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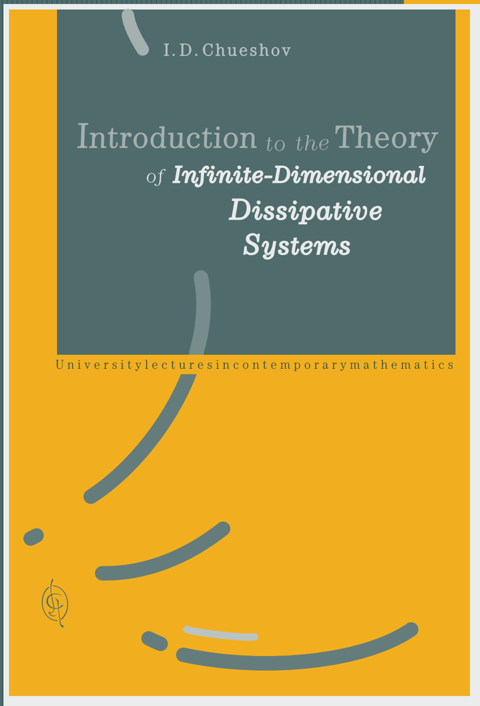

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This book provides an exhaustive introduction to the scope of main ideas and methods of the theory of infinite-dimensional dissipative dynamical systems which has been rapidly developing in recent years. In the examples systems generated by nonlinear partial differential equations arising in the different problems of modern mechanics of continua are considered. The main goal of the book is to help the reader to master the basic strategies used in the study of infinite-dimensional dissipative systems and to qualify him/her for an independent scientific research in the given branch. Experts in nonlinear dynamics will find many fundamental facts in the convenient and practical form in this book.

The core of the book is composed of the courses given by the author at the Department of Mechanics and Mathematics at Kharkov University during a number of years. This book contains a large number of exercises which make the main text more complete. It is sufficient to know the fundamentals of functional analysis and ordinary differential equations to read the book.

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Chapter 1

Basic Concepts of the Theory of Infinite-Dimensional Dynamical Systems

Contents

The mathematical theory of dynamical systems is based on the qualitative theory of ordinary differential equations the foundations of which were laid by Henri Poincaré (1854–1912). An essential role in its development was also played by the works of A. M. Lyapunov (1857–1918) and A. A. Andronov (1901–1952). At present the theory of dynamical systems is an intensively developing branch of mathematics which is closely connected to the theory of differential equations.

In this chapter we present some ideas and approaches of the theory of dynamical systems which are of general-purpose use and applicable to the systems generated by nonlinear partial differential equations.

§ 1 Notion of Dynamical System

In this book ***dynamical system*** is taken to mean the pair of objects (X, S_t) consisting of a complete metric space X and a family S_t of continuous mappings of the space X into itself with the properties

$$S_{t+\tau} = S_t \circ S_\tau, \quad t, \tau \in \mathbb{T}_+, \quad S_0 = I, \tag{1.1}$$

where $\mathbb{T}_+$ coincides with either a set $\mathbb{R}_+$ of nonnegative real numbers or a set $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. If $\mathbb{T}_+ = \mathbb{R}_+$, we also assume that $y(t) = S_t y$ is a continuous function with respect to t for any $y \in X$. Therewith X is called a ***phase space***, or a state space, the family S_t is called an ***evolutionary operator*** (or semigroup), parameter $t \in \mathbb{T}_+$ plays the role of time. If $\mathbb{T}_+ = \mathbb{Z}_+$, then dynamical system is called ***discrete*** (or a system with discrete time). If $\mathbb{T}_+ = \mathbb{R}_+$, then (X, S_t) is frequently called to be dynamical system with ***continuous*** time. If a notion of dimension can be defined for the phase space X (e. g., if X is a lineal), the value $\dim X$ is called a ***dimension*** of dynamical system.

Originally a dynamical system was understood as an isolated mechanical system the motion of which is described by the Newtonian differential equations and which is characterized by a finite set of generalized coordinates and velocities. Now people associate any time-dependent process with the notion of dynamical system. These processes can be of quite different origins. Dynamical systems naturally arise in physics, chemistry, biology, economics and sociology. The notion of dynamical system is the key and uniting element in synergetics. Its usage enables us to cover a rather wide spectrum of problems arising in particular sciences and to work out universal approaches to the description of qualitative picture of real phenomena in the universe.

Let us look at the following examples of dynamical systems.

E x a m p l e *1.1*

Let $f(x)$ be a continuously differentiable function on the real axis posessing the property $xf(x) \ge -C(1+x^2)$, where C is a constant. Consider the Cauchy problem for an ordinary differential equation

$$\dot{x}(t) = -f(x(t)), \qquad t > 0, \quad x(0) = x_0. \tag{1.2}$$

For any $x \in \mathbb{R}$ problem (1.2) is uniquely solvable and determines a dynamical system in $\mathbb{R}$. The evolutionary operator S_t is given by the formula $S_t x_0 = x(t)$, where $x(t)$ is a solution to problem (1.2). Semigroup property (1.1) holds by virtue of the theorem of uniqueness of solutions to problem (1.2). Equations of the type (1.2) are often used in the modeling of some ecological processes. For example, if we take $f(x) = \alpha \cdot x(x-1)$, $\alpha > 0$, then we get a logistic equation that describes a growth of a population with competition (the value $x(t)$ is the population level; we should take $\mathbb{R}_+$ for the phase space).

E x a m p l e *1.2*

Let $f(x)$ and $g(x)$ be continuously differentiable functions such that

$$F(x) = \int_0^x f(\xi)\,d\xi \;\ge\; -c, \qquad g(x) \ge -c$$

with some constant c. Let us consider the Cauchy problem

$$\begin{cases} \ddot{x} + g(x)\dot{x} + f(x) = 0, \quad \mathrm{t} > 0, \\ x(0) = x_0, \quad \dot{x}(0) = x_1. \end{cases} \tag{1.3}$$

For any $y_0 = (x_0, x_1) \in \mathbb{R}^2$, problem (1.3) is uniquely solvable. It generates a two-dimensional dynamical system $(\mathbb{R}^2, S_t)$, provided the evolutionary operator is defined by the formula

$$S_t(x_0; x_1) = (x(t); \dot{x}(t)),$$

where $x(t)$ is the solution to problem (1.3). It should be noted that equations of the type (1.3) are known as Liénard equations in literature. The van der Pol equation:

$$g(x) = \varepsilon(x^2 - 1), \quad \varepsilon > 0, \qquad f(x) = x$$

and the Duffing equation:

$$g(x) = \varepsilon, \quad \varepsilon > 0, \qquad f(x) = x^3 - a \cdot x - b$$

which often occur in applications, belong to this class of equations.

Example *1.3*

Let us now consider an autonomous system of ordinary differential equations

$$\dot{x}_k(t) = f_k(x_1,\ x_2,\ \dots,\ x_N), \qquad k = 1,\ 2,\ \dots,\ N\ . \tag{1.4}$$

Let the Cauchy problem for the system of equations (1.4) be uniquely solvable over an arbitrary time interval for any initial condition. Assume that a solution continuously depends on the initial data. Then equations (1.4) generate an N- dimensional dynamical system $(\mathbb{R}^N,\ S_t)$ with the evolutionary operator S_t acting in accordance with the formula

$$S_t y_0 = (x_1(t),\ \dots,\ x_N(t)), \qquad y_0 = (x_{10},\ x_{20},\ \dots,\ x_{N0})\ ,$$

where $\{x_i(t)\}$ is the solution to the system of equations (1.4) such that $x_i(0) = x_{i\,0},\quad i = 1,\ 2,\ \dots,\ N$. Generally, let X be a linear space and F be a continuous mapping of X into itself. Then the Cauchy problem

$$\dot{x}(t) = F(x(t)), \quad t > 0, \quad x(0) = x_0 \in X \tag{1.5}$$

generates a dynamical system $(X,\ S_t)$ in a natural way provided this problem is well-posed, i.e. theorems on existence, uniqueness and continuous dependence of solutions on the initial conditions are valid for (1.5).

Example *1.4*

Let us consider an ordinary retarded differential equation

$$\dot{x}(t) + \alpha x(t) = f(x(t-1)), \quad t > 0, \tag{1.6}$$

where f is a continuous function on $\mathbb{R}^1$, $\alpha > 0$. Obviously an initial condition for (1.6) should be given in the form

$$x(t)\big|_{t\,\in\,[-1,\ 0]} = \phi(t)\,. \tag{1.7}$$

Assume that $\phi(t)$ lies in the space $C[-1,\ 0]$ of continuous functions on the segment $[-1,\ 0]$. In this case the solution to problem (1.6) and (1.7) can be constructed by step-by-step integration. For example, if $0 \le t \le 1$, the solution $x(t)$ is given by

$$x(t) = e^{-\alpha t}\phi(0) + \int_0^t e^{-\alpha(t-\tau)} f(\phi(\tau-1))\,d\tau\ ,$$

and if $t \in [1,\ 2]$, then the solution is expressed by the similar formula in terms of the values of the function $x(t)$ for $t \in [0,\ 1]$ and so on. It is clear that the solution is uniquely determined by the initial function $\phi(t)$. If we now define an operator S_t in the space $X = C[-1,\ 0]$ by the formula

$$(S_t\phi)(\tau) = x(t+\tau), \qquad \tau \in [-1,\ 0],$$

where $x(t)$ is the solution to problem (1.6) and (1.7), then we obtain an infinite-dimensional dynamical system $(C[-1, 0],\ S_t)$.

Now we give several examples of discrete dynamical systems. First of all it should be noted that any system (X, S_t) with continuous time generates a discrete system if we take $t \in \mathbb{Z}_+$ instead of $t \in \mathbb{R}_+$. Furthermore, the evolutionary operator S_t of a discrete dynamical system is a degree of the mapping S_1, i. e. $S_t = S_1^t$, t $\in \mathbb{Z}_+$. Thus, a dynamical system with discrete time is determined by a continuous mapping of the phase space X into itself. Moreover, a discrete dynamical system is very often defined as a pair (X, S), consisting of the metric space X and the continuous mapping S.

Example *1.5*

Let us consider a one-step difference scheme for problem (1.5):

$$\frac{x_{n+1} - x_n}{\tau} = F(x_n), \quad n = 0,\ 1,\ 2,\ \dots, \quad \tau > 0.$$

There arises a discrete dynamical system (X, S^n), where S is the continuous mapping of X into itself defined by the formula $Sx = x + \tau F(x)$.

Example *1.6*

Let us consider a nonautonomous ordinary differential equation

$$\dot{x}(t) = f(x,\ t), \qquad t > 0, \quad x \in \mathbb{R}^1, \tag{1.9}$$

where $f(x, t)$ is a continuously differentiable function of its variables and is periodic with respect to t, i. e. $f(x, t) = f(x, t + T)$ for some $T > 0$. It is assumed that the Cauchy problem for (1.9) is uniquely solvable on any time interval. We define a ***monodromy*** operator (a period mapping) by the formula $Sx_0 = x(T)$, where $x(t)$ is the solution to (1.9) satisfying the initial condition $x(0) = x_0$. It is obvious that this operator possesses the property

$$S^k x(t) = x(t + kT) \tag{1.10}$$

for any solution $x(t)$ to equation (1.9) and any $k \in \mathbb{Z}_+$. The arising dynamical system $(\mathbb{R}^1, S^k)$ plays an important role in the study of the long-time properties of solutions to problem (1.9).

Example *1.7* (Bernoulli shift)

Let $X = \Sigma_2$ be a set of sequences $x = \{x_i,\ i \in \mathbb{Z}\}$ consisting of zeroes and ones. Let us make this set into a metric space by defining the distance by the formula

$$d(x,\ y) = \inf\{\, 2^{-n} :\ x_i = y_i, \quad |i| < n \,\}.$$

Let S be the shift operator on X, i. e. the mapping transforming the sequence $x = \{x_i\}$ into the element $y = \{y_i\}$, where $y_i = x_{i+1}$. As a result, a dynamical system (X, S^n) comes into being. It is used for describing complicated (quasirandom) behaviour in some quite realistic systems.

In the example below we describe one of the approaches that enables us to connect dynamical systems to nonautonomous (and nonperiodic) ordinary differential equations.

Example *1.8*

Let $h(x, t)$ be a continuous bounded function on $\mathbb{R}^2$. Let us define the hull L_h of the function $h(x, t)$ as the closure of a set

$$\left\{h_\tau(x, t) \equiv h(x, t+\tau), \quad \tau \in \mathbb{R}\right\}$$

with respect to the norm

$$\|h\|_C = \sup\left\{|h(x, t)| : x \in \mathbb{R}, t \in \mathbb{R}\right\}.$$

Let $g(x)$ be a continuous function. It is assumed that the Cauchy problem

$$\dot{x}(t) = g(x) + \tilde{h}(x, t), \quad x(0) = x_0 \tag{1.11}$$

is uniquely solvable over the interval $[0, +\infty)$ for any $\tilde{h} \in L_h$. Let us define the evolutionary operator S_τ on the space $X = \mathbb{R}^1 \times L_h$ by the formula

$$S_\tau(x_0, \tilde{h}) = (x(\tau), \tilde{h}_\tau),$$

where $x(t)$ is the solution to problem (1.11) and $\tilde{h}_\tau = \tilde{h}(x, t+\tau)$. As a result, a dynamical system $(\mathbb{R} \times L_h, S_t)$ comes into being. A similar construction is often used when L_h is a compact set in the space C of continuous bounded functions (for example, if $h(x, t)$ is a quasiperiodic or almost periodic function). As the following example shows, this approach also enables us to use naturally the notion of the dynamical system for the description of the evolution of objects subjected to random influences.

Example *1.9*

Assume that f_0 and f_1 are continuous mappings from a metric space Y into itself. Let Y be a state space of a system that evolves as follows: if y is the state of the system at time k, then its state at time $k+1$ is either $f_0(y)$ or $f_1(y)$ with probability $1/2$, where the choice of f_0 or f_1 does not depend on time and the previous states. The state of the system can be defined after a number of steps in time if we flip a coin and write down the sequence of events from the right to the left using 0 and 1. For example, let us assume that after 8 flips we get the following set of outcomes:

$$\ldots 10\,110010\,.$$

Here 1 corresponds to the head falling, whereas 0 corresponds to the tail falling. Therewith the state of the system at time $t = 8$ will be written in the form:

$$W = (f_1 \circ f_0 \circ f_1 \circ f_1 \circ f_0 \circ f_0 \circ f_1 \circ f_0)(y) .$$

This construction can be formalized as follows. Let Σ_2 be a set of two-sided sequences consisting of zeroes and ones (as in Example 1.7), i.e. a collection of elements of the type

$$\omega = (\ldots\ \omega_{-n} \ldots\ \omega_{-1}\, \omega_0\, \omega_1 \ldots\ \omega_n \ldots) ,$$

where ω_i is equal to either 1 or 0 . Let us consider the space $X = \Sigma_2 \times Y$ consisting of pairs $x = (\omega,\ y)$, where $\omega \in \Sigma_2$, $y \in Y$. Let us define the mapping $F:\ X \to X$ by the formula:

$$F(x) \equiv F(\omega,\ y) = (S\,\omega,\ f_{\omega_0}(y)) ,$$

where S is the left-shift operator in Σ_2 (see Example 1.7). It is easy to see that the n - th degree of the mapping F actcts according to the formula

$$F^n(\omega,\ y) = (S^n\,\omega,\ (f_{\omega_{n-1}} \circ \ldots \circ f_{\omega_1} \circ f_{\omega_0})(y))$$

and it generates a discrete dynamical system $(\Sigma_2 \times Y,\ F^n)$. This system is often called a universal random (discrete) dynamical system.

Examples of dynamical systems generated by partial differential equations will be given in the chapters to follow.

Exercise *1.1* Assume that operators S_t have a continuous inverse for any t. Show that the family of operators $\{\hat{S}_t:\ t \in \mathbb{R}\}$ defined by the equality $\hat{S}_t = S_t$ for $t \geq 0$ and $\hat{S}_t = S^{-1}_{|t|}$ for $t < 0$ form a group, i.e. (1.1) holds for all $t,\ \tau \in \mathbb{R}$.

Exercise *1.2* Prove the unique solvability of problems (1.2) and (1.3) involved in Examples 1.1 and 1.2.

Exercise *1.3* Ground formula (1.10) in Example 1.6.

Exercise *1.4* Show that the mapping S_t in Example 1.8 possesses semigroup property (1.1).

Exercise *1.5* Show that the value $d(x,\ y)$ involved in Example 1.7 is a metric. Prove its equivalence to the metric

$$d^*(x,\ y) = \sum_{i=-\infty}^{\infty} 2^{-|i|}\left|x_i - y_i\right| .$$

§ 2 Trajectories and Invariant Sets

Let (X, S_t) be a dynamical system with continuous or discrete time. Its ***trajectory*** (or ***orbit***) is defined as a set of the type

$$\gamma = \{u(t): \ t \in \mathbb{T}\},$$

where $u(t)$ is a continuous function with values in X such that $S_\tau u(t) = u(t+\tau)$ for all $\tau \in \mathbb{T}_+$ and $t \in \mathbb{T}$. Positive (negative) ***semitrajectory*** is defined as a set $\gamma^+ = \{u(t): \ t \geq 0\}$, $(\gamma^- = \{u(t): \ t \leq 0\}$, respectively), where a continuous on $\mathbb{T}_+$ ($\mathbb{T}_-$, respectively) function $u(t)$ possesses the property $S_\tau u(t) = u(t+\tau)$ for any $\tau > 0$, $t \geq 0$ ($\tau > 0$, $t \leq 0$, $\tau + t \leq 0$, respectively). It is clear that any positive semitrajectory γ^+ has the form $\gamma^+ = \{S_t v: \ t \geq 0\}$, i.e. it is uniquely determined by its initial state v. To emphasize this circumstance, we often write $\gamma^+ = \gamma^+(v)$. In general, it is impossible to continue this semitrajectory $\gamma^+(v)$ to a full trajectory without imposing any additional conditions on the dynamical system.

Exercise *2.1* Assume that an evolutionary operator S_t is invertible for some $t > 0$. Then it is invertible for all $t > 0$ and for any $v \in X$ there exists a negative semitrajectory $\gamma^- = \gamma^-(v)$ ending at the point v.

A trajectory $\gamma = \{u(t): \ t \in \mathbb{T}\}$ is called a ***periodic trajectory*** (or a ***cycle***) if there exists $T \in \mathbb{T}_+$, $T > 0$ such that $u(t+T) = u(t)$. Therewith the minimal number $T > 0$ possessing the property mentioned above is called a ***period*** of a trajectory. Here $\mathbb{T}$ is either $\mathbb{R}$ or $\mathbb{Z}$ depending on whether the system is a continuous or a discrete one. An element $u_0 \in X$ is called a ***fixed point*** of a dynamical system (X, S_t) if $S_t u_0 = u_0$ for all $t \geq 0$ (synonyms: ***equilibrium point***, ***stationary point***).

Exercise *2.2* Find all the fixed points of the dynamical system (R, S_t) generated by equation (1.2) with $f(x) = x(x-1)$. Does there exist a periodic trajectory of this system?

Exercise *2.3* Find all the fixed points and periodic trajectories of a dynamical system in $\mathbb{R}^2$ generated by the equations

$$\begin{cases} \dot{x} = -\alpha y - x[(x^2+y^2)^2 - 4(x^2+y^2) + 1], \\ \dot{y} = \alpha x - y[(x^2+y^2)^2 - 4(x^2+y^2) + 1]. \end{cases}$$

Consider the cases $\alpha \neq 0$ and $\alpha = 0$. *Hint*: use polar coordinates.

Exercise *2.4* Prove the existence of stationary points and periodic trajectories of any period for the discrete dynamical system described in Example 1.7. Show that the set of all periodic trajectories is dense in the phase space of this system. Make sure that there exists a trajectory that passes at a whatever small distance from any point of the phase space.

The notion of invariant set plays an important role in the theory of dynamical systems. A subset Y of the phase space X is said to be:

a) ***positively invariant***, if $S_t Y \subseteq Y$ for all $t \ge 0$;

b) ***negatively invariant***, if $S_t Y \supseteq Y$ for all $t \ge 0$;

c) ***invariant***, if it is both positively and negatively invariant, i.e. if $S_t Y = Y$ for all $t \ge 0$.

The simplest examples of invariant sets are trajectories and semitrajectories.

— Exercise *2.5* Show that γ^+ is positively invariant, γ^- is negatively invariant and γ is invariant.

— Exercise *2.6* Let us define the sets

$$\gamma^+(A) = \bigcup_{t \ge 0} S_t(A) \equiv \bigcup_{t \ge 0} \{v = S_t u : \ u \in A\}$$

and

$$\gamma^-(A) = \bigcup_{t \ge 0} S_t^{-1}(A) \equiv \bigcup_{t \ge 0} \{v : \ S_t v \in A\}$$

for any subset A of the phase space X. Prove that $\gamma^+(A)$ is a positively invariant set, and if the operator S_t is invertible for some $t > 0$, then $\gamma^-(A)$ is a negatively invariant set.

Other important example of invariant set is connected with the notions of ω-limit and α-limit sets that play an essential role in the study of the long-time behaviour of dynamical systems.

Let $A \subset X$. Then the ω-***limit set*** for A is defined by

$$\omega(A) = \bigcap_{s \ge 0} \Big[\bigcup_{t \ge s} S_t(A) \Big]_X ,$$

where $S_t(A) = \{v = S_t u : \ u \in A\}$. Hereinafter $[Y]_X$ is the closure of a set Y in the space X. The set

$$\alpha(A) = \bigcap_{s \ge 0} \Big[\bigcup_{t \ge s} S_t^{-1}(A) \Big]_X ,$$

where $S_t^{-1}(A) = \{v : \ S_t v \in A\}$, is called the α-***limit set*** for A.

Lemma 2.1

For an element y to belong to an ω-limit set $\omega(A)$, it is necessary and sufficient that there exist a sequence of elements $\{y_n\} \subset A$ and a sequence of numbers t_n, the latter tending to infinity such that

$$\lim_{n \to \infty} d(S_{t_n} y_n,\ y) = 0,$$

where $d(x,\ y)$ is the distance between the elements x and y in the space X.

Proof.

Let the sequences mentioned above exist. Then it is obvious that for any $\tau > 0$ there exists $n_0 \geq 0$ such that

$$S_{t_n} y_n \in \bigcup_{t \geq \tau} S_t(A), \quad n \geq n_0.$$

This implies that

$$y = \lim_{n \to \infty} S_{t_n} y_n \in \left[\bigcup_{t \geq \tau} S_t(A)\right]_X$$

for all $\tau > 0$. Hence, the element y belongs to the intersection of these sets, i.e. $y \in \omega(A)$.

On the contrary, if $y \in \omega(A)$, then for all $n = 0,\ 1,\ 2,\ \ldots$

$$y \in \left[\bigcup_{t \geq n} S_t(A)\right]_X.$$

Hence, for any n there exists an element z_n such that

$$z_n \in \bigcup_{t \geq n} S_t(A), \qquad d(y,\ z_n) \leq \frac{1}{n}.$$

Therewith it is obvious that $z_n = S_{t_n} y_n$, $y_n \in A$, $t_n \geq n$. This proves the lemma.

It should be noted that this lemma gives us a description of an ω-limit set but does not guarantee its nonemptiness.

Exercise *2.7* Show that $\omega(A)$ is a positively invariant set. If for any $t > 0$ there exists a continuous inverse to S_t, then $\omega(A)$ is invariant, i.e. $S_t\,\omega(A) = \omega(A)$.

Exercise *2.8* Let S_t be an invertible mapping for every $t > 0$. Prove the counterpart of Lemma 2.1 for an α-limit set:

$$y \in \alpha(A) \Leftrightarrow \left\{\exists\{y_n\} \in A,\quad \exists t_n,\ t_n \to +\infty;\quad \lim_{n \to \infty} d(S_{t_n}^{-1} y_n,\ y) = 0\right\}.$$

Establish the invariance of $\alpha(A)$.

Exercise 2.9 Let $\gamma = \{u(t):\ -\infty < t < \infty\}$ be a periodic trajectory of a dynamical system. Show that $\gamma = \omega(u) = \alpha(u)$ for any $u \in \gamma$.

Exercise 2.10 Let us consider the dynamical system $(\mathbb{R},\ S_t)$ constructed in Example 1.1. Let a and b be the roots of the function $f(x)$: $f(a) = f(b) = 0$, $a < b$. Then the segment $I = \{x:\ a \le x \le b\}$ is an invariant set. Let $F(x)$ be a primitive of the function $f(x)$ ($F'(x) = f(x)$). Then the set $\{x:\ F(x) \le c\}$ is positively invariant for any c.

Exercise 2.11 Assume that for a continuous dynamical system $(X,\ S_t)$ there exists a continuous scalar function $V(y)$ on X such that the value $V(S_t y)$ is differentiable with respect to t for any $y \in X$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(V(S_t y)) + \alpha V(S_t y) \le \rho, \qquad (\alpha > 0,\ \rho > 0,\ y \in X).$$

Then the set $\{y:\ V(y) \le R\}$ is positively invariant for any $R \ge \rho/\alpha$.

§ 3 Definition of Attractor

Attractor is a central object in the study of the limit regimes of dynamical systems. Several definitions of this notion are available. Some of them are given below. From the point of view of infinite-dimensional systems the most convenient concept is that of the global attractor.

A bounded closed set $A_1 \subset X$ is called a ***global attractor*** for a dynamical system $(X,\ S_t)$, if

1) A_1 is an invariant set, i.e. $S_t A_1 = A_1$ for any $t > 0$;
2) the set A_1 uniformly attracts all trajectories starting in bounded sets, i.e. for any bounded set B from X

$$\lim_{t \to \infty} \sup \left\{ \mathrm{dist}(S_t y,\ A_1):\ y \in B \right\} = 0.$$

We remind that the distance between an element z and a set A is defined by the equality:

$$\mathrm{dist}\,(z,\ A) = \inf\{d(z,\ y):\ y \in A\},$$

where $d(z,\ y)$ is the distance between the elements z and y in X.

The notion of a weak global attractor is useful for the study of dynamical systems generated by partial differential equations.

Let X be a complete linear metric space. A bounded weakly closed set A_2 is called a ***global weak attractor*** if it is invariant $(S_t A_2 = A_2,\ t > 0)$ and for any weak vicinity $\mathcal{O}$ of the set A_2 and for every bounded set $B \subset X$ there exists $t_0 = t_0(\mathcal{O},\ B)$ such that $S_t B \subset \mathcal{O}$ for $t \geq t_0$.

We remind that an open set in weak topology of the space X can be described as finite intersection and subsequent arbitrary union of sets of the form

$$U_{l,\ c} = \{x \in X:\ l(x) < c\},$$

where c is a real number and l is a continuous linear functional on X.

It is clear that the concepts of global and global weak attractors coincide in the finite-dimensional case. In general, a global attractor A is also a global weak attractor, provided the set A is weakly closed.

Exercise *3.1* Let A be a global or global weak attractor of a dynamical system $(X,\ S_t)$. Then it is uniquely determined and contains any bounded negatively invariant set. The attractor A also contains the ω-limit set $\omega(B)$ of any bounded $B \subset X$.

Exercise *3.2* Assume that a dynamical system $(X,\ S_t)$ with continuous time possesses a global attractor A_1. Let us consider a discrete system $(X,\ T^n)$, where $T = S_{t_0}$ with some $t_0 > 0$. Prove that A_1 is a global attractor for the system $(X,\ T^n)$. Give an example which shows that the converse assertion does not hold in general.

If the global attractor A_1 exists, then it contains a ***global minimal attractor*** A_3 which is defined as a minimal closed positively invariant set possessing the property

$$\lim_{t \to \infty} \operatorname{dist}(S_t y,\ A_3) = 0 \quad \text{for every} \quad y \in X.$$

By definition minimality means that A_3 has no proper subset possessing the properties mentioned above. It should be noted that in contrast with the definition of the global attractor the uniform convergence of trajectories to A_3 is not expected here.

Exercise *3.3* Show that $S_t A_3 = A_3$, provided A_3 is a compact set.

Exercise *3.4* Prove that $\omega(x) \in A_3$ for any $x \in X$. Therewith, if A_3 is a compact, then $A_3 = \bigcup\{\omega(x):\ x \in X\}$.

By definition the attractor A_3 contains limit regimes of each individual trajectory. It will be shown below that $A_3 \neq A_1$ in general. Thus, a set of real limit regimes (states) originating in a dynamical system can appear to be narrower than the global attractor. Moreover, in some cases some of the states that are unessential from the point of view of the frequency of their appearance can also be "removed" from A_3, for example, such states like absolutely unstable stationary points. The next two definitions take into account the fact mentioned above. Unfortunately, they require additional assumptions on the properties of the phase space. Therefore, these definitions are mostly used in the case of finite-dimensional dynamical systems.

Let a Borel measure μ such that $\mu(X) < \infty$ be given on the phase space X of a dynamical system (X, S_t). A bounded set A_4 in X is called a ***Milnor attractor*** (with respect to the measure μ) for (X, S_t) if A_4 is a minimal closed invariant set possessing the property

$$\lim_{t \to \infty} \operatorname{dist}(S_t y, A_4) = 0$$

for almost all elements $y \in X$ with respect to the measure μ. The Milnor attractor is frequently called a probabilistic global minimal attractor.

At last let us introduce the notion of a statistically essential global minimal attractor suggested by Ilyashenko. Let U be an open set in X and let $X_U(x)$ be its characteristic function: $X_U(x) = 1$, $x \in U$; $X_U(x) = 0$, $x \notin U$. Let us define the average time $\tau(x, U)$ which is spent by the semitrajectory $\gamma^+(x)$ emanating from x in the set U by the formula

$$\tau(x,\ U) = \overline{\lim_{T \to \infty}}\ \frac{1}{T} \int_0^T X_U(S_t x)\,\mathrm{d}t\,.$$

A set U is said to be unessential with respect to the measure μ if

$$M(U) \equiv \mu\{x\colon\ \tau(x,\ U) > 0\} = 0\,.$$

The complement A_5 to the maximal unessential open set is called an ***Ilyashenko attractor*** (with respect to the measure μ).

It should be noted that the attractors A_4 and A_5 are used in cases when the natural Borel measure is given on the phase space (for example, if X is a closed measurable set in $\mathbb{R}^N$ and μ is the Lebesgue measure).

The relations between the notions introduced above can be illustrated by the following example.

E x a m p l e *3.1*

Let us consider a quasi-Hamiltonian system of equations in $\mathbb{R}^2$:

$$\begin{cases} \dot{q} \ = \ \dfrac{\partial H}{\partial p} - \mu H \dfrac{\partial H}{\partial q}\,, \\[2ex] \dot{p} \ = \ -\dfrac{\partial H}{\partial q} - \mu H \dfrac{\partial H}{\partial p}\,, \end{cases} \tag{3.1}$$

where $H(p,\ q) = (1/2)p^2 + q^4 - q^2$ and μ is a positive number. It is easy to ascertain that the phase portrait of the dynamical system generated by equations (3.1) has the form represented on *Fig. 1.*

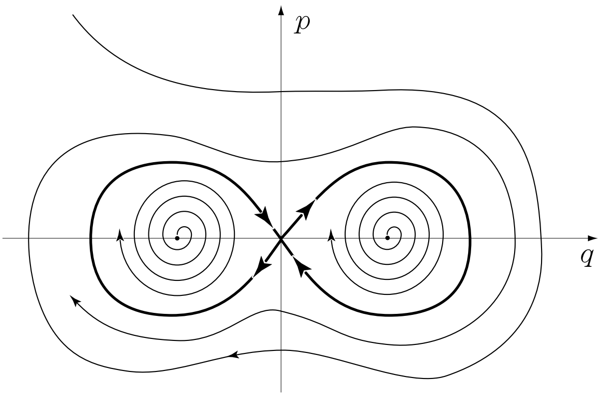

Fig. 1. Phase portrait of system (3.1)

A separatrix ("eight curve") separates the domains of the phase plane with the different qualitative behaviour of the trajectories. It is given by the equation $H(p,q)=0$. The points (p, q) inside the separatrix are characterized by the equation $H(p, q) < 0$. Therewith it appears that

$$A_1 = A_2 = \{(p,\ q):\ H(p,\ q) \le 0\},$$

$$A_3 = \left\{(p,\ q):\ H(p,\ q)=0\right\} \cup \left\{(p,\ q):\ \frac{\partial}{\partial p}H(p,\ q)=\frac{\partial}{\partial q}H(p,\ q)=0\right\},$$

$$A_4 = \{(p,\ q):\ H(p,\ q)=0\}.$$

Finally, the simple calculations show that $A_5 = \{0,\ 0\}$, i.e. the Ilyashenko attractor consists of a single point. Thus,

$$A_1 = A_2 \supset A_3 \supset A_4 \supset A_5,$$

all inclusions being strict.

Exercise *3.5* Display graphically the attractors A_j of the system generated by equations (3.1) on the phase plane.

Exercise *3.6* Consider the dynamical system from Example 1.1 with $f(x) = x(x^2 - 1)$. Prove that $A_1 = \{x:\ -1 \le x \le 1\}$, $A_3 = \{x=0;\ x=\pm 1\}$, and $A_4 = A_5 = \{x=\pm 1\}$.

Exercise *3.7* Prove that $A_4 \subset A_3$ and $A_5 \subset A_3$ in general.

Exercise *3.8* Show that all positive semitrajectories of a dynamical system which possesses a global minimal attractor are bounded sets.

In particular, the result of the last exercise shows that the global attractor can exist only under additional conditions concerning the behaviour of trajectories of the system at infinity. The main condition to be met is the dissipativity discussed in the next section.

§ 4 *Dissipativity and Asymptotic Compactness*

From the physical point of view dissipative systems are primarily connected with irreversible processes. They represent a rather wide and important class of the dynamical systems that are intensively studied by modern natural sciences. These systems (unlike the conservative systems) are characterized by the existence of the accented direction of time as well as by the energy reallocation and dissipation. In particular, this means that limit regimes that are stationary in a certain sense can arise in the system when $t \to +\infty$. Mathematically these features of the qualitative behaviour of the trajectories are connected with the existence of a bounded absorbing set in the phase space of the system.

A set $B_0 \subset X$ is said to be ***absorbing*** for a dynamical system (X, S_t) if for any bounded set B in X there exists $t_0 = t_0(B)$ such that $S_t(B) \subset B_0$ for every $t \ge t_0$. A dynamical system (X, S_t) is said to be ***dissipative*** if it possesses a bounded absorbing set. In cases when the phase space X of a dissipative system (X, S_t) is a Banach space a ball of the form $\{x \in X:\ \|x\|_X \le R\}$ can be taken as an absorbing set. Therewith the value R is said to be a ***radius of dissipativity***.

As a rule, dissipativity of a dynamical system can be derived from the existence of a Lyapunov type function on the phase space. For example, we have the following assertion.

Theorem 4.1.

Let the phase space of a continuous dynamical system (X, S_t) be a Banach space. Assume that:

(a) *there exists a continuous function $U(x)$ on X possessing the properties*

$$\varphi_1(\|x\|) \le U(x) \le \varphi_2(\|x\|)\ , \tag{4.1}$$

where $\varphi_j(r)$ are continuous functions on $\mathbb{R}_+$ and $\varphi_1(r) \to +\infty$ when $r \to \infty$;

(b) *there exist a derivative $\frac{d}{dt}U(S_t y)$ for $t \ge 0$ and positive numbers α and ρ such that*

$$\frac{d}{dt}U(S_t y) \le -\alpha \quad \textit{for} \quad \|S_t y\| > \rho. \tag{4.2}$$

Then the dynamical system (X, S_t) is dissipative.

Proof.

Let us choose $R_0 \ge \rho$ such that $\varphi_1(r) > 0$ for $r \ge R_0$. Let

$$l = \sup\{\varphi_2(r):\ r \le 1 + R_0\}$$

and $R_1 > R_0 + 1$ be such that $\varphi_1(r) > l$ for $r > R_1$. Let us show that

$$\|S_t y\| \le R_1 \quad \text{for all} \quad t \ge 0 \quad \text{and} \quad \|y\| \le R_0. \tag{4.3}$$

Assume the contrary, i.e. assume that for some $y \in X$ such that $\|y\| \le R_0$ there exists a time $\bar{t} > 0$ possessing the property $\|S_{\bar{t}}\, y\| > R_1$. Then the continuity of $S_t y$ implies that there exists $0 < t_0 < \bar{t}$ such that $\rho < \|S_{t_0} y\| \le R_0 + 1$. Thus, equation (4.2) implies that

$$U(S_t\, y) \le U(S_{t_0} y), \qquad t \ge t_0,$$

provided $\|S_t\, y\| > \rho$. It follows that $U(S_t\, y) \le l$ for all $t \ge t_0$. Hence, $\|S_t\, y\| \le R_1$ for all $t \ge t_0$. This contradicts the assumption. Let us assume now that B is an arbitrary bounded set in X that does not lie inside the ball with the radius R_0. Then equation (4.2) implies that

$$U(S_t y) \le U(y) - \alpha t \le l_B - \alpha t, \quad y \in B, \tag{4.4}$$

provided $\|S_t\, y\| > \rho$. Here

$$l_B = \sup\{U(x):\ x \in B\}.$$

Let $y \in B$. If for a time $t^* < (l_B - l)/\alpha$ the semitrajectory $S_t\, y$ enters the ball with the radius ρ, then by (4.3) we have $\|S_t\, y\| \le R_1$ for all $t \ge t^*$. If that does not take place, from equation (4.4) it follows that

$$\varphi_1(\|S_t\, y\|) \le U(S_t\, y) \le l \quad \text{for} \quad t \ge \frac{l_B - l}{\alpha},$$

i.e. $\|S_t\, y\| \le R_1$ for $t \ge \alpha^{-1}(l_B - l)$. Thus,

$$S_t B \subset \{x:\ \|x\| \le R_1\}, \qquad t \ge \frac{l_B - l}{\alpha}.$$

This and (4.3) imply that the ball with the radius R_1 is an absorbing set for the dynamical system (X, S_t). Thus, **Theorem 4.1 is proved**.

Exercise *4.1* Show that hypothesis (4.2) of Theorem 4.1 can be replaced by the requirement

$$\frac{\mathrm{d}}{\mathrm{d}t} U(S_t\, y) + \gamma\, U(S_t\, y) \le C,$$

where γ and C are positive constants.

Exercise *4.2* Show that the dynamical system generated in $\mathbb{R}$ by the differential equation $\dot{x} + f(x) = 0$ (see Example 1.1) is dissipative, provided the function $f(x)$ possesses the property: $xf(x) \ge \delta x^2 - C$, where $\delta > 0$ and C are constants (*Hint*: $U(x) = x^2$). Find an upper estimate for the minimal radius of dissipativity.

Exercise *4.3* Consider a discrete dynamical system $(\mathbb{R}, f^n)$, where f is a continuous function on $\mathbb{R}$. Show that the system $(\mathbb{R}, f)$ is dissipative, provided there exist $\rho > 0$ and $0 < \alpha < 1$ such that $|f(x)| < \alpha |x|$ for $|x| > \rho$.

Exercise 4.4 Consider a dynamical system $(\mathbb{R}^2, S_t)$ generated (see Example 1.2) by the Duffing equation

$$\ddot{x} + \varepsilon \dot{x} + x^3 - a x = b\,,$$

where a and b are real numbers and $\varepsilon > 0$. Using the properties of the function

$$U(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4 - \frac{a}{2}x^2 + \nu\left(x\dot{x} + \frac{\varepsilon}{2}x^2\right)$$

show that the dynamical system $(\mathbb{R}^2, S_t)$ is dissipative for $\nu > 0$ small enough. Find an upper estimate for the minimal radius of dissipativity.

Exercise 4.5 Prove the dissipativity of the dynamical system generated by (1.4) (see Example 1.3), provided

$$\sum_{k=1}^{N} x_k f_k(x_1, x_2, \ldots, x_N) \le -\delta \sum_{k=1}^{N} x_k^2 + C\,, \qquad \delta > 0\,.$$

Exercise 4.6 Show that the dynamical system of Example 1.4 is dissipative if $f(z)$ is a bounded function.

Exercise 4.7 Consider a cylinder Ц with coordinates (x, φ), $x \in \mathbb{R}$, $\varphi \in [0, 1)$ and the mapping T of this cylinder which is defined by the formula $T(x, \varphi) = (x', \varphi')$, where

$$x' = \alpha x + k \sin 2\pi\varphi\,,$$

$$\varphi' = \varphi + x' \pmod 1.$$

Here α and k are positive parameters. Prove that the discrete dynamical system $(Ц, T^n)$ is dissipative, provided $0 < \alpha < 1$. We note that if $\alpha = 1$, then the mapping T is known as the Chirikov mapping. It appears in some problems of physics of elementary particles.

Exercise 4.8 Using Theorem 4.1 prove that the dynamical system $(\mathbb{R}^2, S_t)$ generated by equations (3.1) (see Example 3.1) is dissipative. (*Hint*: $U(x) = [H(p, q)]^2$).

In the proof of the existence of global attractors of infinite-dimensional dissipative dynamical systems a great role is played by the property of asymptotic compactness. For the sake of simplicity let us assume that X is a closed subset of a Banach space. The dynamical system (X, S_t) is said to be ***asymptotically compact*** if for any $t > 0$ its evolutionary operator S_t can be expressed by the form

$$S_t = S_t^{(1)} + S_t^{(2)}\,, \tag{4.5}$$

where the mappings $S_t^{(1)}$ and $S_t^{(2)}$ possess the properties:

a) for any bounded set B in X

$$r_B(t) = \sup_{y \in B} \left\| S_t^{(1)} y \right\|_X \to 0 , \qquad t \to +\infty ;$$

b) for any bounded set B in X there exists t_0 such that the set

$$[\gamma_{t_0}^{(2)}(B)] = \left[\bigcup_{t \geq t_0} S_t^{(2)} B \right] \tag{4.6}$$

is compact in X, where $[\gamma]$ is the closure of the set γ.

A dynamical system is said to be ***compact*** if it is asymptotically compact and one can take $S_t^{(1)} \equiv 0$ in representation (4.5). It becomes clear that any finite-dimensional dissipative system is compact.

Exercise *4.9* Show that condition (4.6) is fulfilled if there exists a compact set K in H such that for any bounded set B the inclusion $S_t^{(2)} B \subset K$, $t \geq t_0(B)$ holds. In particular, a dissipative system is compact if it possesses a compact absorbing set.

Lemma 4.1.

The dynamical system (X, S_t) is asymptotically compact if there exists a compact set K such that

$$\lim_{t \to \infty} \sup \{ \operatorname{dist} (S_t u, K) : \ u \in B \} = 0 \tag{4.7}$$

for any set B bounded in X.

Proof.

The distance to a compact set is reached on some element. Hence, for any $t > 0$ and $u \in X$ there exists an element $v \equiv S_t^{(2)} u \in K$ such that

$$\operatorname{dist} (S_t u, K) = \left\| S_t u - S_t^{(2)} u \right\| .$$

Therefore, if we take $S_t^{(1)} u = S_t u - S_t^{(2)} u$, it is easy to see that in this case decomposition (4.5) satisfies all the requirements of the definition of asymptotic compactness.

Remark 4.1.

In most applications Lemma 4.1 *plays a major role in the proof of the property of asymptotic compactness. Moreover, in cases when the phase space X of the dynamical system (X, S_t) does not possess the structure of a linear space it is convenient to define the notion of the asymptotic compactness using equation* (4.7). *Namely, the system (X, S_t) is said to be asymptotically compact if there exists a compact K possessing property* (4.7) *for any bounded set B in X. For one more approach to the definition of this concept see Exercise* 5.1 *below.*

Exercise *4.10* Consider the infinite-dimensional dynamical system generated by the retarded equation

$$\dot{x}(t) + \alpha x(t) = f(x(t-1)) \ ,$$

where $\alpha > 0$ and $f(z)$ is bounded (see Example 1.4). Show that this system is compact.

Exercise *4.11* Consider the system of Lorentz equations arising as a three-mode Galerkin approximation in the problem of convection in a thin layer of liquid:

$$\begin{cases} \dot{x} \ = \ -\sigma x + \sigma y \,, \\ \dot{y} \ = \ rx - y - xz \,, \\ \dot{z} \ = \ -bz + xy \,. \end{cases}$$

Here σ, r, and b are positive numbers. Prove the dissipativity of the dynamical system generated by these equations in $\mathbb{R}^3$.
Hint: Consider the function

$$V(x,\ y,\ z) \ = \ \frac{1}{2}\left(x^2 + y^2 + (z - r - \sigma)^2\right)$$

on the trajectories of the system.

§ 5 Theorems on Existence of Global Attractor

For the sake of simplicity it is assumed in this section that the phase space X is a Banach space, although the main results are valid for a wider class of spaces (see, e. g., Exercise 5.8). The following assertion is the main result.

Theorem 5.1.

Assume that a dynamical system $(X,\ S_t)$ is dissipative and asymptotically compact. Let B be a bounded absorbing set of the system $(X,\ S_t)$. Then the set $A = \omega(B)$ is a nonempty compact set and is a global attractor of the dynamical system $(X,\ S_t)$. The attractor A is a connected set in X.

In particular, this theorem is applicable to the dynamical systems from Exercises 4.2–4.11. It should also be noted that Theorem 5.1 along with Lemma 4.1 gives the following criterion: a dissipative dynamical system possesses a compact global attractor if and only if it is asymptotically compact.

The proof of the theorem is based on the following lemma.

Lemma 5.1.

Let a dynamical system (X, S_t) be asymptotically compact. Then for any bounded set B of X the ω-limit set $\omega(B)$ is a nonempty compact invariant set.

Proof.

Let $y_n \in B$. Then for any sequence $\{t_n\}$ tending to infinity the set $\{S^{(2)}_{t_n} y_n,\ n = 1, 2, \dots\}$ is relatively compact, i.e. there exist a sequence n_k and an element $y \in X$ such that $S^{(2)}_{t_{n_k}} y_{n_k}$ tends to y as $k \to \infty$. Hence, the asymptotic compactness gives us that

$$\left\| y - S_{t_{n_k}} y_{n_k} \right\| \le \left\| S^{(1)}_{t_{n_k}} y_{n_k} \right\| + \left\| y - S^{(2)}_{t_{n_k}} y_{n_k} \right\| \to 0 \quad \text{as} \quad k \to \infty .$$

Thus, $y = \lim_{k \to \infty} S_{t_{n_k}} y_{n_k}$. Due to Lemma 2.1 this indicates that $\omega(B)$ is nonempty.

Let us prove the invariance of ω-limit set. Let $y \in \omega(B)$. Then according to Lemma 2.1 there exist sequences $\{t_n\}$, $t_n \to \infty$, and $\{z_n\} \subset B$ such that $S_{t_n} z_n \to y$. However, the mapping S_t is continuous. Therefore,

$$S_{t + t_n} z_n = S_t \circ S_{t_n} z_n \to S_t y , \qquad n \to \infty .$$

Lemma 2.1 implies that $S_t y \in \omega(B)$. Thus,

$$S_t\, \omega(B) \subset \omega(B) , \qquad t > 0 .$$

Let us prove the reverse inclusion. Let $y \in \omega(B)$. Then there exist sequences $\{v_n\} \subset B$ and $\{t_n :\ t_n \to \infty\}$ such that $S_{t_n} v_n \to y$. Let us consider the sequence $y_n = S_{t_n - t} v_n$, $t_n \ge t$. The asymptotic compactness implies that there exist a subsequence t_{n_k} and an element $z \in X$ such that

$$z = \lim_{k \to \infty} S^{(2)}_{t_{n_k} - t}\ y_{n_k} .$$

As stated above, this gives us that

$$z = \lim_{k \to \infty} S_{t_{n_k} - t}\ y_{n_k} .$$

Therefore, $z \in \omega(B)$. Moreover,

$$S_t z = \lim_{k \to \infty} S_t \circ S_{t_{n_k} - t} v_{n_k} = \lim_{k \to \infty} S_{t_{n_k}} v_{n_k} = y .$$

Hence, $y \in S_t\, \omega(B)$. Thus, the invariance of the set $\omega(B)$ is proved.

Let us prove the compactness of the set $\omega(B)$. Assume that $\{z_n\}$ is a sequence in $\omega(B)$. Then Lemma 2.1 implies that for any n we can find $t_n \ge n$ and $y_n \in B$ such that $\left\| z_n - S_{t_n} y_n \right\| \le 1/n$. As said above, the property of asymptotic compactness enables us to find an element z and a sequence $\{n_k\}$ such that

$$\left\| S_{t_{n_k}} y_{n_k} - z \right\| \to 0 , \quad k \to \infty .$$

This implies that $z \in \omega(B)$ and $z_{n_k} \to z$. This means that $\omega(B)$ is a closed and compact set in H. Lemma 5.1 is proved completely.

Now we establish Theorem 5.1. Let B be a bounded absorbing set of the dynamical system. Let us prove that $\omega(B)$ is a global attractor. It is sufficient to verify that $\omega(B)$ uniformly attracts the absorbing set B. Assume the contrary. Then the value $\sup\{\operatorname{dist}(S_t y,\ \omega(B)):\ y \in B\}$ does not tend to zero as $t \to \infty$. This means that there exist $\delta > 0$ and a sequence $\{t_n:\ t_n \to \infty\}$ such that

$$\sup\left\{\operatorname{dist}(S_{t_n} y,\ \omega(B)):\ y \in B\right\} \ge 2\delta .$$

Therefore, there exists an element $y_n \in B$ such that

$$\operatorname{dist}(S_{t_n} y_n,\ \omega(B)) \ge \delta, \quad n = 1,\ 2,\ \dots . \tag{5.1}$$

As before, a convergent subsequence $\{S_{t_{n_k}} y_{n_k}\}$ can be extracted from the sequence $\{S_{t_n} y_n\}$. Therewith Lemma 2.1 implies

$$z \equiv \lim_{k \to \infty} S_{t_{n_k}} y_{n_k} \in \omega(B)$$

which contradicts estimate (5.1). Thus, $\omega(B)$ is a global attractor. Its compactness follows from the easily verifiable relation

$$A \equiv \omega(B) = \bigcap_{\tau > 0} \left[\bigcap_{t \ge \tau} S_t^{(2)} B \right].$$

Let us prove the connectedness of the attractor by reductio ad absurdum. Assume that the attractor A is not a connected set. Then there exists a pair of open sets U_1 and U_2 such that

$$U_i \cap A \ne \varnothing, \qquad i = 1,\ 2, \quad A \subset U_1 \cup U_2, \quad U_1 \cap U_2 = \varnothing .$$

Let $A^c = \operatorname{conv}(A)$ be a convex hull of the set A, i.e.

$$A^c = \left[\left\{ \sum_{i=1}^{N} \lambda_i v_i:\ v_i \in A,\ \lambda_i \ge 0,\ \ \sum_{i=1}^{N} \lambda_i = 1, \quad N = 1,\ 2,\ \dots \right\}\right].$$

It is clear that A^c is a bounded connected set and $A^c \supset A$. The continuity of the mapping S_t implies that the set $S_t A^c$ is also connected. Therewith $A = S_t A \subset S_t A^c$. Therefore, $U_i \cap S_t A^c \ne \varnothing$, $i = 1,\ 2$. Hence, for any $t > 0$ the pair U_1, U_2 cannot cover $S_t A^c$. It follows that there exists a sequence of points $x_n = S_n y_n \in S_n A^c$ such that $x_n \notin U_1 \cup U_2$. The asymptotic compactness of the dynamical system enables us to extract a subsequence $\{n_k\}$ such that $x_{n_k} = S_{n_k} y_{n_k}$ tends in X to an element y as $k \to \infty$. It is clear that $y \notin U_1 \cup U_2$ and $y \in \omega(A^c)$. These equations contradict one another since $\omega(A^c) \subset \omega(B) = A \subset U_1 \cup U_2$. Therefore, **Theorem 5.1 is proved** completely.

It should be noted that the connectedness of the global attractor can also be proved without using the linear structure of the phase space (do it yourself).

— Exercise *5.1* Show that the assumption of asymptotic compactness in Theorem 5.1 can be replaced by the Ladyzhenskaya assumption: the sequence $\{S_{t_n} u_n\}$ contains a convergent subsequence for any bounded sequence $\{u_n\} \subset X$ and for any increasing sequence $\{t_n\} \subset \mathbb{T}_+$ such that $t_n \to +\infty$. Moreover, the Ladyzhenskaya assumption is equivalent to the condition of asymptotic compactness.

— Exercise *5.2* Assume that a dynamical system (X, S_t) possesses a compact global attractor A. Let A^* be a minimal closed set with the property

$$\lim_{t \to \infty} \operatorname{dist}(S_t y, A^*) = 0 \quad \text{for every} \quad y \in X.$$

Then $A^* \subset A$ and $A^* = \bigcup\{\omega(x) : x \in X\}$, i.e. A^* coincides with the global minimal attractor (cf. Exercise 3.4).

— Exercise *5.3* Assume that equation (4.7) holds. Prove that the global attractor A possesses the property $A = \omega(K) \subset K$.

— Exercise *5.4* Assume that a dissipative dynamical system possesses a global attractor A. Show that $A = \omega(B)$ for any bounded absorbing set B of the system.

The fact that the global attractor A has the form $A = \omega(B)$, where B is an absorbing set of the system, enables us to state that the set $S_t B$ not only tends to the attractor A, but is also uniformly distributed over it as $t \to \infty$. Namely, the following assertion holds.

Theorem 5.2.

Assume that a dissipative dynamical system (X, S_t) ***possesses a compact global attractor*** A. ***Let*** B ***be a bounded absorbing set for*** (X, S_t). ***Then***

$$\lim_{t \to \infty} \sup\{\operatorname{dist}(a, S_t B) : a \in A\} = 0. \tag{5.2}$$

Proof.

Assume that equation (5.2) does not hold. Then there exist sequences $\{a_n\} \subset A$ and $\{t_n : t_n \to \infty\}$ such that

$$\operatorname{dist}(a_n, S_{t_n} B) \geq \delta \quad \text{for some} \quad \delta > 0. \tag{5.3}$$

The compactness of A enables us to suppose that $\{a_n\}$ converges to an element $a \in A$. Therewith (see Exercise 5.4)

$$a = \lim_{m \to \infty} S_{\tau_m} y_m, \quad \{y_m\} \subset B,$$

where $\{\tau_m\}$ is a sequence such that $\tau_m \to \infty$. Let us choose a subsequence $\{m_n\}$ such that $\tau_{m_n} \ge t_n + t_B$ for every $n = 1, 2, \ldots$. Here t_B is chosen such that $S_t B \subset \subset B$ for all $t \ge t_B$. Let $z_n = S_{\tau_{m_n} - t_n} y_{m_n}$. Then it is clear that $\{z_n\} \subset B$ and

$$a = \lim_{n \to \infty} S_{\tau_{m_n}} y_{m_n} = \lim_{n \to \infty} S_{t_n} z_n .$$

Equation (5.3) implies that

$$\operatorname{dist}(a_n, S_{t_n} z_n) \ge \operatorname{dist}(a_n, S_{t_n} B) \ge \delta .$$

This contradicts the previous equation. **Theorem 5.2 is proved**.

For a description of convergence of the trajectories to the global attractor it is convenient to use the ***Hausdorff metric*** that is defined on subsets of the phase space by the formula

$$\rho(C, D) = \max\{h(C, D);\quad h(D, C)\}, \tag{5.4}$$

where $C, D \in X$ and

$$h(C, D) = \sup\{\operatorname{dist}(c, D):\ c \in C\}. \tag{5.5}$$

Theorems 5.1 and 5.2 give us the following assertion.

Corollary 5.1.

Let (X, S_t) be an asymptotically compact dissipative system. Then its global attractor A possesses the property $\lim_{t \to \infty} \rho(S_t B, A) = 0$ for any bounded absorbing set B of the system (X, S_t).

In particular, this corollary means that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for every $t > t_\varepsilon$ the set $S_t B$ gets into the ε-vicinity of the global attractor A; and vice versa, the attractor A lies in the ε-vicinity of the set $S_t B$. Here B is a bounded absorbing set.

The following theorem shows that in some cases we can get rid of the requirement of asymptotic compactness if we use the notion of the global weak attractor.

Theorem 5.3.

Let the phase space H of a dynamical system (H, S_t) be a separable Hilbert space. Assume that the system (H, S_t) is dissipative and its evolutionary operator S_t is weakly closed, i.e. for all $t > 0$ the weak convergence $y_n \to y$ and $S_t y_n \to z$ imply that $z = S_t y$. Then the dynamical system (H, S_t) possesses a global weak attractor.

The proof of this theorem basically repeats the reasonings used in the proof of Theorem 5.1. The weak compactness of bounded sets in a separable Hilbert space plays the main role instead of the asymptotic compactness.

Lemma 5.2.

Assume that the hypotheses of Theorem 5.3 *hold. For* $B \subset H$ *we define the weak* ω*-limit set* $\omega_w(B)$ *by the formula*

$$\omega_w(B) = \bigcap_{s \geq 0} \Big[\bigcup_{t \geq s} S_t(B) \Big]_w, \tag{5.6}$$

where $[Y]_w$ *is the weak closure of the set* Y. *Then for any bounded set* $B \subset H$ *the set* $\omega_w(B)$ *is a nonempty weakly closed bounded invariant set.*

Proof.

The dissipativity implies that each of the sets $\gamma_w^s(B) = [\bigcup_{t \geq s} S_t(B)]_w$ is bounded and therefore weakly compact. Then the Cantor theorem on the collection of nested compact sets gives us that $\omega_w(B) = \bigcap_{s \geq 0} \gamma_w^s(B)$ is a nonempty weakly closed bounded set. Let us prove its invariance. Let $y \in \omega_w(B)$. Then there exists a sequence $y_n \in \bigcup_{t \geq n} S_t(B)$ such that $y_n \to y$ weakly. The dissipativity property implies that the set $\{S_t y_n\}$ is bounded when t is large enough. Therefore, there exist a subsequence $\{y_{n_k}\}$ and an element z such that $y_{n_k} \to y$ and $S_t y_{n_k} \to z$ weakly. The weak closedness of S_t implies that $z = S_t y$. Since $S_t y_{n_k} \in \gamma_w^s(B)$ for $n_k \geq s$, we have that $z \in \gamma_w^s(B)$ for all s. Hence, $z \in \omega_w(B)$. Therefore, $S_t\, \omega_w(B) \subset \omega_w(B)$. The proof of the reverse inclusion is left to the reader as an exercise.

For the proof of Theorem 5.3 it is sufficient to show that the set

$$A_w = \omega_w(B), \tag{5.7}$$

where B is a bounded absorbing set of the system $(H,\ S_t)$, is a global weak attractor for the system. To do that it is sufficient to verify that the set B is uniformly attracted to $A_w = \omega_w(B)$ in the weak topology of the space H. Assume the contrary. Then there exist a weak vicinity $\mathcal{O}$ of the set A_w and sequences $\{y_n\} \subset B$ and $\{t_n :\ t_n \to \to \infty\}$ such that $S_{t_n} y_n \notin \mathcal{O}$. However, the set $\{S_{t_n} y_n\}$ is weakly compact. Therefore, there exist an element $z \notin \mathcal{O}$ and a sequence $\{n_k\}$ such that

$$z = w - \lim_{k \to \infty} S_{t_{n_k}} y_{n_k}.$$

However, $S_{t_{n_k}} y_{n_k} \in \gamma_w^s(B)$ for $t_{n_k} \geq s$. Thus, $z \in \gamma_w^s(B)$ for all $s \geq 0$ and $z \in \in \omega_w(B)$, which is impossible. **Theorem 5.3 is proved**.

Exercise *5.5* Assume that the hypotheses of Theorem 5.3 hold. Show that the global weak attractor A_w is a connected set in the weak topology of the phase space H.

Exercise *5.6* Show that the global weak minimal attractor $A_w^* = \bigcup\{\omega_w(x) : x \in H\}$ is a strictly invariant set.

— Exercise *5.7* Prove the existence and describe the structure of global and global minimal attractors for the dynamical system generated by the equations

$$\begin{cases} \dot{x} = \mu x - y - x(x^2+y^2), \\ \dot{y} = x + \mu y - y(x^2+y^2) \end{cases}$$

for every real μ.

— Exercise *5.8* Assume that X is a metric space and (X, S_t) is an asymptotically compact (in the sense of the definition given in Remark 4.1) dynamical system. Assume also that the attracting compact K is contained in some bounded connected set. Prove the validity of the assertions of Theorem 5.1 in this case.

In conclusion to this section, we give one more assertion on the existence of the global attractor in the form of exercises. This assertion uses the notion of the asymptotic smoothness (see [3] and [9]). The dynamical system (X, S_t) is said to be ***asymptotically smooth*** if for any bounded positively invariant $(S_t B \subset B,\ t \ge 0)$ set $B \subset X$ there exists a compact K such that $h(S_t B, K) \to 0$ as $t \to \infty$, where the value $h(\cdot, \cdot)$ is defined by formula (5.5).

— Exercise *5.9* Prove that every asymptotically compact system is asymptotically smooth.

— Exercise *5.10* Let (X, S_t) be an asymptotically smooth dynamical system. Assume that for any bounded set $B \subset X$ the set $\gamma^+(B) = \bigcup_{t \ge 0} S_t(B)$ is bounded. Show that the system (X, S_t) possesses a global attractor A of the form

$$A = \bigcup \{ w(B) \colon\ B \subset X,\ B \text{ is bounded} \}.$$

— Exercise *5.11* In addition to the assumptions of Exercise 5.10 assume that (X, S_t) is pointwise dissipative, i.e. there exists a bounded set $B_0 \subset X$ such that $\mathrm{dist}_X(S_t y, B_0) \to 0$ as $t \to \infty$ for every point $y \in X$. Prove that the global attractor A is compact.

§ 6 *On the Structure of Global Attractor*

The study of the structure of global attractor of a dynamical system is an important problem from the point of view of applications. There are no universal approaches to this problem. Even in finite-dimensional cases the attractor can be of complicated structure. However, some sets that undoubtedly belong to the attractor can be pointed out. It should be first noted that every stationary point of the semigroup S_t belongs to the attractor of the system. We also have the following assertion.

Lemma 6.1.

Assume that an element z lies in the global attractor A of a dynamical system (X, S_t). Then the point z belongs to some trajectory γ that lies in A wholly.

Proof.

Since $S_t A = A$ and $z \in A$, then there exists a sequence $\{z_n\} \subset A$ such that $z_0 = z$, $S_1 z_n = z_{n-1}$, $n = 1, 2, \ldots$. Therewith for discrete time the required trajectory is $\gamma = \{u_n : n \in \mathbb{Z}\}$, where $u_n = S_n z$ for $n \ge 0$ and $u_n = z_{-n}$ for $n \le 0$. For continuous time let us consider the value

$$u(t) = \begin{cases} S_t z, & t \ge 0, \\ S_{t+n} z_n, & -n \le t \le -n+1, \quad n = 1, 2, \ldots \end{cases}$$

Then it is clear that $u(t) \in A$ for all $t \in \mathbb{R}$ and $S_\tau u(t) = u(t+\tau)$ for $\tau \ge 0$, $t \in \mathbb{R}$. Therewith $u(0) = z$. Thus, the required trajectory is also built in the continuous case.

— Exercise *6.1* Show that an element z belongs to a global attractor if and only if there exists a bounded trajectory $\gamma = \{u(t) : -\infty < t < \infty\}$ such that $u(0) = z$.

Unstable sets also belong to the global attractor. Let Y be a subset of the phase space X of the dynamical system (X, S_t). Then the ***unstable set emanating from*** Y is defined as the set $\mathbb{M}_+(Y)$ of points $z \in X$ for every of which there exists a trajectory $\gamma = \{u(t) : t \in \mathbb{T}\}$ such that

$$u(0) = z, \qquad \lim_{t \to -\infty} \operatorname{dist}(u(t), Y) = 0 .$$

— Exercise *6.2* Prove that $\mathbb{M}_+(Y)$ is invariant, i.e. $S_t \mathbb{M}_+(Y) = \mathbb{M}_+(Y)$ for all $t > 0$.

Lemma 6.2.

Let $\mathcal{N}$ be a set of stationary points of the dynamical system (X, S_t) possessing a global attractor A. Then $\mathbb{M}_+(\mathcal{N}) \subset A$.

Proof.

It is obvious that the set $\mathcal{N} = \{z : S_t z = z, \quad t > 0\}$ lies in the attractor of the system and thus it is bounded. Let $z \in \mathbb{M}_+(\mathcal{N})$. Then there exists a trajectory $\gamma_z = \{u(t), t \in \mathbb{T}\}$ such that $u(0) = z$ and

$$\operatorname{dist}(u(\tau), \mathcal{N}) \to 0, \qquad \tau \to -\infty .$$

Therefore, the set $B_s = \{u(\tau):\ \tau \le -s\}$ is bounded when $s > 0$ is large enough. Hence, the set $S_t B_s$ tends to the attractor of the system as $t \to +\infty$. However, $z \in S_t B_s$ for $t \ge s$. Therefore,

$$\operatorname{dist}(z, A) \le \sup\{\operatorname{dist}(S_t y, A):\ y \in B_s\} \to 0, \qquad t \to +\infty .$$

This implies that $z \in A$. The lemma is proved.

Exercise *6.3* Assume that the set $\mathcal{N}$ of stationary points is finite. Show that

$$\mathbb{M}_+(\mathcal{N}) = \bigcup_{k=1}^{l} \mathbb{M}_+(z_k),$$

where z_k are the stationary points of S_t (the set $\mathbb{M}_+(z_k)$ is called an unstable manifold emanating from the stationary point z_k).

Thus, the global attractor A includes the unstable set $\mathbb{M}_+(\mathcal{N})$. It turns out that under certain conditions the attractor includes nothing else. We give the following definition. Let Y be a positively invariant set of a semigroup S_t: $S_t Y \subset Y$, $t > 0$. The continuous functional $\Phi(y)$ defined on Y is called the ***Lyapunov function*** of the dynamical system (X, S_t) on Y if the following conditions hold:

a) for any $y \in Y$ the function $\Phi(S_t y)$ is a nonincreasing function with respect to $t \ge 0$;

b) if for some $t_0 > 0$ and $y \in X$ the equation $\Phi(y) = \Phi(S_{t_0} y)$ holds, then $y = S_t y$ for all $t \ge 0$, i.e. y is a stationary point of the semigroup S_t.

Theorem 6.1.

Let a dynamical system (X, S_t) ***possess a compact attractor*** A***. Assume also that the Lyapunov function*** $\Phi(y)$ ***exists on*** A***. Then*** $A = \mathbb{M}_+(\mathcal{N})$***, where*** $\mathcal{N}$ ***is the set of stationary points of the dynamical system.***

Proof.

Let $y \in A$. Let us consider a trajectory γ passing through y (its existence follows from Lemma 6.1). Let

$$\gamma = \{u(t):\ t \in \mathbb{T}\} \quad \text{and} \quad \gamma_\tau^- = \{u(t):\ t \le \tau\}.$$

Since $\gamma_\tau^- \subset A$, the closure $[\gamma_\tau^-]$ is a compact set in X. This implies that the α-limit set

$$\alpha(\gamma) = \bigcap_{\tau < 0} [\gamma_\tau^-]$$

of the trajectory γ is nonempty. It is easy to verify that the set $\alpha(\gamma)$ is invariant: $S_t\,\alpha(\gamma) = \alpha(\gamma)$. Let us show that the Lyapunov function $\Phi(y)$ is constant on $\alpha(\gamma)$. Indeed, if $u \in \alpha(\gamma)$, then there exists a sequence $\{t_n\}$ tending to $-\infty$ such that

$$\lim_{t_n \to -\infty} u(t_n) = u .$$

Consequently,

$$\Phi(u) = \lim_{n \to \infty} \Phi(u(t_n)) .$$

By virtue of monotonicity of the function $\Phi(u)$ along the trajectory we have

$$\Phi(u) = \sup\{\Phi(u(\tau)): \ \tau < 0\} .$$

Therefore, the function $\Phi(u)$ is constant on $\alpha(\gamma)$. Hence, the invariance of the set $\alpha(\gamma)$ gives us that $\Phi(S_t u) = \Phi(u)$, $t > 0$ for all $u \in \alpha(\gamma)$. This means that $\alpha(\gamma)$ lies in the set $\mathcal{N}$ of stationary points. Therewith (verify it yourself)

$$\lim_{t \to -\infty} \operatorname{dist}(u(t), \ \alpha(\gamma)) = 0 .$$

Hence, $y \in \mathbb{M}_+(\mathcal{N})$. **Theorem 6.1 is proved**.

Exercise *6.4* Assume that the hypotheses of Theorem 6.1 hold. Then for any element $y \in A$ its ω-limit set $\omega(y)$ consists of stationary points of the system.

Thus, the global attractor coincides with the set of all full trajectories connecting the stationary points.

Exercise *6.5* Assume that the system $(X, \ S_t)$ possesses a compact global attractor and there exists a Lyapunov function on X. Assume that the Lyapunov function is bounded below. Show that any semitrajectory of the system tends to the set $\mathcal{N}$ of stationary points of the system as $t \to +\infty$, i.e. the global minimal attractor coincides with the set $\mathcal{N}$.

In particular, this exercise confirms the fact realized by many investigators that the global attractor is a too wide object for description of actually observed limit regimes of a dynamical system.

Exercise *6.6* Assume that $(\mathbb{R}, \ S_t)$ is a dynamical system generated by the logistic equation (see Example 1.1): $\dot{x} + \alpha x(x-1) = 0, \quad \alpha > 0$. Show that $V(x) = x^3/3 - x^2/2$ is a Lyapunov function for this system.

Exercise *6.7* Show that the total energy

$$E(x, \ \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4 - \frac{a}{2}x^2 - bx$$

is a Lyapunov function for the dynamical system generated (see Example 1.2) by the Duffing equation

$$\ddot{x} + \varepsilon\dot{x} + x^3 - ax = b , \quad \varepsilon > 0 .$$

If in the definition of a Lyapunov functional we omit the second requirement, then a minor modification of the proof of Theorem 6.1 enables us to get the following assertion.

Theorem 6.2.

Assume that a dynamical system (X, S_t) ***possesses a compact global attractor*** A ***and there exists a continuous function*** $\Psi(y)$ ***on*** X ***such that*** $\Psi(S_t y)$ ***does not increase with respect to*** t ***for any*** $y \in X$***. Let*** $\mathcal{L}$ ***be a set of elements*** $u \in A$ ***such that*** $\Psi(u(t)) = \Psi(u)$ ***for all*** $-\infty < t < \infty$***. Here*** $\{u(t)\}$ ***is a trajectory of the system passing through*** u $(u(0) = u)$***. Then*** $\mathbb{M}_+(\mathcal{L}) = A$ ***and*** $\mathcal{L}$ ***contains the global minimal attractor*** $A^* = \bigcup_{x \in X} \omega(x)$.

Proof.

In fact, the property $\mathbb{M}_+(\mathcal{L}) = A$ was established in the proof of Theorem 6.1. As to the property $A^* \subset \mathcal{L}$, it follows from the constancy of the function $\Psi(u)$ on the ω-limit set $\omega(x)$ of any element $x \in X$.

— Exercise *6.8* Apply Theorem 6.2 to justify the results of Example 3.1 (see also Exercise 4.8).

If the set $\mathcal{N}$ of stationary points of a dynamical system (X, S_t) is finite, then Theorem 6.1 can be extended a little. This extension is described below in Exercises 6.9–6.12. In these exercises it is assumed that the dynamical system (X, S_t) is continuous and possesses the following properties:

(a) there exists a compact global attractor A;

(b) there exists a Lyapunov function $\Phi(x)$ on A;

(c) the set $\mathcal{N} = \{z_1, \dots z_N\}$ of stationary points is finite, therewith $\Phi(z_i) \neq \Phi(z_j)$ for $i \neq j$ and the indexing of z_j possesses the property

$$\Phi(z_1) < \Phi(z_2) < \dots < \Phi(z_N). \tag{6.1}$$

We denote

$$A_j = \bigcup_{k=1}^{j} \mathbb{M}_+(z_k), \quad j = 1, 2, \dots, N, \quad A_0 = \varnothing.$$

— Exercise *6.9* Show that $S_t A_j = A_j$ for all $j = 1, 2, \dots N$.

— Exercise *6.10* Assume that $B \subset A_j \setminus \{z_j\}$. Then

$$\lim_{t \to \infty} \sup\{\operatorname{dist}(S_t y, A_{j-1}) : y \in B\} = 0. \tag{6.2}$$

— Exercise *6.11* Assume that the function Φ is defined on the whole X. Then (6.2) holds for any bounded set $B \subset \{x : \Phi(x) < \Phi(z_j) - \delta\}$, where δ is a positive number.

Exercise *6.12* Assume that $[\mathbb{M}_+(z_j)]$ is the closure of the set $\mathbb{M}_+(z_j)$ and $\partial\mathbb{M}_+(z_j) = [\mathbb{M}_+(z_j)] \setminus \mathbb{M}_+(z_j)$ is its boundary. Show that $\partial\mathbb{M}_+ z_j \subset$ $\subset A_{j-1}$ and

$$S_t[\mathbb{M}_+(z_j)] = [\mathbb{M}_+(z_j)], \qquad S_t\,\partial\mathbb{M}_+(z_j) = \partial\mathbb{M}_+(z_j).$$

It can also be shown (see the book by A. V. Babin and M. I. Vishik [1]) that under some additional conditions on the evolutionary operator S_t the unstable manifolds $\mathbb{M}_+(z_j)$ are surfaces of the class C^1, therewith the facts given in Exercises 6.9–6.12 remain true if strict inequalities are substituted by nonstrict ones in (6.1). It should be noted that a global attractor possessing the properties mentioned above is frequently called ***regular***.

Let us give without proof one more result on the attractor of a system with a finite number of stationary points and a Lyapunov function. This result is important for applications.

At first let us remind several definitions. Let S be an operator acting in a Banach space X. The operator S is called ***Frechét differentiable at a point*** $x \in X$ provided that there exists a linear bounded operator $S'(x)\colon\ X \to X$ such that

$$\|S(y) - S(x) - S'(x)(y-x)\| \;\le\; \gamma(\|x-y\|)\|x-y\|$$

for all y from some vicinity of the point x, where $\gamma(\xi) \to 0$ as $\xi \to 0$. Therewith, the operator S is said to belong to the class $C^{1+\alpha}$, $0 < \alpha < 1$, on a set Y if it is ***differentiable at every point*** $x \in Y$ and

$$\|S'(x) - S'(y)\|_{L(X,\,X)} \;\le\; C\|x-y\|^{\alpha}$$

for all y from some vicinity of the point $x \in Y$. A stationary point z of the mapping S is called ***hyperbolic*** if $S \in C^{1+\alpha}$ in some vicinity of the point z, the spectrum of the linear operator $S'(z)$ does not cross the unit circle $\{\lambda\colon\ |\lambda| = 1\}$ and the spectral subspace of the operator corresponding to the set $\{\lambda\colon\ |\lambda| > 1\}$ is finite-dimensional.

Theorem 6.3.

Let X be a Banach space and let a continuous dynamical system $(X,\ S_t)$ possess the properties:

1) *there exists a global attractor A;*
2) *there exists a vicinity Ω of the attractor A such that*

$$\|S_t x - S_t y\| \le C e^{\alpha(t-\tau)}\|S_\tau x - S_\tau y\|$$

for all $t \ge \tau \ge 0$, provided $S_t x$ and $S_t y$ belong to Ω for all $t \ge 0$;

3) *there exists a Lyapunov function continuous on X;*
4) *the set $\mathcal{N} = \{z_1,\ \dots,\ z_N\}$ of stationary points is finite and all the points are hyperbolic;*

5) ***the mapping*** $(t, u) \to S_t u$ ***is continuous.***
Then for any compact set B ***in*** X ***the estimate***

$$\sup \left\{ \operatorname{dist}(S_t y, A) : \ y \in B \right\} \le C_B e^{-\eta t} \tag{6.3}$$

holds for all $t \ge 0$***, where*** $\eta > 0$ ***does not depend on*** B.

The proof of this theorem as well as other interesting results on the asymptotic behaviour of a dynamical system possessing a Lyapunov function can be found in the book by A. V. Babin and M. I. Vishik [1].

To conclude this section, we consider a finite-dimensional example that shows how the Lyapunov function method can be used to prove the existence of periodic trajectories in the attractor.

E x a m p l e *6.1* (on the theme by E. Hopf)

Studying Galerkin approximations in a model suggested by E. Hopf for the description of possible mechanisms of turbulence appearence, we obtain the following system of ordinary differential equations

$$\begin{cases} \dot{u} + \mu u + v^2 + w^2 = 0\,, & (6.4) \\ \dot{v} + \nu v - v u - \beta w = 0\,, & (6.5) \\ \dot{w} + \nu w - w u + \beta v = 0\,. & (6.6) \end{cases}$$

Here μ is a positive parameter, ν and β are real parameters. It is clear that the Cauchy problem for (6.4)–(6.6) is solvable, at least locally for any initial condition. Let us show that the dynamical system generated by equations (6.4)–(6.6) is dissipative. It will also be sufficient for the proof of global solvability. Let us introduce a new unknown function $u^* = u + \mu/2 - \nu$. Then equations (6.4)–(6.6) can be rewritten in the form

$$\begin{cases} \dot{u}^* + \mu u^* + v^2 + w^2 = \mu\left(\dfrac{\mu}{2} - \nu\right), \\ \dot{v} + \dfrac{1}{2}\,\mu v - v u^* - \beta w = 0\,, \\ \dot{w} + \dfrac{1}{2}\,\mu w - w u^* + \beta v = 0\,. \end{cases}$$

These equations imply that

$$\frac{1}{2}\frac{d}{dt}\left(|u^*|^2 + |v|^2 + |w|^2\right) + \mu|u^*|^2 + \frac{\mu}{2}\left(|v|^2 + |w|^2\right) = \mu\left(\frac{\mu}{2} - \nu\right)u^*$$

on any interval of existence of solutions. Hence,

$$\frac{d}{dt}\left(\left(|u^*|^2 + |v|^2 + |w|^2\right)\right) + \mu\left(|u^*|^2 + |v|^2 + |w|^2\right) \le \mu\left(\frac{\mu}{2} - \nu\right)^2 .$$

Thus,

$$|u^*(t)|^2 + |v(t)|^2 + |w(t)|^2 \le$$
$$\le \left(|u^*(0)|^2 + |v(0)|^2 + |w(0)|^2\right) e^{-\mu t} + \left(\frac{\mu}{2} - \nu\right)^2 (1 - e^{-\mu t}) \; .$$

Firstly, this equation enables us to prove the global solvability of problem (6.4)–(6.6) for any initial condition and, secondly, it means that the set

$$B_0 = \left\{(u,\ v,\ w) \colon \ \left(u + \frac{\mu}{2} - \nu\right)^2 + v^2 + w^2 \le 1 + \left(\frac{\mu}{2} - \nu\right)^2\right\}$$

is absorbing for the dynamical system $(\mathbb{R}^3,\ S_t)$ generated by the Cauchy problem for equations (6.4)–(6.6). Thus, Theorem 5.1 guarantees the existence of a global attractor A. It is a connected compact set in $\mathbb{R}^3$.

Exercise *6.13* Verify that B_0 is a positively invariant set for $(\mathbb{R}^3,\ S_t)$.

In order to describe the structure of the global attractor A we introduce the polar coordinates

$$v(t) = r(t)\cos\varphi(t), \qquad w(t) = r(t)\sin\varphi(t)$$

on the plane of the variables $\{v;\ w\}$. As a result, equations (6.4)–(6.6) are transformed into the system

$$\begin{cases} \dot{u} + \mu u + r^2 = 0, & (6.7) \\ \dot{r} + \nu r - u r = 0, & (6.8) \end{cases}$$

therewith, $\varphi(t) = -\beta t + \varphi_0$. System (6.7) and (6.8) has a stationary point $\{u = 0,\ r = 0\}$ for all $\mu > 0$ and $\nu \in \mathbb{R}$. If $\nu < 0$, then one more stationary point $\{u = \nu,\ r = \sqrt{-\mu\nu}\}$ occurs in system (6.7) and (6.8). It corresponds to a periodic trajectory of the original problem (6.4)–(6.6).

Exercise *6.14* Show that the point $(0;\ 0)$ is a stable node of system (6.7) and (6.8) when $\nu > 0$ and it is a saddle when $\nu < 0$.

Exercise *6.15* Show that the stationary point $\{u = \nu,\ r = \sqrt{-\mu\nu}\}$ is stable ($\nu < 0$) being a node if $-\mu/8 < \nu < 0$ and a focus if $\nu < -\dfrac{\mu}{8}$.

If $\nu > 0$, then (6.7) and (6.8) imply that

$$\frac{1}{2}\frac{d}{dt}(u^2 + r^2) + \min(\mu,\ \nu)(u^2 + r^2) \le 0 \, .$$

Therefore,

$$|u(t)|^2 + |r(t)|^2 \le |u(0)|^2 + |r(0)|^2 \, e^{-2\min(\mu,\ \nu)t} \, .$$

Hence, for $\nu > 0$ the global attractor A of the system $(\mathbb{R}^3, S_t)$ consists of the single stationary exponentially attracting point

$$\{u = 0, \ v = 0, \ w = 0\}.$$

Exercise *6.16* Prove that for $\nu = 0$ the global attractor of problem (6.4)–(6.6) consists of the single stationary point $\{u = 0, v = 0, w = 0\}$. Show that it is not exponentially attracting.

Now we consider the case $\nu < 0$. Let us again refer to problem (6.7) and (6.8). It is clear that the line $r = 0$ is a stable manifold of the stationary point $\{u = 0, \ r = 0\}$. Moreover, it is obvious that if $r(t_0) > 0$, then the value $r(t)$ remains positive for all $t > t_0$. Therefore, the function

$$V(u, \ r) = \frac{1}{2}(u - \nu)^2 + \frac{1}{2}r^2 + \mu\nu \ln r \tag{6.9}$$

is defined on all the trajectories, the initial point of which does not lie on the line $\{r = 0\}$. Simple calculations show that

$$\frac{d}{dt}(V(u(t), \ r(t))) + \mu(u(t) - \nu)^2 = 0 \tag{6.10}$$

and

$$V(u, r) \ \ge \ V(\nu, \sqrt{-\mu\nu}) + \frac{1}{2}\left(|u - \nu|^2 + |r - \sqrt{-\mu\nu}|^2\right); \tag{6.11}$$

therewith, $V(\nu, \sqrt{-\mu\nu}) = (1/2)\mu|\nu| \ln(e/(\mu|\nu|))$. Equation (6.10) implies that the function $V(u, \ r)$ does not increase along the trajectories. Therefore, any semi-trajectory $\{(u(t); \ r(t)), \ t \in \mathbb{R}_+\}$ emanating from the point $\{u_0, \ r_0; \ \ r_0 \neq 0\}$ of the system $(\mathbb{R} \times \mathbb{R}_+, \ S_t)$ generated by equations (6.7) and (6.8) possesses the property $V(u(t), \ r(t)) \le V(u_0, \ r_0)$ for $t \ge 0$. Therewith, equation (6.9) implies that this semitrajectory can not approach the line $\{r = 0\}$ at a distance less then $\exp\{[1/(\mu\nu)] \cdot V(u_0, \ r_0)\}$. Hence, this semitrajectory tends to $\bar{y} = \{u = \nu, \ r = \sqrt{-\mu\nu}\}$. Moreover, for any $\xi \in \mathbb{R}$ the set

$$B_\xi = \{y = (u, \ r): \ V(u, \ r) \le \xi\}$$

is uniformly attracted to $\bar{y}$, i.e. for any $\varepsilon > 0$ there exists $t_0 = t_0(\xi, \ \varepsilon)$ such that

$$S_t B_\xi \subset \{y: \ |y - \bar{y}| \le \varepsilon\}.$$

Indeed, if it is not true, then there exist $\varepsilon_0 > 0$, a sequence $t_n \to +\infty$, and $z_n \in B_\xi$ such that $\left|S_{t_n} z_n - \bar{y}\right| > \varepsilon_0$. The monotonicity of $V(y)$ and property (6.11) imply that

$$V(S_t z_n) \ \ge \ V(S_{t_n} z_n) \ \ge \ V(\nu, \sqrt{-\mu\nu}) + \frac{1}{2}\varepsilon_0^2$$

for all $0 \le t \le t_n$. Let z be a limit point of the sequence $\{z_n\}$. Then after passing to the limit we find out that

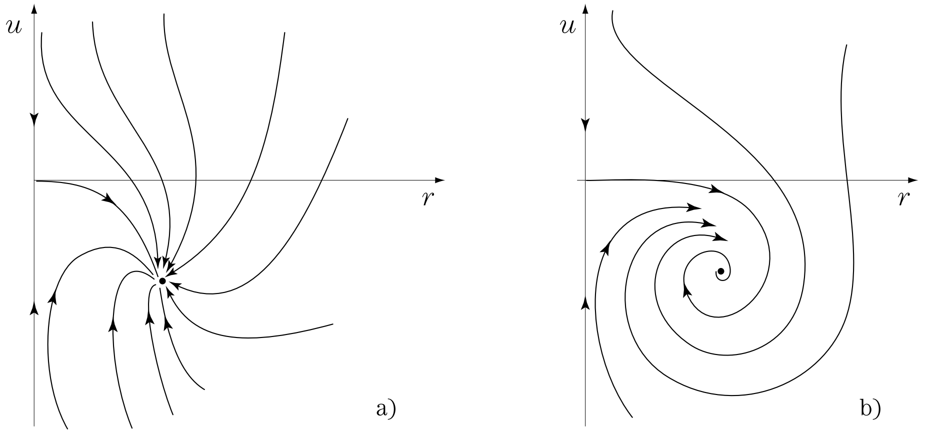

Fig. 2. Qualitative behaviour of solutions to problem (6.7), (6.8):
a) $-\mu/8 < \nu < 0$, b) $\nu < -\mu/8$

$$V(S_t z) \ \geq \ V(\nu, \ \sqrt{-\mu\nu}) + \frac{1}{2}\varepsilon_0^2, \qquad t \geq 0$$

with $z \notin \{r = 0\}$. Thus, the last inequality is impossible since $S_t z \to \overline{y} = \{u = \nu,$ $r = \sqrt{-\mu\nu}\}$. Hence

$$\lim_{t \to \infty} \sup\{\operatorname{dist}(S_t y, \ \overline{y}) \colon \ y \in B_\xi\} = 0\,. \tag{6.12}$$

The qualitative behaviour of solutions to problem (6.7) and (6.8) on the semiplane is shown on *Fig. 2*.

In particular, the observations above mean that the global minimal attractor $A_{\min}$ of the dynamical system $(\mathbb{R}^3, \ S_t)$ generated by equations (6.4)–(6.6) consists of the saddle point $\{u = 0, \ v = 0, \ w = 0\}$ and the stable limit cycle

$$C_\nu = \{u = \nu, \quad v^2 + w^2 = -\mu\nu\} \tag{6.13}$$

for $\nu < 0$. Therewith, equation (6.12) implies that the cycle C_ν uniformly attracts all bounded sets B in $\mathbb{R}^3$ possessing the property

$$d \ \equiv \ \inf\{v^2 + w^2 \colon \ (u, \ v, \ w) \in B\} > 0\,, \tag{6.14}$$

i.e. which lie at a positive distance from the line $\{v = 0, \ w = 0\}$.

Exercise *6.17* Using the structure of equations (6.7) and (6.8) near the stationary point $\{u = \nu, \ r = \sqrt{-\mu\nu}\}$, prove that a bounded set B possessing property (6.14) is uniformly and exponentially attracted to the cycle C_ν, i.e.

$$\sup\{\operatorname{dist}(S_t y, \ C_\nu), \quad y \in B\} \leq C e^{-\gamma(t - t_B)}$$

for $t \geq t_B$, where γ is a positive constant.

Now let $y_0 = (u_0,\ v_0,\ w_0)$ lie in the global attractor A of the system $(\mathbb{R}^3,\ S_t)$. Assume that $r_0 \neq 0$ and $r_0^2 = v_0^2 + w_0^2 \neq -\mu\nu$. Then (see Lemma 6.1) there exists a trajectory $\gamma = \{y(t) = (u(t);\ v(t);\ w(t)),\quad t \in \mathbb{R}\}$ lying in A such that $y(0) = y_0$. The analysis given above shows that $y(t) \to C_\nu$ as $t \to +\infty$. Let us show that $y(t) \to 0$ when $t \to -\infty$. Indeed, the function $V(u(t),\ r(t))$ is monotonely nondecreasing as $t \to -\infty$. If we argue by contradiction and use the fact that $|y(t)|$ is bounded we can easily find out that

$$\lim_{t \to -\infty} V(u(t),\ r(t)) = \infty$$

and therefore

$$r(t) = \left(|v(t)|^2 + |w(t)|^2\right)^{1/2} \to 0 \quad \text{as} \quad t \to -\infty\,. \tag{6.16}$$

Equation (6.7) gives us that

$$u(t) = e^{-\mu(t-\tau)}u(s) - \int_s^t e^{-\mu(t-\tau)}[r(\tau)]^2\,d\tau\,. \tag{6.17}$$

Since $u(s)$ is bounded for all $s \in \mathbb{R}$, we can get the equation

$$u(t) = -\int_{-\infty}^t e^{-\mu(t-\tau)}[r(\tau)]^2\,d\tau$$

by tending $s \to -\infty$ in (6.17). Therefore, by virtue of (6.16) we find that $u(t) \to 0$ as $t \to -\infty$. Thus, $y(t) \to 0$ as $t \to -\infty$. Hence, for $\nu < 0$ the global attractor A

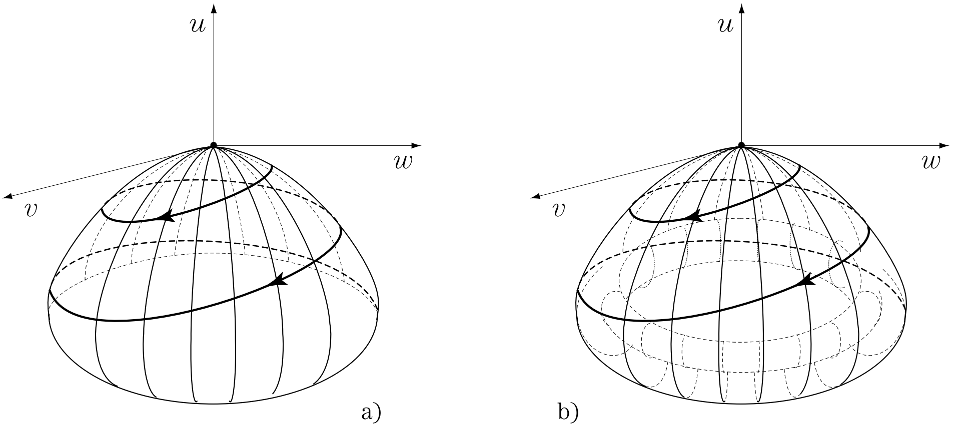

Fig. 3. Attractor of the system (6.4)–(6.6);
a) $-\mu/8 < \nu < 0$, b) $\nu < -\mu/8$

of the system $(\mathbb{R}^3, S_t)$ coincides with the union of the unstable manifold $\mathbb{M}_+(0)$ emanating from the point $\{u=0,\ v=0,\ w=0\}$ and the limit cycle (6.13). The attractor is shown on *Fig. 3*.

§ 7 Stability Properties of Attractor and Reduction Principle

A positively invariant set M in the phase space of a dynamical system (X, S_t) is said to be ***stable (in Lyapunov's sense)*** in X if its every vicinity $\mathcal{O}$ contains some vicinity $\mathcal{O}'$ such that $S_t(\mathcal{O}') \subset \mathcal{O}$ for all $t \ge 0$. Therewith, M is said to be ***asymptotically stable*** if it is stable and $S_t y \to M$ as $t \to \infty$ for every $y \in \mathcal{O}'$. A set M is called ***uniformly asymptotically stable*** if it is stable and

$$\lim_{t \to \infty} \sup\{\operatorname{dist}(S_t y, M):\ y \in \mathcal{O}'\} = 0. \tag{7.1}$$

The following simple assertion takes place.

Theorem 7.1.

Let A be the compact global attractor of a continuous dynamical system (X, S_t). Assume that there exists its bounded vicinity U such that the mapping $(t, u) \to S_t u$ is continuous on $\mathbb{R}_+ \times U$. Then A is a stable set.

Proof.

Assume that $\mathcal{O}$ is a vicinity of A. Then there exists $T > 0$ such that $S_t U \subset \mathcal{O}$ for $t \ge T$. Let us show that there exists a vicinity $\mathcal{O}'$ of the attractor A such that $S_t \mathcal{O}' \subset \mathcal{O}$ for all $t \in [0, T]$. Assume the contrary. Then there exist sequences $\{u_n\}$ and $\{t_n\}$ such that $\operatorname{dist}(u_n, A) \to 0$, $\{t_n\} \in [0, T]$ and $S_{t_n} u_n \notin \mathcal{O}$. The set A being compact, we can choose a subsequence $\{n_k\}$ such that $u_{n_k} \to u \in A$ as $t_{n_k} \to$ $\to t \in [0, T]$. Therefore, the continuity property of the function $(t, u) \to S_t u$ gives us that $S_{t_{n_k}} u_{n_k} \to S_t u \in A$. This contradicts the equation $S_{t_n} u_n \notin \mathcal{O}$. Thus, there exists $\mathcal{O}'$ such that $S_t \mathcal{O}' \subset \mathcal{O}$ for $t \in [0, T]$. We can choose T such that $S_t(\mathcal{O}' \cap U) \subset \mathcal{O}$ for all $t \ge 0$. Therefore, the attractor A is stable. **Theorem 7.1 is proved**.

It is clear that the stability of the global attractor implies its uniform asymptotic stability.

Exercise *7.1* Assume that M is a positively invariant set of a system (X, S_t). Prove that if there exists an element $y \notin M$ such that its α-limit set $\alpha(y)$ possesses the property $\alpha(y) \cap M \ne \varnothing$, then M is not stable.

In particular, the result of this exercise shows that the global minimal attractor can appear to be an unstable set.

— Exercise *7.2* Let us return to Example 3.1 (see also Exercises 4.8 and 6.8). Show that:

(a) the global attractor A_1 and the Milnor attractor A_4 are stable;

(b) the global minimal attractor A_3 and the Ilyashenko attractor A_5 are unstable.

Now let us consider the question concerning the stability of the attractor with respect to perturbations of a dynamical system. Assume that we have a family of dynamical systems (X, S_t^λ) with the same phase space X and with an evolutionary operator S_t^λ depending on a parameter λ which varies in a complete metric space Λ. The following assertion was proved by L. V. Kapitansky and I. N. Kostin [6].

Theorem 7.2.

Assume that a dynamical system (X, S_t^λ) ***possesses a compact global attractor*** A^λ ***for every*** $\lambda \in \Lambda$. ***Assume that the following conditions hold:***

(a) ***there exists a compact*** $K \subset X$ ***such that*** $A^\lambda \subset K$ ***for all*** $\lambda \in \Lambda$***;***

(b) ***if*** $\lambda_k \to \lambda_0$, $x_k \in A^{\lambda_k}$ ***and*** $x_k \to x_0$, ***then*** $S_{t_0}^{\lambda_k} x_k \to S_{t_0} x_0$ ***for some*** $t_0 > 0$.

Then the family of attractors A^λ ***is upper semicontinuous at the point*** λ_0, ***i.e.***

$$h(A^{\lambda_k}, A^{\lambda_0}) \equiv \sup\{\operatorname{dist}(y, A^{\lambda_0}) : y \in A^{\lambda_k}\} \to 0 \tag{7.2}$$

as $\lambda_k \to \lambda_0$.

Proof.

Assume that equation (7.2) does not hold. Then there exist a sequence $\lambda_k \to \to \lambda_0$ and a sequence $x_k \in A^{\lambda_k}$ such that $\operatorname{dist}(x_k, A^{\lambda_0}) \ge \delta$ for some $\delta > 0$. But the sequence x_k lies in the compact K. Therefore, without loss of generality we can assume that $x_k \to x_0 \in K$ for some $x_0 \in K$ and $x_0 \notin A^{\lambda_0}$. Let us show that this result leads to contradiction. Let $\gamma_k = \{u_k(t) : -\infty < t < \infty\}$ be a trajectory of the dynamical system $(X, S_t^{\lambda_k})$ passing through the element x_k $(u_k(0) = x_k)$. Using the standard diagonal process it is easy to find that there exist a subsequence $\{k(n)\}$ and a sequence of elements $\{u_m\} \subset K$ such that

$$\lim_{n\to\infty} u_{k(n)}(-m t_0) = u_m \quad \text{for all} \quad m = 0,\ 1,\ 2,\ \ldots\ ,$$

where $u_0 = x_0$. Here $t_0 > 0$ is a fixed number. Sequential application of condition (b) gives us that

$$u_{m-l} = \lim_{n \to \infty} u_{k(n)}(-(m-l)t_0) = \lim_{n \to \infty} S_{lt_0}^{\lambda_{k(n)}} u_{k(n)}(-mt_0) = S_{lt_0}^{\lambda_0} u_m$$

for all $m = 1, 2, \ldots$ and $l = 1, 2, \ldots, m$. It follows that the function

$$u(t) = \begin{cases} S_t^{\lambda_0} u_0, & t \ge 0, \\ S_{t+t_0 m}^{\lambda_0} u_m, & -t_0 m \le t < -t_0(m-1), \quad m = 1, 2, \ldots \end{cases}$$

gives a full trajectory γ passing through the point x_0. It is obvious that the trajectory γ is bounded. Therefore (see Exercise 6.1), it wholly belongs to A^{λ_0}, but that contradicts the equation $x_0 \notin A^{\lambda_0}$. **Theorem 7.2 is proved**.

— Exercise *7.3* Following L. V. Kapitansky and I. N. Kostin [6], for $\lambda \to \lambda_0$ define the upper limit $A(\lambda_0; \Lambda)$ of the attractors A^λ along Λ by the equality

$$A(\lambda_0, \Lambda) = \bigcap_{\delta > 0} \left[\bigcup \{ A^\lambda : \lambda \in \Lambda, \quad 0 < \operatorname{dist}(\lambda, \lambda_0) < \delta \} \right],$$

where $[\,\cdot\,]$ denotes the closure operation. Prove that if the hypotheses of Theorem 7.2 hold, then $A(\lambda_0, \Lambda)$ is a nonempty compact invariant set lying in the attractor A^{λ_0}.

Theorem 7.2 embraces only the upper semicontinuity of the family of attractors $\{A^\lambda\}$. In order to prove their continuity (in the Hausdorff metric defined by equation (5.4)), additional conditions should be imposed on the family of dynamical systems (X, S_t^λ). For example, the following assertion proved by A. V. Babin and M. I. Vishik concerning the power estimate of the deviation of the attractors A^λ and A^{λ_0} in the Hausdorff metric holds.

Theorem 7.3.

Assume that a dynamical system (X, S_t^λ) ***possesses a global attractor*** A^λ ***for every*** $\lambda \in \Lambda$. ***Let the following conditions hold:***

(a) ***there exists a bounded set*** $B_0 \subset X$ ***such that*** $A^\lambda \subset B_0$ ***for all*** $\lambda \in \Lambda$ ***and***

$$h(S_t^\lambda B_0, A^\lambda) \le C_0 e^{-\eta t}, \qquad \lambda \in \Lambda, \tag{7.3}$$

with constants $C_0 > 0$ ***and*** $\eta > 0$ ***independent of*** λ ***and with***

$$h(B, A) = \sup\{ \operatorname{dist}(b, A) : b \in B \};$$

(b) ***for any*** $\lambda_1, \lambda_2 \in \Lambda$ ***and*** $u_1, u_2 \in B_0$ ***the estimate***

$$\operatorname{dist}(S_t^{\lambda_1} u_1, S_t^{\lambda_2} u_2) \le C_1 e^{\alpha t} (\operatorname{dist}(u_1, u_2) + \operatorname{dist}(\lambda_1, \lambda_2)) \tag{7.4}$$

***holds, with constants** C_1 **and** α **independent of** λ.*

Then there exists** $C_2 > 0$ **such that

$$\rho(A^{\lambda_1}, A^{\lambda_2}) \le C_2 [\operatorname{dist}(\lambda_1, \lambda_2)]^q, \qquad q = \frac{\eta}{\eta + \alpha}. \tag{7.5}$$

Here** $\rho(\cdot\,,\cdot)$ **is the Hausdorff metric defined by the formula

$$\rho(B, A) = \max\{h(B, A);\;\; h(A, B)\}.$$

Proof.

By virtue of the symmetry of (7.5) it is sufficient to find out that

$$h(A^{\lambda_1}, A^{\lambda_2}) \le C_2 [\operatorname{dist}(\lambda_1, \lambda_2)]^q. \tag{7.6}$$

Equation (7.3) implies that for any $\varepsilon > 0$

$$S_t^{\lambda} B_0 \subset \mathcal{O}_\varepsilon(A^{\lambda}) \quad \text{for all} \quad \lambda \in \Lambda \tag{7.7}$$

when $t \ge t^*(\varepsilon, C_0) \equiv \eta^{-1}(\ln 1/\varepsilon + \ln C_0)$. Here $\mathcal{O}_\varepsilon(A^{\lambda})$ is an ε-vicinity of the set A^{λ}. It follows from equation (7.4) that

$$h(S_t^{\lambda_1} B_0, S_t^{\lambda_2} B_0) = \sup_{x \in B_0} \inf_{y \in B_0} \operatorname{dist}(S_t^{\lambda_1} x, S_t^{\lambda_2} y) \le$$

$$\le \sup_{x \in B_0} \operatorname{dist}(S_t^{\lambda_1} x, S_t^{\lambda_2} x) \le C_1 e^{\alpha t} \operatorname{dist}(\lambda_1, \lambda_2). \tag{7.8}$$

Since $A^{\lambda} \subset B_0$, we have $A^{\lambda} = S_t^{\lambda} A^{\lambda} \subset S_t^{\lambda} B_0$. Therefore, with $t \ge t^*(\varepsilon, C_0)$, equation (7.7) gives us that

$$A^{\lambda} \subset S_t^{\lambda} B_0 \subset \mathcal{O}_\varepsilon(A^{\lambda}). \tag{7.9}$$

For any $x, z \in X$ the estimate

$$\operatorname{dist}(x, A^{\lambda}) \le \operatorname{dist}(x, z) + \operatorname{dist}(z, A^{\lambda})$$

holds. Hence, we can find that

$$\operatorname{dist}(x, A^{\lambda}) \le \operatorname{dist}(x, z) + \varepsilon$$

for all $x \in X$ and $z \in \mathcal{O}_\varepsilon(A^{\lambda})$. Consequently, equation (7.9) implies that

$$\operatorname{dist}(x, A^{\lambda}) \le \operatorname{dist}(x, S_t^{\lambda} B_0) + \varepsilon, \qquad x \in X$$

for $t \ge t^*(\varepsilon, C_0)$. It means that

$$h(A^{\lambda_1}, A^{\lambda_2}) = \sup_{x \in A^{\lambda_1}} \operatorname{dist}(x, A^{\lambda_2}) \le$$

$$\le \sup_{x \in A^{\lambda_1}} \operatorname{dist}(x, S_t^{\lambda_2} B_0) + \varepsilon \le h(S_t^{\lambda_1} B_0, S_t^{\lambda_2} B_0) + \varepsilon .$$

Thus, equation (7.8) gives us that for any $\varepsilon > 0$

$$h(A^{\lambda_1}, A^{\lambda_2}) \le C_1 e^{\alpha t}\,\mathrm{dist}(\lambda_1, \lambda_2) + \varepsilon$$

for $t \ge t^*(\varepsilon, C_0)$. By taking $\varepsilon = [\mathrm{dist}(\lambda_1, \lambda_2)]^q$, $q = \dfrac{\eta}{\eta+\alpha}$ and $t = t^*(\varepsilon, C_0) \equiv$ $\equiv \eta^{-1}(\ln 1/\varepsilon + \ln C_0)$ in this formula we find estimate (7.6). **Theorem 7.3 is proved**.

It should be noted that condition (7.3) in Theorem 7.3 is quite strong. It can be verified only for a definite class of systems possessing the Lyapunov function (see Theorem 6.3).

In the theory of dynamical systems an important role is also played by the notion of the Poisson stability. A trajectory $\gamma = \{u(t):\ -\infty < t < \infty\}$ of a dynamical system (X, S_t) is said to be ***Poisson stable*** if it belongs to its ω-limit set $\omega(\gamma)$. It is clear that stationary points and periodic trajectories of the system are Poisson stable.

— Exercise *7.4* Show that any Poisson stable trajectory is contained in the global minimal attractor if the latter exists.

— Exercise *7.5* A trajectory γ is Poisson stable if and only if any point x of this trajectory is recurrent, i.e. for any vicinity $\mathcal{O} \ni x$ there exists $t > 0$ such that $S_t x \in \mathcal{O}$.

The following exercise testifies to the fact that not only periodic (and stationary) trajectories can be Poisson stable.

— Exercise *7.6* Let $C_b(\mathbb{R})$ be a Banach space of continuous functions bounded on the real axis. Let us consider a dynamical system $(C_b(\mathbb{R}), S_t)$ with the evolutionary operator defined by the formula

$$(S_t f)(x) = f(x+t), \qquad f(x) \in C_b(\mathbb{R}).$$

Show that the element $f_0(x) = \sin\omega_1 x + \sin\omega_2 x$ is recurrent for any real ω_1 and ω_2 (in particular, when ω_1/ω_2 is an irrational number). Therewith the trajectory $\gamma = \{f_0(x+t):\ -\infty < t < \infty\}$ is Poisson stable.

In conclusion to this section we consider a theorem that is traditionally associated with the stability theory. Sometimes this theorem enables us to significantly decrease the dimension of the phase space, this fact being very important for the study of infinite-dimensional systems.

Theorem 7.4. (reduction principle).

Assume that in a dissipative dynamical system (X, S_t) there exists a positively invariant locally compact set M possessing the property of uniform attraction, i.e. for any bounded set $B \subset X$ the equation

$$\lim_{t \to \infty} \sup_{y \in B} \operatorname{dist}(S_t y, M) = 0 \tag{7.10}$$

holds. Let A ***be a global attractor of the dynamical system*** (M, S_t). ***Then*** A ***is also a global attractor of*** (X, S_t).

Proof.

It is sufficient to verify that

$$\lim_{t \to \infty} \sup_{y \in B} \operatorname{dist}(S_t y, A) = 0 \tag{7.11}$$

for any bounded set $B \subset X$. Assume that there exists a set B such that (7.11) does not hold. Then there exist sequences $\{y_n\} \subset B$ and $\{t_n : t_n \to \infty\}$ such that

$$\operatorname{dist}(S_{t_n} y_n, A) \ge \delta \tag{7.12}$$

for some $\delta > 0$. Let B_0 be a bounded absorbing set of (X, S_t). We choose a moment t_0 such that

$$\sup\left\{\operatorname{dist}(S_{t_0} y, A) : y \in M \cap B_0\right\} \le \frac{\delta}{2}. \tag{7.13}$$

This choice is possible because A is a global attractor of (M, S_t). Equation (7.10) implies that

$$\operatorname{dist}(S_{t_n - t_0} y_n, M) \to 0, \qquad t_n \to \infty .$$

The dissipativity property of (X, S_t) gives us that $S_{t_n - t_0} y_n \in B_0$ when n is large enough. Therefore, local compactness of the set M guarantees the existence of an element $z \in M \cap B_0$ and a subsequence $\{n_k\}$ such that

$$z = \lim_{k \to \infty} S_{t_{n_k} - t_0} y_{n_k} .$$

This implies that $S_{t_{n_k}} y_{n_k} \to S_{t_0} z$. Therefore, equation (7.12) gives us that $\operatorname{dist}(S_{t_0} z, A) \ge \delta$. By virtue of the fact that $z \in M \cap B_0$ this contradicts equation (7.13). **Theorem 7.4 is proved**.

E x a m p l e *7.1*

We consider a system of ordinary differential equations

$$\begin{cases} \dot{y} + y^3 - y = y z^2, & y|_{t=0} = y_0, \\ \dot{z} + z(1 + y^2) = 0, & z|_{t=0} = z_0 . \end{cases} \tag{7.14}$$

It is obvious that for any initial condition (y_0, z_0) problem (7.14) is uniquely solvable over some interval $(0, t^*(y_0, z_0))$. If we multiply the first equation by y and the second equation by z and if we sum the results obtained, then we get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(y^2+z^2)+y^4-y^2+z^2=0\,, \qquad t\in(0,\ t^*(y_0,\ z_0))\,.$$

This implies that the function $V(y,\ z)=y^2+z^2$ possesses the property

$$\frac{\mathrm{d}}{\mathrm{d}t}V(y(t),\ z(t))+2\,V(y(t),\ z(t))\le 2\,, \qquad t\in[0,\ t^*(y_0,\ z_0))\,.$$

Therefore,

$$V(y(t),\ z(t))\ \le\ V(y_0,\ z_0)e^{-2t}+1\,, \qquad t\in[0,\ t^*(y_0,\ z_0))\,.$$

This implies that any solution to problem (7.14) can be extended to the whole semiaxis $\mathbb{R}_+$ and the dynamical system $(\mathbb{R}^2,\ S_t)$ generated by equation (7.14) is dissipative. Obviously, the set $M=\{(y,\ 0)\colon\ y\in\mathbb{R}\}$ is positively invariant. Therewith the second equation in (7.14) implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}z^2+z^2\le 0\,, \qquad t>0\,.$$

Hence, $|z(t)|^2\le z_0^2e^{-2t}$. Thus, the set M exponentially attracts all the bounded sets in $\mathbb{R}^2$. Consequently, Theorem 7.4 gives us that the global attractor of the dynamical system $(M,\ S_t)$ is also the attractor of the system $(\mathbb{R}^2,\ S_t)$. But on the set M system of equations (7.14) is reduced to the differential equation

$$\dot{y}+y^3-y=0, \qquad y\big|_{t=0}=y_0\,. \tag{7.15}$$

Thus, the global attractors of the dynamical systems generated by equations (7.14) and (7.15) coincide. Therewith the study of dynamics on the plane is reduced to the investigation of the properties of the one-dimensional dynamical system.

Exercise *7.7* Show that the global attractor A of the dynamical system $(\mathbb{R}^2,\ S_t)$ generated by equations (7.14) has the form

$$A=\{(y,\ z)\colon\ -1\le y\le 1,\quad z=0\}\,.$$

Figure the qualitative behaviour of the trajectories on the plane.

Exercise *7.8* Consider the system of ordinary differential equations

$$\begin{cases}\dot{y}-y^5+y^3(1+2z)-y(1+z^2)=0\\[2ex] \dot{z}+z(1+4y^4)-2y^2(z^2+y^4-y^2+3/2)=0\,.\end{cases} \tag{7.16}$$

Show that these equations generate a dissipative dynamical system in $\mathbb{R}^2$. Verify that the set $M=\{(y,\ z)\colon\ z=y^2,\ y\in\mathbb{R}\}$ is invariant and exponentially attracting. Using Theorem 7.4, prove that the global attractor A of problem (7.16) has the form

$$A=\{(y,\ z)\colon\ z=y^2,\quad -1\le y\le 1\}\,.$$

Hint: Consider the variable $w=z-y^2$ instead of the variable z.

§ 8 Finite Dimensionality of Invariant Sets

Finite dimensionality is an important property of the global attractor which can be established in many situations interesting for applications. There are several approaches to the proof of this property. The simplest of them seems to be the one based on Ladyzhenskaya's theorem on the finite dimensionality of the invariant set. However, it should be kept in mind that the estimates of dimension based on Ladyzhenskaya's theorem usually turn out to be too overstated. Stronger estimates can be obtained on the basis of the approaches developed in the books by A. V. Babin and M. I. Vishik, and by R. Temam (see the references at the end of the chapter).

Let M be a compact set in a metric space X. Then its ***fractal dimension*** is defined by

$$\dim_f M = \overline{\lim_{\varepsilon \to 0}} \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)} ,$$

where $n(M, \varepsilon)$ is the minimal number of closed balls of the radius ε which cover the set M.

Let us illustrate this definition with the following examples.

Example *8.1*

Let M be a segment of the length l. It is evident that

$$\frac{l}{2\varepsilon} - 1 \le n(M, \varepsilon) \le \frac{l}{2\varepsilon} + 1 .$$

Therefore,

$$\ln\frac{1}{\varepsilon} + \ln\frac{l-2\varepsilon}{2} \le \ln n(M, \varepsilon) \le \ln\frac{1}{\varepsilon} + \ln\frac{l+2\varepsilon}{2} .$$

Hence, $\dim_f M = 1$, i.e. the fractal dimension coincides with the value of the standard geometric dimension.

Example *8.2*

Let M be the Cantor set obtained from the segment $[0, 1]$ by the sequential removal of the centre thirds. First we remove all the points between $1/3$ and $2/3$. Then the centre thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the two remaining segments $[0, 1/3]$ and $[2/3, 1]$ are deleted. After that the centre parts $(1/27, 2/27)$, $(7/27, 8/27)$, $(19/27, 20/27)$ and $(25/27, 26/27)$ of the four remaining segments $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$ and $[8/9, 1]$, respectively, are deleted. If we continue this process to infinity, we obtain the Cantor set M. Let us calculate its fractal dimension. First of all it should be noted that

$$M = \bigcap_{k=0}^{\infty} J_k ,$$

$$J_0 = [0,\ 1], \quad J_1 = [0,\ 1/3] \cup [2/3,\ 1],$$

$$J_2 = [0,\ 1/9] \cup [2/9,\ 1/3] \cup [2/3,\ 7/9] \cup [8/9,\ 1]$$

and so on. Each set J_k can be considered as a union of 2^k segments of the length 3^{-k}. In particular, the cardinality of the covering of the set M with the segment of the length 3^{-k} equals to 2^k. Therefore,

$$\dim_f M = \lim_{k \to \infty} \frac{\ln 2^k}{\ln(2 \cdot 3^k)} = \frac{\ln 2}{\ln 3} .$$

Thus, the fractal dimension of the Cantor set is not an integer (if a set possesses this property, it is called fractal).

It should be noted that the fractal dimension is often referred to as the metric order of a compact. This notion was first introduced by L. S. Pontryagin and L. G. Shnirelman in 1932. It can be shown that any compact set with the finite fractal dimension is homeomorphic to a subset of the space $\mathbb{R}^d$ when $d > 0$ is large enough.

To obtain the estimates of the fractal dimension the following simple assertion is useful.

Lemma 8.1.

The following equality holds:

$$\dim_f M = \overline{\lim_{\varepsilon \to 0}} \frac{\ln N(M,\ \varepsilon)}{\ln(1/\varepsilon)} ,$$

where $N(M,\ \varepsilon)$ is the cardinality of the minimal covering of the compact M with closed sets diameter of which does not exceed 2ε (the diameter of a set X is defined by the value $d(X) = \sup\{\|x-y\| :\ x,\ y \in X\}$).

Proof.

It is evident that $N(M,\ \varepsilon) \le n(M,\ \varepsilon)$. Since any set of the diameter d lies in a ball of the radius d, we have that $n(M,\ 2\varepsilon) \le N(M,\ \varepsilon)$. These two inequalities provide us with the assertion of the lemma.

All the sets are expected to be compact in Exercises 8.1–8.4 given below.

— Exercise *8.1* Prove that if $M_1 \subseteq M_2$, then $\dim_f M_1 \le \dim_f M_2$.

— Exercise *8.2* Verify that $\dim_f (M_1 \cup M_2) \le \max\{\dim_f M_1;\ \dim_f M_2\}$.

— Exercise *8.3* Assume that $M_1 \times M_2$ is a direct product of two sets. Then

$$\dim_f (M_1 \times M_2) \le \dim_f M_1 + \dim_f M_2 .$$

— Exercise *8.4* Let g be a Lipschitzian mapping of one metric space into another. Then $\dim_f g(M) \le \dim_f M$.

The notion of the dimension by Hausdorff is frequently used in the theory of dynamical systems along with the fractal dimension. This notion can be defined as follows. Let M be a compact set in X. For positive d and ε we introduce the value

$$\mu(M,\ d,\ \varepsilon) = \inf \sum (r_j)^d ,$$

where the infimum is taken over all the coverings of the set M with the balls of the radius $r_j \le \varepsilon$. It is evident that $\mu(M,\ d,\ \varepsilon)$ is a monotone function with respect to ε. Therefore, there exists

$$\mu(M,\ d) = \lim_{\varepsilon \to 0} \mu(M,\ d,\ \varepsilon) = \sup_{\varepsilon > 0} \mu(M,\ d,\ \varepsilon) .$$

The ***Hausdorff dimension*** of the set M is defined by the value

$$\dim_H M = \inf\{d:\ \mu(M,\ d) = 0\}.$$

— Exercise *8.5* Show that the Hausdorff dimension does not exceed the fractal one.

— Exercise *8.6* Show that the fractal dimension coincides with the Hausdorff one in Example 8.1, the same is true for Example 8.2.

— Exercise *8.7* Assume that $M = \{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$, where a_n monotonically tends to zero. Prove that $\dim_H M = 0$ (*Hint*: $\mu(M,\ d,\ \varepsilon) \le \le a_{n+1}^d + n\,2^{-dn}$ when $a_{n+1} \le \varepsilon \le a_n$).

— Exercise *8.8* Let $M = \{1/n\}_{n=1}^{\infty} \subset \mathbb{R}$. Show that $\dim_f M = 1/2$. $\left(\text{Hint: } n < n(M,\ \varepsilon) < n+1+\dfrac{1}{(n+1)\varepsilon} \quad \text{when} \quad \dfrac{1}{(n+1)(n+2)} \le \varepsilon < < \dfrac{1}{n(n+1)}\right).$

— Exercise *8.9* Let $M = \left\{\dfrac{1}{\ln n}\right\}_{n=2}^{\infty} \subset \mathbb{R}$. Prove that $\dim_f M = 1$.

— Exercise *8.10* Find the fractal and Hausdorff dimensions of the global minimal attractor of the dynamical system in $\mathbb{R}$ generated by the differential equation

$$\dot{y} + y \sin\frac{1}{|y|} = 0 .$$

The facts presented in Exercises 8.7–8.9 show that the notions of the fractal and Hausdorff dimensions do not coincide. The result of Exercise 8.5 enables us to restrict ourselves to the estimates of the fractal dimension when proving the finite dimensionality of a set.

The main assertion of this section is the following variant of Ladyzhenskaya's theorem. It will be used below in the proof of the finite dimensionality of global attractors of a number of infinite-dimensional systems generated by partial differential equations.

Theorem 8.1.

Assume that M is a compact set in a Hilbert space H. Let V be a continuous mapping in H such that $V(M) \supset M$. Assume that there exists a finite-dimensional projector P in the space H such that

$$\|P(Vv_1 - Vv_2)\| \le l\,\|v_1 - v_2\|, \qquad v_1,\ v_2 \in M, \tag{8.1}$$

$$\|(1-P)(Vv_1 - Vv_2)\| \le \delta\,\|v_1 - v_2\|, \qquad v_1,\ v_2 \in M, \tag{8.2}$$

where $\delta < 1$. We also assume that $l \ge 1-\delta$. Then the compact M possesses a finite fractal dimension and

$$\dim_f M \le \dim P \cdot \ln\frac{9l}{1-\delta} \cdot \left[\ln\frac{2}{1+\delta}\right]^{-1}. \tag{8.3}$$

We remind that a projector in a space H is defined as a bounded operator P with the property $P^2 = P$. A projector P is said to be finite-dimensional if the image PH is a finite-dimensional subspace. The dimension of a projector P is defined as a number $\dim P \equiv \dim PH$.

The following lemmata are used in the proof of Theorem 8.1.

Lemma 8.2.

Let B_R be a ball of the radius R in $\mathbb{R}^d$. Then

$$N(B_R,\ \varepsilon) \le n(B_R,\ \varepsilon) \le \left(1 + \frac{2R}{\varepsilon}\right)^d. \tag{8.4}$$

Proof.

Estimate (8.4) is self-evident when $\varepsilon \ge R$. Assume that $\varepsilon < R$. Let $\{\xi_1, \ldots, \xi_l\}$ be a maximal set in B_R with the property $|\xi_i - \xi_j| > \varepsilon$, $i \ne j$. By virtue of its maximality for every $x \in B_R$ there exists ξ_i such that $|x - \xi_i| \le \varepsilon$. Hence, $n(B_R,\ \varepsilon) \le l$. It is clear that

$$B_{\varepsilon/2}(\xi_i) \subset B_{R+\varepsilon/2}, \qquad B_{\varepsilon/2}(\xi_i) \cap (B_{\varepsilon/2}(\xi_j)) = \varnothing, \qquad i \ne j.$$

Here $B_r(\xi)$ is a ball of the radius r centred at ξ. Therefore,

$$l\,\mathrm{Vol}(B_{\varepsilon/2}) = \sum_{i=1}^{l} \mathrm{Vol}(B_{\varepsilon/2}(\xi_i)) \le \mathrm{Vol}(B_{R+\varepsilon/2}).$$

This implies the assertion of the lemma.

Exercise *8.11* Show that

$$n(B_R,\ \varepsilon) \ge \left(\frac{R}{\varepsilon}\right)^d, \qquad \dim_f B_R = d.$$

Lemma 8.3.

Let $\mathcal{F}$ be a closed subset in H such that equations (8.1) *and* (8.2) *hold for all its elements. Then for any $q > 0$ and $\varepsilon > 0$ the following estimate holds:*

$$N(V\mathcal{F},\ \varepsilon(q+\delta)) \le \left(1+\frac{4l}{q}\right)^n N(\mathcal{F},\ \varepsilon)\ , \tag{8.5}$$

where $n = \dim P$ is the dimension of the projector P.

Proof.

Let $\{\mathcal{F}_i\}$ be a minimal covering of the set $\mathcal{F}$ with its closed subsets the diameter of which does not exceed 2ε. Equation (8.1) implies that in PH there exist balls B_i with radius $2l\varepsilon$ such that $PV\mathcal{F}_i \subset B_i$. By virtue of Lemma 8.1 there exists a covering $\{B_{ij}\}_{j=1}^{N_i}$ of the set $PV\mathcal{F}_i$ with the balls of the diameter $2q\varepsilon$, where $N_i \le (1+(4l/q))^n$. Therefore, the collection

$$\left\{G_{ij} = B_{ij} + (1-P)V\mathcal{F}_i:\ \ i = 1,\ 2,\ \dots,\ N(\mathcal{F},\ \varepsilon),\ \ \ j = 1,\ 2,\ \dots,\ N_i\right\}$$

is a covering of the set $V\mathcal{F}$. Here the sum of two sets A and B is defined by the equality

$$A + B = \{a+b:\ \ a \in A,\ b \in B\}.$$

It is evident that

$$\operatorname{diam} G_{ij} \le \operatorname{diam} B_{ij} + \operatorname{diam}(1-P)V\mathcal{F}_i\,.$$

Equation (8.2) implies that $\operatorname{diam}(1-P)V\mathcal{F}_i \le 2\delta\varepsilon$. Therefore, $\operatorname{diam} G_{ij} \le$ $\le 2(q+\delta)\varepsilon$. Hence, estimate (8.5) is valid. Lemma 8.3 is proved.

Let us return to the proof of Theorem 8.1. Since $M \subset VM$, Lemma 8.3 gives us that

$$N(M,\ \varepsilon(q+\delta)) \le N(M,\ \varepsilon)\cdot\left(1+\frac{4l}{q}\right)^n .$$

It follows that

$$N(M,\ (q+\delta)^m) \le N(M,\ 1)\cdot\left(1+\frac{4l}{q}\right)^{nm}, \qquad m = 1,\ 2,\ \dots\ .$$

We choose q and $m = m(\varepsilon)$ such that

$$\delta + q < 1, \qquad (\delta+q)^m \le \varepsilon\,,$$

where $0 < \varepsilon < 1$. Then

$$N(M,\ \varepsilon)\ \le\ N(M,\ (\delta+q)^m)\ \le\ N(M,\ 1)\cdot\left(1+\frac{4l}{q}\right)^{nm(\varepsilon)} .$$

Consequently,

$$\dim_f M = \varlimsup_{\varepsilon\to 0} \frac{\ln N(M,\ \varepsilon)}{\ln(1/\varepsilon)}\ \le\ n\cdot\ln\left(1+\frac{4l}{q}\right)\cdot \varlimsup_{\varepsilon\to 0} \frac{m(\varepsilon)}{\ln(1/\varepsilon)}\ .$$

Obviously, the choice of $m(\varepsilon)$ can be made to fulfil the condition

$$m(\varepsilon) \le \frac{\ln\varepsilon}{\ln(q+\delta)} + 1 .$$

Thus,

$$\dim_f M \le n \ln\left(1+\frac{4l}{q}\right)\left[\ln\frac{1}{q+\delta}\right]^{-1} .$$

By taking $q = 1/2(1-\delta)$ we obtain estimate (8.3). **Theorem 8.1 is proved**.

Exercise *8.12* Assume that the hypotheses of Theorem 8.1 hold and $l < 1-\delta$. Prove that $\dim_f M = 0$.

Of course, in the proof of Theorem 8.1 a principal role is played by equations (8.1) and (8.2). Roughly speaking, they mean that the mapping V squeezes sets along the space $(1-P)H$ while it does not stretch them too much along PH. Negative invariance of M gives us that $M \subset V^k M$ for all $k = 1, 2, \dots$. Therefore, the set M should be initially squeezed. This property is expressed by the assertion of its finite dimensionality. As to positively invariant sets, their finite dimensionality is not guaranteed by conditions (8.1) and (8.2). However, as the next theorem states, they are attracted to finite-dimensional compacts at an exponential velocity.

Theorem 8.2.

Let V be a continuous mapping defined on a compact set M in a Hilbert space H such that $VM \subset M$. Assume that there exists a finite-dimensional projector P such that equations (8.1) and (8.2) hold with $0 < \delta < 1/2$ and $l + \delta \ge 1$. Then for any $\theta \in (\delta, 1)$ there exists a positively invariant closed set $A_\theta \subset M$ such that

$$\sup\{\operatorname{dist}(V^k y, A_\theta) : y \in M\} \le \theta^k , \qquad k = 1, 2, \dots \tag{8.6}$$

and

$$\dim_f A_\theta \le \dim P \cdot \max\left\{ \frac{\ln\left(1+\frac{4l}{\theta-\delta}\right)}{\ln\frac{1}{\theta}},\ \frac{\ln\left(1+\frac{4l}{q}\right)}{\ln\frac{1}{2(q+\delta)}} \right\} , \tag{8.7}$$

where q is an arbitrary number from the interval $(0, 1/2-\delta)$.

Proof.

The pair (M, V^k) is a discrete dynamical system. Since M is compact, Theorem 5.1 gives us that there exists a global attractor $M_0 = \bigcap_{k \ge 0} V^k M$ with the properties $VM_0 = M_0$ and

$$h(V^k M, M_0) \equiv \sup\{\operatorname{dist}(V^k y, M_0) : y \in M\} \to 0 . \tag{8.8}$$

We construct a set A_θ as an extension of M_0. Let E_j be a maximal set in V^jM possessing the property $\operatorname{dist}(a, b) \ge \theta^j$ for $a, b \in E_j$, $a \ne b$. The existence of such a set follows from the compactness of V^jM. It is obvious that

$$L_j \equiv \operatorname{Card} E_j = N\Big(E_j, \tfrac{1}{3}\theta^j\Big) \le N\Big(V^jM, \tfrac{1}{3}\theta^j\Big).$$

Lemma 8.3 with $\mathcal{F} = M$, $q = \theta - \delta$, and $\varepsilon = (1/3)\,\theta^{j-1}$ gives us that

$$N\Big(V^jM, \tfrac{1}{3}\theta^j\Big) \le \left(1 + \frac{4l}{\theta - \delta}\right)^n N\Big(V^{j-1}M, \tfrac{1}{3}\theta^{j-1}\Big)$$

with $\theta > \delta$. Hereinafter $n = \dim P$. Therefore,

$$L_j \equiv \operatorname{Card} E_j \le \left(1 + \frac{4l}{\theta - \delta}\right)^{nj} N\Big(M, \tfrac{1}{3}\Big), \quad \theta > \delta. \tag{8.9}$$

Let us prove that the set

$$A_\theta = M_0 \cup \{\cup\{V^kE_j : j = 1, 2, \ldots, \quad k = 0, 1, 2, \ldots\}\} \tag{8.10}$$

possesses the properties required. It is evident that $VA_\theta \subset A_\theta$. Since $V^kE_j \subset \subset V^{k+j}M$, by virtue of (8.8) all the limit points of the set

$$\bigcup\{V^kE_j : j = 1, 2, \ldots, \quad k = 0, 1, 2, \ldots\}$$

lie in M_0. Thus, A_θ is a closed subset in M. The evident inequality

$$h(V^kM, A_\theta) \le h(V^kM, E_k) \le \theta^k \tag{8.11}$$

implies (8.6). Here and below $h(X, Y) = \sup\{\operatorname{dist}(x, Y) : x \in X\}$. Let us prove (8.7). It is clear that

$$A_\theta = VA_\theta \cup \{\cup\{E_j : j = 1, 2, \ldots\}\}. \tag{8.12}$$

Let $\{F_i\}$ be a minimal covering of the set A_θ with the closed sets the diameter of which is not greater than 2ε. By virtue of Lemma 8.3 there exists a covering $\{G_i\}$ of the set VA_θ with closed subsets of the diameter $2\varepsilon(q + \delta)$. The cardinality of this covering can be estimated as follows

$$N(\varepsilon, q, \delta) \equiv N(VA_\theta, \varepsilon(q + \delta)) \le \left(1 + 4\frac{l}{q}\right)^n N(A_\theta, \varepsilon). \tag{8.13}$$

Using the covering $\{G_i\}$, we can construct a covering of the same cardinality of the set VA_θ with the balls $B(x_i, 2\varepsilon(q + \delta))$ of the radius $2\varepsilon(q + \delta)$ centered at the points x_i, $i = 1, 2, \ldots, N(\varepsilon, q, \delta)$. We increase the radius of every ball up to the value $2\varepsilon(q + \delta + \gamma)$. The parameter $\gamma > 0$ will be chosen below. Thus, we consider the covering

$$\{B(x_i, 2\varepsilon(q + \delta + \gamma)), \quad i = 1, 2, \ldots, N(\varepsilon, q, \delta)\}$$

of the set VA_θ. It is evident that every point $x \in VA_\theta$ belongs to this covering together with the ball $B(x, 2\gamma\varepsilon)$. If $j \ge 2$, the inequalities

$$h(E_j,\ VA_\theta)\ \le\ h(V^jM,\ VA_\theta)\ \le\ h(V^jM,\ VE_{j-1})$$

hold. By virtue of equation (8.11) with the help of (8.1) and (8.2) we have that

$$h(V^jM,\ VE_{j-1})\ \le\ (l+\delta)h(V^{j-1}M,\ E_{j-1})\ \le\ (l+\delta)\,\theta^{j-1}\,.$$

Therefore, $h(E_j,\ VA_\theta)\le 2\gamma\varepsilon$, provided $2\gamma\varepsilon\le(l+\delta)\,\theta^{j-1}$, i.e. if

$$j\ge j_0\equiv 2+\left[\frac{\ln\dfrac{l+\delta}{2\gamma\varepsilon}}{\ln\dfrac{1}{\theta}}\right].$$

Here $[z]$ is an integer part of the number z. Consequently,

$$VA_\theta\cup\left\{\bigcup_{j\ge j_0}E_j\right\}\subset\bigcup_{i=1}^{N(\varepsilon,\ q,\ \delta)}B(x_i,\ 2\varepsilon(q+\delta+\gamma))\,.$$

Therefore, equation (8.12) gives us that

$$N(A_\theta,\ \varepsilon\xi)\le N(\varepsilon,\ q,\ \delta)+\sum_{j=0}^{j_0-1}\mathrm{Card}\,E_j\,,$$

where $\xi=2(q+\delta+\gamma)$. Using (8.9) and (8.13) we find that

$$N(A_\theta,\ \varepsilon\xi)\le\ \eta N(A_\theta,\ \varepsilon)+N\Big(M,\ \frac{1}{3}\Big)\sum_{j=1}^{j_0-1}\Big(1+\frac{4l}{\theta-\delta}\Big)^{nj}$$

for $\theta>\delta$. Here and further $\eta=\Big(1+\dfrac{4l}{q}\Big)^n$. Since

$$j_0\le\frac{\ln\dfrac{1}{\varepsilon}}{\ln\dfrac{1}{\theta}}+C(l,\ \delta,\ \gamma,\ \theta),$$

it is easy to find that

$$\sum_{j=0}^{j_0-1}\Big(1+\frac{4l}{\theta-\delta}\Big)^{nj}\le\beta\cdot\Big(\frac{1}{\varepsilon}\Big)^{\alpha}\quad\text{for}\quad\alpha=\frac{n}{\ln(1/\theta)}\cdot\ln\Big(1+\frac{4l}{\theta-\delta}\Big),$$

where the constant $\beta>0$ does not depend on ε (its value is unessential further). Therefore,

$$N(A_\theta,\ \varepsilon\xi)\le\ \eta N(A_\theta,\ \varepsilon\xi)+\beta\varepsilon^{-\alpha}\,.$$

If we take $\varepsilon=\xi^{m-1}$, then after iterations we get

$$N(A_\theta,\ \xi^m)\le\ \eta N(A_\theta,\ \xi^{m-1})+\beta\xi^{-(m-1)\alpha}\le$$

$$\le\ \eta^m N(A_\theta,\ 1)+\beta\,\eta^{m-1}\sum_{i=0}^{m-1}(\xi^\alpha\,\eta)^{-i}\,.$$

Let us fix $\delta \in (0,\ 1/2)$, $\theta \in (\delta,\ 1)$ and $q \in (0,\ 1/2-\delta)$ and choose $\gamma > 0$ such that $\xi = 2(q+\delta+\gamma) < 1$ and $\xi^{\alpha}\eta \neq 1$. Then summarizing the geometric progression we obtain

$$N(A_\theta,\ \xi^m) \le \eta^m N(A_\theta,\ 1) + \beta\frac{\xi^{-\alpha m} - \eta^m}{\xi^{-\alpha} - \eta} \le$$

$$\le \eta^m\left(N(A_\theta,\ 1) + \frac{\beta}{|\xi^{-\alpha} - \eta|}\right) + \frac{\beta}{|\xi^{-\alpha} - \eta|}\xi^{-\alpha m} \quad . \tag{8.14}$$

Let $\varepsilon > 0$ be small enough and

$$m(\varepsilon) = 1 + \left[\frac{\ln(1/\varepsilon)}{\ln(1/\xi)}\right],$$

where, as mentioned above, $[z]$ is an integer part of the number z. Since $\varepsilon \le \xi^{m(\varepsilon)}$, equation (8.14) gives us that

$$N(A_\theta,\ \varepsilon) \le N(A_\theta,\ \xi^m) \le a_1\left(1+\frac{4l}{q}\right)^{n\,m(\varepsilon)} + a_2\left(\frac{1}{\xi^{\alpha}}\right)^{m(\varepsilon)},$$

where a_1 and a_2 are positive numbers which do not depend on ε. Therefore,

$$\dim_f A_\theta = \overline{\lim_{\varepsilon\to 0}}\frac{\ln N(A_\theta,\ \varepsilon)}{\ln\frac{1}{\varepsilon}} \le$$

$$\le \overline{\lim_{\varepsilon\to 0}}\frac{m(\varepsilon)}{\ln\frac{1}{\varepsilon}}\ \lim_{m\to\infty}\ \frac{1}{m}\left\{a_1\left(1+\frac{4e}{q}\right)^{nm} + a_2\left(\frac{1}{\xi^{\alpha}}\right)^{m}\right\}.$$

Simple calculations give us that

$$\dim_f A_\theta \le \frac{1}{\ln\frac{1}{\xi}}\cdot\ln\left\{\max\left(\left(1+\frac{4e}{q}\right)^n,\ \xi^{-\alpha}\right)\right\}.$$

This easily implies estimate (8.7). Thus, **Theorem 8.2 is proved**.

Exercise *8.13* Show that for $\delta < \theta < 1/2$ formula (8.7) for the dimension of the set A_θ can be rewritten in the form

$$\dim_f A_\theta = \dim P\cdot\frac{\ln\left(1+\frac{4l}{\theta-\delta}\right)}{\ln\frac{1}{2\theta}}\ . \tag{8.15}$$

If the hypotheses of Theorem 8.2 hold, then the discrete dynamical system $(M,\ V^k)$ possesses a finite-dimensional global attractor M_0. This attractor uniformly attracts all the trajectories of the system. Unfortunately, the speed of its convergence to the attractor cannot be estimated in general. This speed can appear to be small. However, Theorem 8.2 implies that the global attractor is contained in a finite-dimensional positively invariant set possessing the property of uniform exponential attraction. From the applied point of view the most interesting corollary of this fact is that the dynamics of a system becomes ***finite-dimensional*** exponentially fast independent of the speed of convergence of the trajectories to the global attractor. Moreover, the reduction principle (see Theorem 7.4) is applicable in this case. Thus, finite-dimensional invariant exponentially attracting sets can be used to describe the qualitative behaviour of infinite-dimensional systems. These sets are frequently referred to as ***inertial sets***, or ***fractal exponential attractors***. In some cases they turn out to be surfaces in the phase space. In contrast with the global attractor, the inertial set of a dynamical system can not be uniquely determined. The construction in the proof of Theorem 8.2 shows it.

§ 9 Existence and Properties of Attractors of a Class of Infinite-Dimensional Dissipative Systems

The considerations given in the previous sections are mainly of general character. They are related to a dissipative dynamical system of the generic structure. Therewith, we inevitably make additional assumptions on the behaviour of trajectories of these systems (e.g., the asymptotic compactness, the existence of a Lyapunov function, the squeezing property along a subspace, etc.). Thereby it is natural to ask what properties of the original objects of a particular dynamical system guarantee the fulfilment of the assumptions mentioned above. In this section we discuss this question in terms of the dynamical system generated by a differential equation of the form

$$\frac{d}{dt} y + A y = B(y), \qquad y\big|_{t=0} = y_0 \tag{9.1}$$

in a separable Hilbert space $\mathscr{H}$, where A is a linear operator and B is a nonlinear mapping which is coordinated with A in some sense. Our main goal is to demonstrate the *generic line of arguments* as well as to describe those properties of the operators A and B which provide the applicability of general theorems proved in the previous sections. The main attention is paid to the questions of existence and finite dimensionality of a global attractor. Nowadays the presented line of arguments (or a modification of it) is one of the main components of a great number of works on global attractors.

It is assumed below that the following conditions are fulfilled.

(A) There exists a strongly continuous semigroup S_t of continuous mappings in $\mathcal{H}$ such that $y(t) = S_t y_0$ is a solution to problem (9.1) in the sense that the following identity holds:

$$S_t y_0 = T_t y_0 + G(t, y_0) \equiv T_t y_0 + \int_0^t T_{t-\tau} B(S_\tau y_0) \, d\tau, \tag{9.2}$$

where $T_t = \exp(-At)$ (see condition **(B)** below). The semigroup S_t is dissipative, i.e. there exists $R > 0$ such that for any B from the collection $\mathcal{B}(\mathcal{H})$ of all bounded subsets of the space $\mathcal{H}$ the estimate $\|S_t y\| < R$ holds when $y \in B$ and $t \ge t_0(B)$. We also assume that the set $\gamma^+(B) = \bigcup_{t \ge 0} S_t B$ is bounded for any $B \in \mathcal{B}(\mathcal{H})$.

(B) The linear closed operator A generates a semigroup $T_t = \exp(-At)$ which admits the estimate $\|T_t\| \le L_1 \exp(\omega t)$ (L_1 and ω are some constants). There exists a sequence of finite-dimensional projectors $\{P_n\}$ which strongly converges to the identity operator such that

1) A commutes with P_n, i.e. $P_n A \subset A P_n$ for any n;
2) there exists n_0 such that $\|T_t(1-P_n)\| \le L_2 \exp(-\varepsilon t)$ for $n > n_0$, where ε, $L_2 > 0$;
3) $r_n = \|A^{-1}(1-P_n)\| \to 0$ as $n \to \infty$.

(C) For any $R' > 0$ the nonlinear operator $B(u)$ possesses the properties:

1) $\|B(u_1) - B(u_2)\| \le C_1(R')\|u_1 - u_2\|$ if $\|u_i\| \le R'$, $i = 1, 2$;
2) for $n \ge n_0$, $\|u_i\| \le R'$, $i = 1, 2$, and for some $\sigma > 0$ the following equations hold:

$$\|A^\sigma(1-P_n)B(u_1)\| \le C_2(R'),$$

$$\|A^\sigma(1-P_n)(B(u_1) - B(u_2))\| \le C_3(R')\|u_1 - u_2\|$$

(the existence of the operator $A^\sigma(1-P_n)$ follows from **(B2)**).

It should be noted that although conditions **(A)–(C)** seem a little too lengthy, they are valid for a class of problems of the theory of nonlinear oscillations as well as for a number of systems generated by parabolic partial differential equations.

The following assertion should be mainly interpreted as a principal result which testifies to the fact that the asymptotic behaviour of the system is determined by a finite set of parameters.

Theorem 9.1.

If conditions **(A)–(C)** *are fulfilled, then the semigroup S_t possesses a compact global attractor $\mathcal{A}$. The attractor has a finite fractal dimension which can be estimated as follows:*

$$\dim_f \mathcal{A} = a_1(1 + \ln\|P_n\| L_1 L_2)(1 + D(R)\varepsilon^{-1}) \dim P_n, \tag{9.3}$$

where $D(R) = \omega + L_1 C_1(R)$ ***and*** $n \geq n_0$ ***is determined from the condition***

$$r_n^{\sigma} \leq a_2 D(R)\left[L_1 L_2 C_3(R)\right]^{-1} \cdot \exp\{-a_3 D(R)(1+\ln L_2)\varepsilon^{-1}\}. \tag{9.4}$$

Here a_1, a_2, ***and*** a_3 ***are some absolute constants.***

When proving the theorem, we mainly rely on decomposition (9.2) and the lemmata below.

Lemma 9.1.

Let K_n *be a set of elements which for some* $t \geq 0$ *have the form* $v = u_0 + (1-P_n)G(t, u)$, *where* $u_0 \in P_n \mathcal{H}$, $\|u_0\| \leq c_0 R$ *with the constant* c_0 *determined by the condition* $\|P_n\| \leq c_0$ *for* $n = 1, 2, \ldots$. *Here the value* $G(t, u)$ *is the same as in* (9.2) *with the element* $u \in \mathcal{H}$ *being such that* $\|S_t u\| \leq R$ *for all* $t \geq 0$. *Then the set* K_n *is precompact in* $\mathcal{H}$ *for* $n \geq n_0$.

Proof.

Properties **(B2)** and **(C2)** imply that

$$\|A^{\sigma}(1-P_n)G(t, u)\| \leq L_2 \int_0^t e^{-\varepsilon(t-\tau)} \|A^{\sigma}(1-P_n)B(S_\tau u)\| d\tau \leq \frac{C_2(R)L_2}{\varepsilon}$$

when $\|S_\tau u\| \leq R$ for $\tau \geq 0$. Therefore, the set

$$\{v: \ v = (1-P_n)G(t, u), \ t > 0\}, \tag{9.5}$$

where $\|S_t u\| \leq R$ for all $t \geq 0$, is bounded in the space $D(A^{\sigma} \upharpoonright (1-P_n)\mathcal{H})$ with the norm $\|A^{\sigma} \cdot\|$. The symbol $\upharpoonright$ denotes the restriction of an operator on a subspace. However, property **(B3)** implies that

$$\lim_{m \to \infty} \|P_m A^{-1}(1-P_n) - A^{-1}(1-P_n)\| = 0.$$

Therefore, the operator $A^{-1} \upharpoonright (1-P_n)\mathcal{H}$ is compact. Hence, $D(A^{\sigma} \upharpoonright (1-P_n)\mathcal{H})$ is compactly embedded into $(1-P_n)\mathcal{H}$. It means that the set (9.5) is precompact in $(1-P_n)\mathcal{H}$. This implies the precompactness of K_n.

Lemma 9.2.

There exists a compact set $\overline{K}$ *in the space* $\mathcal{H}$ *such that*

$$h(S_t B, \overline{K}) \equiv \sup\{\operatorname{dist}(S_t y, \overline{K}): \ y \in B\} \leq L_2 R e^{-\varepsilon(t-t_0)} \tag{9.6}$$

for any bounded set $B \subset \mathcal{H}$ *and* $t \geq t_0 \equiv t_0(B)$.

Proof.

Let $u \in B$, where B is a bounded set in $\mathcal{H}$. Then $\|S_t u\| \leq R$ for $t \geq t_0 = t_0(B)$. By virtue of (9.2) we have that

$$S_t u = (1-P_n)T_{t-t_0} S_{t_0} u + W_n(t, t_0, u),$$

where

$$W_n(t, t_0, u) = P_n S_t u + (1-P_n) G(t-t_0, S_{t_0} u).$$

It is evident that $W_n(t, t_0, u) \in K_n$ for $t \ge t_0$. Therefore,

$$\operatorname{dist}(S_t u, K_n) \le \|(1-P_n)T_{t-t_0} S_{t_0} u\| \le L_2 R e^{-\varepsilon(t-t_0)}.$$

This implies (9.6) with $\overline{K} = [K_n]$, where $[K_n]$ is the closure in $\mathscr{H}$ of the set K_n described in Lemma 9.1.

Exercise *9.1* Show that $\overline{K} = [K_n]$ lies in the set

$$K_{n,\sigma} = \{v_1 + v_2 : \ v_1 \in P_n \mathscr{H}, \ v_2 \in (1-P_n)\mathscr{H},$$
$$\|v_1\| \le C_1, \ \|A^\sigma v_2\| \le C_2\}, \tag{9.7}$$

where C_1 and C_2 are some constants.

In particular, Lemma 9.2 means that the system $(\mathscr{H}, S_t)$ is asymptotically compact. Therefore, we can use Theorem 5.1 (see also Exercise 5.3) to guarantee the existence of the global attractor $\mathscr{A}$ lying in $\overline{K} = [K_n]$.

Let us use Theorem 8.1 to prove the finite dimensionality of the attractor. Verification of the hypotheses of the theorem is based on the following assertion.

Lemma 9.3.

Let $\|S_t u_i\| \le R, \quad t \ge 0, \ i = 1, 2$. *Then*

$$\|S_t u_1 - S_t u_2\| \le L_1 \exp(D(R)t)\|u_1 - u_2\| \tag{9.8}$$

and

$$\|(1-P_n)(S_t u_1 - S_t u_2)\| \le L_2 e^{-\varepsilon t} \cdot \left(1 + C_0 r_n^\sigma \frac{L_1 C_3(R)}{D(R)} e^{\alpha t}\right) \|u_1 - u_2\| \tag{9.9}$$

for $n \ge n_0$ *and* $\alpha = \varepsilon + D(R)$.

Proof.

Decomposition (9.2) and condition **(C1)** imply that

$$\|S_t u_1 - S_t u_2\| \le L_1 \left(\|u_1 - u_2\| + C_1(R) \int_0^t e^{-w\tau} \|S_\tau u_1 - S_\tau u_2\| \, d\tau \right) e^{wt}.$$

With the help of Gronwall's lemma we obtain (9.8).

To prove (9.9) it should be kept in mind that decomposition (9.2) and equations **(B2)** and **(C2)** imply that for $n \ge n_0$

$$\|(1-P_n)(S_t u_1 - S_t u_2)\| \le L_2 e^{-\varepsilon t}\Bigg(\|u_1-u_2\| +$$

$$+ C_0 r_n^{\sigma} C_3(R) \int_0^t e^{\varepsilon\tau}\|S_\tau u_1 - S_\tau u_2\| \,d\tau\Bigg). \tag{9.10}$$

Here the inequality $\|A^{-\sigma}(1-P_n)\| \le C_0 r_n^{\sigma}$ is used . If we put (9.8) in the right-hand side of formula (9.10), we obtain estimate (9.9).

The following simple argument completes the proof of Theorem 9.1. Let us fix an arbitrary number $0 < \delta < 1$ and choose t_0 and n such that

$$L_2 e^{-\varepsilon t_0} = \frac{\delta}{2} \quad \text{and} \quad C_0 r_n^{\sigma} L_1 \frac{C_3(R)}{D(R)}\; e^{\alpha t_0} \le 1 .$$

Then the hypotheses of Theorem 8.1 with $M = \mathcal{A}$, $V = S_{t_0}$, $P = P_n$, and $l =$ $= L\|P_n\| \exp(D(R)t_0)$ hold for the attractor $\mathcal{A}$. Hence, it is finite-dimensional with estimate (9.3) holding for its fractal dimension. **Theorem 9.1 is proved**.

Exercise *9.2* Prove that the global attractor $\mathcal{A}$ of problem (9.1) is stable (*Hint*: verify that the hypotheses of Theorem 7.1 hold).

Properties **(A)–(C)** also enable us to prove that the system generated by equation (9.1) possesses an inertial set. A compact set $A_{\exp}$ in the phase space $\mathcal{H}$ is said to be an ***inertial set*** (or a fractal exponential attractor) if it is positively invariant $(S_t A_{\exp} \subset A_{\exp})$, its fractal dimension is finite $(\dim_f A_{\exp} < \infty)$ and it possesses the property

$$h(S_t B,\, A_{\exp}) \equiv \sup\{\operatorname{dist}(S_t y,\, A_{\exp}) :\; y \in B\} \le C_B\, e^{-\gamma(t-t_0)} \tag{9.11}$$

for any bounded set $B \subset \mathcal{H}$ and for $t \ge t_0 \ge t_0(B)$, where C_B and γ are positive numbers. (The importance of this notion for the theory of infinite-dimensional dynamical systems has been discussed at the end of Section 8).

Lemma 9.4.

Assume that properties **(A)–(C)** *hold. Then the dynamical system* $(\mathcal{H}, S_t)$ *generated by equation* (9.1) *possesses the following properties:*

1) *there exist a compact positively invariant set* K *and constants* $C, \gamma > 0$ *such that*

$$\sup\{\operatorname{dist}(S_t y,\, K) :\; y \in B\} \le C e^{-\gamma(t-t_B)} \tag{9.12}$$

for any bounded set B *in* $\mathcal{H}$ *and for* $t \ge t_B > 0$ *;*

2) *there exist a vicinity* $\mathcal{O}$ *of the compact* K *and numbers* Δ_1 *and* $\alpha_1 > 0$ *such that*

$$\|S_t y_1 - S_t y_2\| \le \Delta_1 e^{\alpha_1 t} \|y_1 - y_2\| , \tag{9.13}$$

provided that for all $t \ge 0$ *the semitrajectories* $S_t y_i$ *lie in the closure* $[\mathcal{O}]$ *of the set* $\mathcal{O}$;

3) *there exist a sequence of finite-dimensional projectors* $\{P_n\}$ *in the space* $\mathcal{H}$, *constants* Δ_2, α_2, $\beta > 0$, *and a sequence of positive numbers* $\{\rho_n\}$ *tending to zero as* $n \to \infty$ *such that*

$$\|(1-P_n)(S_t y_1 - S_t y_2)\| \le \Delta_2 e^{-\beta t}(1 + \rho_n e^{\alpha_2 t})\|y_1 - y_2\| \tag{9.14}$$

for any $y_1, y_2 \in K$.

Proof.

Let $\overline{K}$ be a compact set from Lemma 9.2. Let

$$K^* = \gamma^+(\overline{K}) \equiv \bigcup_{t \ge 0} S_t \overline{K} .$$

It is clear that $S_t K^* \subset K^*$ and equation (9.12) holds for $K = K^*$ with $C = L_2 R$ and $\gamma = \varepsilon$. Let us prove that K^* is a compact set. Let $\{z_n\}$ be a sequence of elements of $K^* = \gamma^+(\overline{K})$. Then $z_n = S_{t_n} y_n$ for some $t_n > 0$ and $y_n \in \overline{K}$. If there exists an infinitely increasing subsequence $\{t_{n_k}\}$, then equation (9.6) gives us that

$$\lim_{k \to \infty} \operatorname{dist}\left(S_{t_{n_k}} y_{n_k},\ \overline{K}\right) = 0 .$$

Therefore, the sequence $\{z_n\}$ possesses a limit point in $\overline{K} \subset K^*$. If $\{t_n\}$ is a bounded sequence, then by virtue of the compactness of $\overline{K}$ there exist a number $t_0 > 0$, an element $y \in \overline{K}$ and a sequence $\{n_k\}$ such that $y_{n_k} \to y$ and $t_{n_k} \to t$. Therewith

$$\left\|S_{t_{n_k}} y_{n_k} - S_{t_0} y\right\| \le \left\|S_{t_n} y - S_{t_0} y\right\| + \left\|S_{t_{n_k}} y_{n_k} - S_{t_{n_k}} y\right\| .$$

The first term in the right-hand side of this inequality evidently tends to zero. As for the second term, our argument is the same as in the proof of formula (9.8). We use the boundedness of the set $\gamma^+(\overline{K})$ (see property **(A)**) and properties **(B)** and **(C2)** to obtain the estimate

$$\|S_t y_1 - S_t y_2\| \le C e^{C_{\overline{K}} t} \|y_1 - y_2\| , \qquad y_1, y_2 \in \overline{K} . \tag{9.15}$$

It follows that

$$\lim_{k \to \infty} \left\|S_{t_{n_k}} y_{n_k} - S_{t_{n_k}} y\right\| = 0 .$$

Therefore,

$$S_{t_{n_k}} y_{n_k} \to S_{t_0} y \in K^* = \gamma^+(\overline{K}).$$

The closedness of the set K^* can be established with the help of similar arguments. Thus, property (9.12) is proved for $K = K^*$. Now we suppose that $K = S_{t_0} K^*$, where t_0 is chosen such that $\|y\| < R$ for all $y \in K$. It is obvious that K is a compact positively invariant set. As it is proved above, it is easy to find the estimate of form (9.15) for all y_1 and y_2 from an arbitrary bounded set B. Here an important role is played by the boundedness of the set $\gamma^+(B)$ (see property **(A)**). Therefore, for any $B \in \mathcal{B}(\mathcal{H})$ there exists a constant $C_0 = C(B, K^*, t_0) > 0$ such that

$$\left\|S_{t_0} y_1 - S_{t_0} y_2\right\| \le C_0 \left\|y_1 - y_2\right\|, \qquad y_1, y_2 \in B \cup K^*.$$

Hence, for $y \in B$ we have that

$$\operatorname{dist}(S_t y, K) = \operatorname{dist}(S_{t_0} \cdot S_{t-t_0} y, S_{t_0} K^*) \le C_0 \operatorname{dist}(S_{t-t_0} y, K^*)$$

for $t > t_0$. This implies estimate (9.12) with the constant C depending on K and B. However, if we change the moment t_B in equation (9.12), we can presume that, for example, $C = 1$. Therewith $\gamma = \varepsilon$. Thus, the first assertion of the lemma is proved.

Since the set K lies in the ball of dissipativity $\{z \in \mathcal{H}: \|z\| \le R\}$, estimates (9.13) and (9.14) follow from Lemma 9.3. Moreover,

$$\Delta_1 = L_1, \quad \alpha_1 = D(R), \qquad \Delta_2 = L_2, \quad \alpha_2 = \varepsilon + D(R),$$

$$\beta = \gamma = \varepsilon, \quad \rho_n = C_0 r_n^{\sigma} C_3(R) \cdot D(R)^{-1}. \tag{9.16}$$

Thus, Lemma 9.4 is proved.

Lemma 9.4 along with the theorem given below enables us to verify the existence of an inertial set for the dynamical system generated by equation (9.1).

Theorem 9.2.

Let the phase space $\mathcal{H}$ ***of a dynamical system*** $(\mathcal{H}, S_t)$ ***be a Hilbert space. Assume that in*** $\mathcal{H}$ ***there exists a compact positively invariant set*** K ***possessing properties*** (9.12)–(9.14). ***Then for any*** $\nu > \ln 2$ ***there exists an inertial set*** $A_{\exp}^{\nu}$ ***of the dynamical system*** $(\mathcal{H}, S_t)$ ***such that***

$$h(S_t B, A_{\exp}^{\nu}) \le C(B, \nu) \cdot \exp\left\{-\gamma\left(1 - \frac{\gamma + \alpha_1}{\nu + \gamma + \alpha_1}\right)(t - t_B)\right\} \tag{9.17}$$

for any bounded set B ***and*** $t \ge t_B$. ***Here, as above,*** $h(X, Y) = \sup\{\operatorname{dist}(x, Y): x \in X\}$. ***Moreover,***

$$\dim_f A_{\exp}^{\nu} \le C_0 \cdot (1 + \ln\left\|P_n\right\|) \dim P_n, \tag{9.18}$$

where the number n is determined from the condition

$$\rho_n \le (4\Delta_2)^{\frac{\alpha_2}{\beta}} \cdot \exp\left\{-\nu\frac{\alpha_2}{\beta}\right\} \tag{9.19}$$

and constant C_0 does not depend on ν and n.

The proof of the theorem is based on the following preliminary assertions.

Lemma 9.5.

Let (K, S_t) be a dynamical system, its phase space being a compact in a Hilbert space $\mathcal{H}$. Assume that for all $(y_1, y_2) \in K$ equations (9.13) *and* (9.14) *are valid. Then for any $\nu > \ln 2$ there exists an inertial set $A^{\nu}_{\exp}$ of the system (K, S_t) such that*

$$h(S_t K, A^{\nu}_{\exp}) = \sup\{\operatorname{dist}(S_t y, A^{\nu}_{\exp}):\ y \in K\} \le C_{\nu} e^{-\nu t}. \tag{9.20}$$

Moreover, estimate (9.18) *holds for the value* $\dim_f A^{\nu}_{\exp}$.

Proof.

We use Theorem 8.2 with $M = K$, $V = S_{t_0}$, and $\delta = \frac{1}{2}e^{-\nu}$, where t_0 and n are chosen to fulfil

$$\Delta_2\, e^{-\beta t_0} = \frac{\delta}{2} = \frac{1}{4}e^{-\nu} \quad \text{and} \quad \rho_n\, e^{\alpha_2 t_0} \le 1.$$

In this case conditions (8.1) and (8.2) are valid for $V = S_{t_0}$ with $\delta = \frac{1}{2}e^{-\nu}$ and $l = \|P_n\|\,\Delta_1\, e^{\alpha_1 t_0}$. Therefore, there exists a bounded closed positively invariant set A_θ with $\delta < \theta < 1/2$ such that (see (8.6) and (8.15))

$$\sup\{\operatorname{dist}(V^m y, A_\theta):\ y \in K\} \le \theta^m, \quad m = 1, 2, \ldots \tag{9.21}$$

and

$$\dim_f A_\theta \le \ln\left(1 + \frac{4l}{\theta - \delta}\right)\left[\ln\frac{1}{2\theta}\right]^{-1} \cdot \dim P_n \tag{9.22}$$

Assume that $\theta = 2\delta = e^{-\nu}$ and consider the set

$$A^{\nu}_{\exp} = \bigcup\{S_t A_\theta:\ 0 \le 0 \le t_0\}.$$

Here $\nu = \ln\frac{1}{\theta} > \ln 2$. It is easy to see that

$$\dim_f A^{\nu}_{\exp} \le 1 + \dim_f A^{\nu}_{\exp}.$$

Therefore, equations (9.20) and (9.18) follow from (9.21) and (9.20) after some simple calculations.

Lemma 9.6.

Assume that in the phase space $\mathcal{H}$ of a dynamical system $(\mathcal{H}, S_t)$ there exist compact sets K and K_0 such that (a) $K_0 \subset K$; (b) *properties*

(9.12) *and* (9.13) *are valid for* K; *and* (c) *the set* K_0 *possesses the property*

$$h(S_t K,\, K_0) \le C e^{-\gamma_0 t}, \tag{9.23}$$

where $h(X, Y) = \sup\{\operatorname{dist}(x, Y) \colon x \in X\}$. *Then for any bounded set* $B \subset \mathscr{H}$ *and* $t \ge t_B$ *the following inequality holds*

$$h(S_t B,\, K_0) \le C_B \exp\left\{-\frac{\gamma\gamma_0}{\gamma+\gamma_0+\alpha_1}\, t\right\}. \tag{9.24}$$

Proof.

By virtue of (9.12) every bounded set B reaches the vicinity $\mathscr{O}$ in finite time and stays in it. Therefore, it is sufficient to prove the lemma for a set $B \in \mathscr{B}(\mathscr{H})$ such that $S_t B \subset [\mathscr{O}]$ for $t \ge 0$, where $[\mathscr{O}]$ denotes the closure of $\mathscr{O}$. Let $k_0 \in K_0$ and $y \in B$. Evidently,

$$\|S_t y - k_0\| \le \|S_{\varkappa t} S_{(1-\varkappa)t}\, y - S_{\varkappa t} k\| + \|S_{\varkappa t} k - k_0\|$$

for any $0 \le \varkappa \le 1$ and $k \in K$. With the help of (9.13) we have that

$$\|S_t y - k_0\| \le \Delta_1 e^{\alpha_1 \varkappa t} \|S_{(1-\varkappa)t}\, y - k\| + \|S_{\varkappa t} k - k_0\| .$$

Therefore, for any $0 < \varkappa < 1$ and $k \in K$ we have that

$$\operatorname{dist}(S_t y,\, K_0) \le \Delta_1 e^{\alpha_1 \varkappa t} \|S_{(1-\varkappa)t}\, y - k\| + \operatorname{dist}(S_{\varkappa t} k,\, K_0) \le$$

$$\le \Delta_1 e^{\alpha_1 \varkappa t} \|S_{(1-\varkappa)t}\, y - k\| + h(S_{\varkappa t} K,\, K_0).$$

If we take an infimum over $k \in K$ and a supremum over $y \in B$, we find that

$$h(S_t B,\, K_0) \le \Delta_1 e^{\alpha_1 \varkappa t} h(S_{(1-\varkappa)t} B,\, K) + h(S_{\varkappa t} K,\, K_0)$$

for all $0 < \varkappa < 1$. Hence, equations (9.12) and (9.23) give us that

$$h(S_t B,\, K_0) \le C_{\boldsymbol{B}} e^{(\alpha_1 \varkappa - \gamma(1-\varkappa))t} + C_K e^{-\varkappa\gamma_0 t}$$

for $t \ge t_B$. If we choose $\varkappa = \gamma(\gamma + \gamma_0 + \alpha_1)^{-1}$, we obtain (9.24). Lemma 9.6 is proved.

If we now use Lemma 9.6 with $K_0 = A^{\nu}_{\exp}$ and estimate (9.20), we get equation (9.17). This **completes the proof of Theorem 9.2**.

Thus, by virtue of Lemma 9.4 and Theorem 9.2 the dynamical system $(\mathscr{H},\, S_t)$ generated by equation (9.1) possesses an inertial set $A^{\nu}_{\exp}$ for which equations (9.17)–(9.19) hold with relations (9.16).

It should be noted that a slightly different approach to the construction of inertial sets is developed in the book by A. Eden, C. Foias, B. Nicolaenko, and R. Temam (see the list of references). This book contains further developments and applications of the theory of inertial sets.

To conclude this section, we outline the results on the behaviour of the projection onto the finite-dimensional subspace $P_n \mathcal{H}$ of the trajectories of the system $(\mathcal{H}, S_t)$ generated by equation (9.1).

Assume that an element y_0 belongs to the global attractor $\mathcal{A}$ of a dynamical system $(\mathcal{H}, S_t)$. Lemma 6.1 implies that there exists a trajectory $\gamma = \{y(t),\ t \in \mathbb{R}\}$ lying in $\mathcal{A}$ wholly such that $y(0) = y_0$. Therewith the following assertion is valid.

Lemma 9.7.

Assume that properties **(A)–(C)** *are fulfilled and let* $y_0 \in \mathcal{A}$. *Then the following equation holds:*

$$(1-P_n)y_0 = \int_{-\infty}^{0} (1-P_n)T_{-\tau}B(y(\tau))\,d\tau\ , \qquad n \ge n_0\ , \tag{9.25}$$

where $\{y(\tau)\}$ *is a trajectory passing through* y_0, *the number* n_0 *can be found from* **(B2)** *and the integral in* (9.25) *converges in the norm of the space* $\mathcal{H}$.

Proof.

Since $y_0 = S_t\, y(-t)$, equation (9.2) gives us that

$$(1-P_n)y_0 = (1-P_n)\left(T_t\, y(-t) + \int_{-t}^{0} (1-P_n)T_{-\tau}B(y(\tau))\,d\tau\right). \tag{9.26}$$

A trajectory in the attractor possesses the property $\|y(t)\| \le R$, $t \in (-\infty, \infty)$. Therefore, property **(B2)** implies that

$$\|(1-P_n)T_t\, y(-t)\| \le L_2 \cdot R e^{-\varepsilon t} \quad \text{and} \quad \|(1-P_n)T_{-\tau}B(y(\tau))\| \le L_2 C_R\, e^{-\varepsilon|\tau|}\ .$$

These estimates enable us to pass to the limit in (9.26) as $t \to -\infty$. Thereupon we obtain (9.25).

The following assertion is valid under the hypotheses of Theorem 9.1.

Theorem 9.3.

There exists $N_0 \ge n_0$ ***such that for all*** $N \ge N_0$ ***the following assertions are valid:***

1) ***for any two trajectories*** $y_1(t)$ ***and*** $y_2(t)$ ***lying in the attractor of the system generated by equation*** (9.1) ***the equality*** $P_N y_1(t) = P_N y_2(t)$ ***for all*** $t \in \mathbb{R}$ ***implies that*** $y_1(t) \equiv y_2(t)$***;***
2) ***for any two solutions*** $u_1(t)$ ***and*** $u_2(t)$ ***of the system*** (9.1) ***the equation***

$$\lim_{t \to -\infty} P_N(u_1(t) - u_2(t)) = 0 \tag{9.27}$$

implies that $\|u_1(t) - u_2(t)\| \to 0$ ***as*** $t \to \infty$.

We can also obtain an upper estimate of the number N_0 from the inequality $r_{N_0}^{\sigma} \le a_0 \varepsilon (L_2 C_3(R))^{-1}$.

Proof.

Equation (9.25) implies that for any trajectory $y_i(t)$ lying in the attractor of system (9.1) the equation

$$(1-P_N)\, y_i(t) = \int_{-\infty}^{t} (1-P_N) T_{t-\tau} B(y_i(\tau))\, d\tau, \quad i = 1, 2,$$

holds. Therefore, if $P_N y_1(t) = P_N y_2(t)$, then properties **(B2)**, **(B3)**, and **(C2)** give us that

$$\|y_1(t) - y_2(t)\| \le C_0 r_N^{\sigma} L_2 C_3(R) \int_{-\infty}^{t} e^{-\varepsilon(t-\tau)} \|y_1(\tau) - y_2(\tau)\|\, d\tau .$$

It follows that the estimate

$$\|y_1(t) - y_2(t)\| \le A_N \exp\{-\varepsilon t + A_N(t-t_0)\} \int_{-\infty}^{t_0} e^{\varepsilon\tau} \|y_1(\tau) - y_2(\tau)\|\, d\tau$$

holds for $t \ge t_0$, where $A_N = C_0 r_N^{\sigma} L_2 C_3(R)$. If we tend $t_0 \to -\infty$, we obtain the first assertion, provided $A_N < \varepsilon$.

Now let us prove the second assertion of the theorem. Let

$$\alpha_N(t) = \|P_N(u_1(t) - u_2(t))\| .$$

Then

$$\|u_1(t) - u_2(t)\| \le \alpha_N(t) + \|(1-P_N)(u_1(t) - u_2(t))\| .$$

Therefore, equation (9.10) for the function $\psi(t) = \|u_1 - u_2\| \exp(\varepsilon t)$ gives us that

$$\psi(t) \le \alpha_N(t)\, e^{\varepsilon t} + L_2 \|u_1(0) - u_2(0)\| + A_N \int_0^t \psi(\tau)\, d\tau .$$

This and Gronwall's lemma imply that

$$\|u_1(t) - u_2(t)\| \le \alpha_N(t) + L_2 \exp(-(\varepsilon - A_N)t) \|u_1(0) - u_2(0)\| +$$

$$+ A_N \int_0^t \alpha_N(\tau) \exp(-(\varepsilon - A_N)(t-\tau))\, d\tau .$$

Therefore, if $A_N < \varepsilon$, then equation (9.27) gives us that $\|u_1(t) - u_2(t)\| \to 0$. Thus, the second assertion of **Theorem 9.3 is proved**.

Theorem 9.3 can be presented in another form. Let $\{e_k:\ k=1,\ \ldots\ ,\ d_N\}$ be a basis in the space $P_N\mathscr{H}$. Let us define linear functionals $l_j(u)=(u\,,\ e_j)$ on $\mathscr{H}$, $j=1,\ \ldots,\ d_N$. Theorem 9.3 implies that the asymptotic behaviour of trajectories of the system $(\mathscr{H},\ S_t)$ is uniquely determined by its values on the functionals l_j. Therefore, it is natural that the family of functionals $\{l_j\}$ is said to be the determining collection. At present some general approaches have been worked out which enable us to define whether a particular set of functionals is determining. Chapter 5 is devoted to the exposition of these approaches. It should be noted that for the first time Theorem 9.3 was proved for the two-dimensional Navier-Stokes system by C. Foias and D. Prodi (the second assertion) and by O. A. Ladyzhenskaya (the first assertion).

Concluding the chapter, we would like to note that the list of references given below does not claim to be full. It contains only references to some monographs and reviews devoted to the developments of the questions touched on here and comprising intensive bibliography.

References

1. BABIN A. V., VISHIK M. I. *Attractors of evolution equations*. –Amsterdam: North-Holland, 1992.
2. TEMAM R. *Infinite-dimensional dynamical systems in Mechanics and Physics*. –New York: Springer, 1988.
3. HALE J. K. *Asymptotic behavior of dissipative systems*. –Providence: Amer. Math. Soc., 1988.
4. EDEN A., FOIAS C., NICOLAENKO B., TEMAM R. *Exponential attractors for dissipative evolution equations*. –New York: Willey, 1994.
5. LADYZHENSKAYA O. A. *On the determination of minimal global attractors for the Navier-Stokes equations and other partial differential equations* // Russian Math. Surveys. –1987. –Vol. 42. –# 6. –P. 27–73.
6. KAPITANSKY L. V., KOSTIN I. N. *Attractors of nonlinear evolution equations and their approximations* // Leningrad Math. J. –1991. –Vol. 2. –P. 97–117.
7. CHUESHOV I. D. *Global attractors for nonlinear problems of mathematical physics* // Russian Math. Surveys. –1993. –Vol. 48. –# 3. –P. 133–161.
8. PLISS V. A. *Integral sets of periodic systems of differential equations*. –Moskow: Nauka, 1977 (in Russian).
9. HALE J. K. *Theory of functional differential equations.* –Berlin; Heidelberg; New York: Springer, 1977.