,C}(O) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -2b \\ -2c \end{bmatrix} + \begin{bmatrix} 2b \\ 2c \end{bmatrix} = \begin{bmatrix} -2c+2b \\ 2b+2c \end{bmatrix},$$

$$\begin{bmatrix} x_{A_8} \\ y_{A_8} \end{bmatrix} = \rho_{90^\circ,O}(C) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2b \\ 2c \end{bmatrix} = \begin{bmatrix} -2c \\ 2b \end{bmatrix},$$

$$B_2 = \rho_{-90^\circ,A_2}(A_3) = (4a-2b, -2c-2a).$$

C_1, the center of the square $A_2A_3B_3B_7$ is the middle point of B_2A_3, therefore $C_1 = (3a-b+c, -a-b-c)$.

$$\begin{bmatrix} x_{A_7} \\ y_{A_7} \end{bmatrix} = \rho_{90^\circ,A_7}(A_6) = \begin{bmatrix} -2a+4b-2c \\ 4c+2b \end{bmatrix} \Rightarrow$$
$$\Rightarrow A_7 = (-2a+4b-2c, 4c+2b).$$

C_3 is the center of $A_7A_6B_6B_7$ therefore $C_3 = (3b-a-c, 3c+a+b)$.

We denote by Q the middle point of C_1C_3, then $Q = (a+b, c)$.

Now

$$\begin{bmatrix} x_{B_1} \\ y_{B_1} \end{bmatrix} = \rho_{90^\circ,A_1}(A_8) = \begin{bmatrix} -2b-2a \\ -2c-2a \end{bmatrix} \Rightarrow$$
$$\Rightarrow B_1 = (-2b-2a, -2c-2a).$$

C_4 is the center of square $B_1A_1A_8B_8$. We obtain $C_4 = (-a-b-c, -a+b-c)$.

$$\begin{bmatrix} x_{B_5} \\ y_{B_5} \end{bmatrix} = \rho_{90°,A_5}(A_4) = \begin{bmatrix} 4a+4b \\ 4c+2a \end{bmatrix} \Rightarrow$$
$$\Rightarrow B_5 = (4a+4b, 4c+2a)$$

C_2 is the center of square $A_5B_5A_4B_4$, $C_2 = (c+3a+3b, 3c+a-b)$. Now it can be easily seen that the middle point of C_2C_4 is Q, therefore $C_1C_2C_3C_4$ is a parallelogram. The other assertions of the problem now easily follow.

Many other problems and theorems in which similar constructions are involved can be proved using the tools exposed in *Basics.* For example: Some of the problems of the Olympiad corner from the Canadian Journal *Crux Mathematicorum,* the examples and theorems from [3], problems 8.3.5, 8.3.13–8.3.17 from [5].

5. References

[1] Artzy, R. (1992): Geometry: an algebraic approach. – Mannheim: BI-Wiss.-Verl.
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Vorschau auf Analysethemen der nächsten Hefte

Für die Analysen der Jahrgänge 31 (1999) bis 32 (2000) sind folgende Themen geplant:
– TIMSS
– Computergestütztes Lösen offener Probleme im Mathematikunterricht
– Mathematikdidaktische Forschung im Primarbereich
– Mathematik an Hochschulen lehren und lernen
– Analysis an Hochschulen
– Mathematik in der Ingenieurausbildung
– Theoretische Betrachtungen zu Schulbuchanalysen.

Vorschläge für Beiträge zu o.g. Themen erbitten wir an die Schriftleitung.

Outlook on Future Topics

The following subjects are intended for the analysis sections of Vol. 31 (1999) to Vol. 32 (2000):
– TIMSS
– Computer-aided solution of open problems in mathematics teaching
– Research in primary mathematics education
– Teaching and learning mathematics at university level
– Calculus at universities
– Mathematics and engineering education
– Concepts and issues in textbook analyses.

Suggestions for contributions to these subjects are welcome and should be addressed to the editor.