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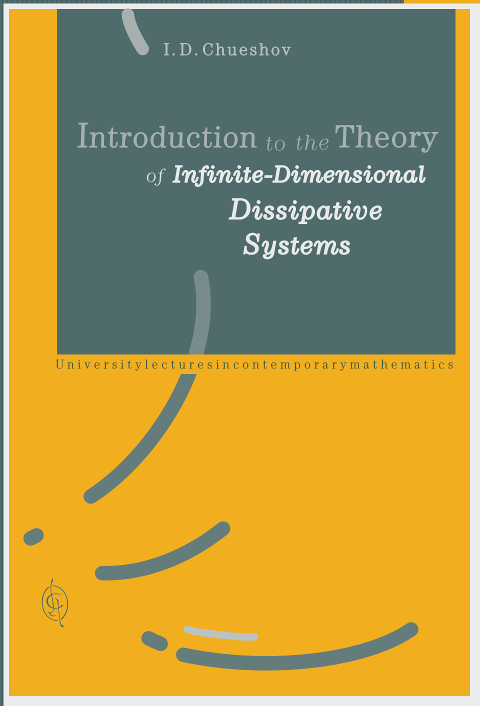

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This book provides an exhaustive introduction to the scope of main ideas and methods of the theory of infinite-dimensional dissipative dynamical systems which has been rapidly developing in recent years. In the examples systems generated by nonlinear partial differential equations arising in the different problems of modern mechanics of continua are considered. The main goal of the book is to help the reader to master the basic strategies used in the study of infinite-dimensional dissipative systems and to qualify him/her for an independent scientific research in the given branch. Experts in nonlinear dynamics will find many fundamental facts in the convenient and practical form in this book.

The core of the book is composed of the courses given by the author at the Department of Mechanics and Mathematics at Kharkov University during a number of years. This book contains a large number of exercises which make the main text more complete. It is sufficient to know the fundamentals of functional analysis and ordinary differential equations to read the book.

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Chapter 2

Long-Time Behaviour of Solutions to a Class of Semilinear Parabolic Equations

Contents

In this chapter we study well-posedness and the asymptotic behaviour of solutions to a class of abstract nonlinear parabolic equations. A typical representative of this class is the nonlinear heat equation

$$\frac{\partial u}{\partial t} = \nu \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} + f(x,\ u)$$

considered in a bounded domain Ω of $\mathbb{R}^d$ with appropriate boundary conditions on the border $\partial\Omega$. However, the class also contains a number of nonlinear partial differential equations arising in Mechanics and Physics that are interesting from the applied point of view. The main feature of this class of equations lies in the fact that the corresponding dynamical systems possess a compact absorbing set.

The first three sections of this chapter are devoted to the questions of existence and uniqueness of solutions and a brief description of examples. They are independent of the results of Chapter 1. In the other sections containing the discussion of asymptotic properties of solutions we use general results on the existence and properties of global attractors proved in Chapter 1. In Sections 6 and 7 we present two quite simple infinite-dimensional systems for which the asymptotic behaviour of the trajectories can be explicitly described. In Section 8 we consider a class of systems generated by infinite-dimensional retarded equations.

The list of references at the end of the chapter consists only of the books recommended for further reading.

§ 1 Positive Operators with Discrete Spectrum

This section contains some auxiliary facts that play an important role in the subsequent considerations related to the study of the asymptotic properties of solutions to abstract semilinear parabolic equations.

Assume that H is a separable Hilbert space with the inner product $(\cdot,\ \cdot)$ and the norm $\|\cdot\|$. Let A be a selfadjoint positive linear operator with the domain $D(A)$. An operator A is said to have a ***discrete spectrum*** if in the space H there exists an orthonormal basis $\{e_k\}$ of the eigenvectors:

$$(e_k,\ e_j) = \delta_{kj}\,, \quad A e_k = \lambda_k e_k\,, \qquad k,\ j = 1,\ 2,\ \dots\,, \tag{1.1}$$

such that

$$0 < \lambda_1 \le \lambda_2 \le \dots\,, \qquad \lim_{k \to \infty} \lambda_k = \infty\,. \tag{1.2}$$

The following exercise contains a simple example of an operator with discrete spectrum.

Exercise *1.1* Let $H = L^2(0,\ 1)$ and let A be an operator defined by the equation $Au = -u''$ with the domain $D(A)$ which consists of continuously differentiable functions $u(x)$ such that (a) $u(0)=u(1)=0$, (b) $u'(x)$ is absolutely continuous and (c) $u'' \in L^2(0,\ 1)$. Show that A is a positive operator with discrete spectrum. Find its eigenvectors and eigenvalues.

The above-mentioned structure of the operator A enables us to define an operator $f(A)$ for a wide class of functions $f(\lambda)$ defined on the positive semiaxis. It can be done by supposing that

$$D(f(A)) = \left\{h = \sum_{k=1}^{\infty} c_k e_k \in H\colon\ \sum_{k=1}^{\infty} c_k^2 [f(\lambda_k)]^2 < \infty\right\},$$

$$f(A)h = \sum_{k=1}^{\infty} c_k f(\lambda_k) e_k\,, \qquad h \in D(f(A))\,. \tag{1.3}$$

In particular, one can define operators A^α with $\alpha \in \mathbb{R}$. For $\alpha = -\beta < 0$ these operators are bounded. However, in this case it is also convenient to introduce the lineals $D(A^\alpha)$ if we regard $D(A^{-\beta})$ as a completion of the space H with respect to the norm $\|A^{-\beta}\cdot\|$.

Exercise *1.2* Show that the space $\mathcal{F}_{-\beta} = D(A^{-\beta})$ with $\beta > 0$ can be identified with the space of formal series $\sum c_k e_k$ such that

$$\sum_{k=1}^{\infty} c_k^2 \lambda_k^{-2\beta} < \infty\,.$$

Exercise *1.3* Show that for any $\beta \in \mathbb{R}$ the operator A^β can be defined on every space $D(A^\alpha)$ as a bounded mapping from $D(A^\alpha)$ into $D(A^{\alpha-\beta})$ such that

$$A^\beta D(A^\alpha) = D(A^{\alpha-\beta})\,, \qquad A^{\beta_1+\beta_2} = A^{\beta_1}\cdot A^{\beta_2}\,. \tag{1.4}$$

Exercise *1.4* Show that for all $\alpha \in \mathbb{R}$ the space $\mathcal{F}_\alpha \equiv D(A^\alpha)$ is a separable Hilbert space with the inner product $(u,\ v)_\alpha = (A^\alpha u,\ A^\alpha v)$ and the norm $\|u\|_\alpha = \|A^\alpha u\|$.

Exercise *1.5* The operator A with the domain $\mathcal{F}_{1+\sigma}$ is a positive operator with discrete spectrum in each space $\mathcal{F}_\sigma$.

Exercise *1.6* Prove the continuity of the embedding of the space $\mathcal{F}_\alpha$ into $\mathcal{F}_\beta$ for $\alpha > \beta$, i.e. verify that $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ and $\|u\|_\beta \le C\|u\|_\alpha$.

Exercise *1.7* Prove that $\mathcal{F}_\alpha$ is dense in $\mathcal{F}_\beta$ for any $\alpha > \beta$.

— Exercise *1.8* Let $f \in \mathcal{F}_\sigma$ for $\sigma > 0$. Show that the linear functional $F(g) \equiv \equiv (f, g)$ can be continuously extended from the space H to $\mathcal{F}_{-\sigma}$ and $|(f, g)| \le \|f\|_\sigma \cdot \|g\|_{-\sigma}$ for any $f \in \mathcal{F}_\sigma$ and $g \in \mathcal{F}_{-\sigma}$.

— Exercise *1.9* Show that any continuous linear functional F on $\mathcal{F}_\sigma$ has the form: $F(f) = (f, g)$, where $g \in \mathcal{F}_{-\sigma}$. Thus, $\mathcal{F}_{-\sigma}$ is the space of continuous linear functionals on $\mathcal{F}_\sigma$.

The collection of Hilbert spaces with the properties mentioned in Exercises 1.7–1.9 is frequently called a ***scale*** of Hilbert spaces. The following assertion on the compactness of embedding is valid for the scale of spaces $\{\mathcal{F}_\sigma\}$.

Theorem 1.1.

Let $\sigma_1 > \sigma_2$. Then the space $\mathcal{F}_{\sigma_1}$ is compactly embedded into $\mathcal{F}_{\sigma_2}$, i.e. every sequence bounded in $\mathcal{F}_{\sigma_1}$ is compact in $\mathcal{F}_{\sigma_2}$.

Proof.

It is well known that every bounded set in a separable Hilbert space is weakly compact, i.e. it contains a weakly convergent sequence. Therefore, it is sufficient to prove that any sequence weakly tending to zero in $\mathcal{F}_{\sigma_1}$ converges to zero with respect to the norm of the space $\mathcal{F}_{\sigma_2}$. We remind that a sequence $\{f_n\}$ in $\mathcal{F}_\sigma$ weakly converges to an element $f \in \mathcal{F}_\sigma$ if for all $g \in \mathcal{F}_\sigma$

$$\lim_{n \to \infty} (f_n, g)_\sigma = (f, g)_\sigma .$$

Let the sequence $\{f_n\}$ be weakly convergent to zero in $\mathcal{F}_{\sigma_1}$ and let

$$\|f_n\|_{\sigma_1} \le C, \qquad n = 1, 2, \dots . \tag{1.5}$$

Then for any N we have

$$\|f_n\|^2_{\sigma_2} \le \sum_{k=1}^{N-1} \lambda_k^{2\sigma_2} (f_n, e_k)^2 + \frac{1}{N^{2(\sigma_1 - \sigma_2)}} \sum_{k=N}^{\infty} \lambda_k^{2\sigma_1} (f_n, e_k)^2 . \tag{1.6}$$

Here we applied the fact that for $k \ge N$

$$\lambda_k^{2\sigma_2} \le \frac{1}{\lambda_N^{2(\sigma_1 - \sigma_2)}} \lambda_k^{2\sigma_1} .$$

Equations (1.5) and (1.6) imply that

$$\|f_n\|^2_{\sigma_2} \le \sum_{k=1}^{N-1} \lambda_k^{2\sigma_2} (f_n, e_k)^2 + C \lambda_N^{-2(\sigma_1 - \sigma_2)} .$$

We fix $\varepsilon > 0$ and choose N such that

$$\|f_n\|^2_{\sigma_2} \le \sum_{k=1}^{N-1} \lambda_k^{2\sigma_2} (f_n, e_k)^2 + \varepsilon . \tag{1.7}$$

Let us fix the number N. The weak convergence of f_n to zero gives us

$$\lim_{n\to\infty}(f_n,\, e_k)=0\,,\qquad k=1,\,2,\,\ldots,\,N-1\,.$$

Therefore, it follows from (1.7) that

$$\overline{\lim_{n\to\infty}}\|f_n\|_{\sigma_2}\le\varepsilon\,.$$

By virtue of the arbitrariness of ε we have

$$\lim_{n\to\infty}\|f_n\|_{\sigma_2}=0\,.$$

Thus, **Theorem 1.1 is proved**.

— Exercise *1.10* Show that the resolvent $R_\lambda(A)=(A-\lambda)^{-1}$, $\lambda\ne\lambda_k$, is a compact operator in each space $\mathcal{F}_\sigma$.

We point out several properties of the scale of spaces $\{\mathcal{F}_\sigma\}$ that are important for further considerations.

— Exercise *1.11* Show that in each space $\mathcal{F}_\sigma$ the equation

$$P_l u=\sum_{k=1}^{l}(u,\, e_k)e_k\,,\qquad u\in\mathcal{F}_\sigma\,,\quad -\infty<\sigma<\infty$$

defines an orthoprojector onto the finite-dimensional subspace generated by the set of elements $\{e_k,\ k=1,\,2,\,\ldots,\,l\}$. Moreover, for each σ we have

$$\lim_{l\to\infty}\|P_l u-u\|_\sigma=0\,.$$

— Exercise *1.12* Using the Hölder inequality

$$\sum_k a_k b_k\le\left(\sum_k a_k^p\right)^{1/p}\left(\sum_k b_k^q\right)^{1/q},\qquad \frac{1}{p}+\frac{1}{q}=1\,,\quad a_k,\, b_k>0\,,$$

prove the interpolation inequality

$$\|A^\theta u\|\le\|Au\|^\theta\cdot\|u\|^{1-\theta}\,,\qquad 0\le\theta\le1\,,\quad u\in D(A)\,.$$

— Exercise *1.13* Relying on the result of the previous exercise verify that for any $\sigma_1,\ \sigma_2\in\mathbb{R}$ the following interpolation estimate holds:

$$\|u\|_{\sigma(\theta)}\le\|u\|_{\sigma_1}^{\theta}\,\|u\|_{\sigma_2}^{1-\theta}\,,$$

where $\sigma(\theta)=\theta\sigma_1+(1-\theta)\sigma_2$, $0\le\theta\le1$, and $u\in\mathcal{F}_{\max(\sigma_1,\,\sigma_2)}$. Prove the inequality

$$\|u\|_{\sigma(\theta)}^2\le\varepsilon\|u\|_{\sigma_1}^2+C_\theta\,\varepsilon^{-\frac{\theta}{1-\theta}}\|u\|_{\sigma_2}^2\,,$$

where $0<\theta<1$ and ε is a positive number.

Equations (1.3) enable us to define an exponential operator $\exp(-tA)$, $t \geq 0$, in the scale $\{\mathcal{F}_\sigma\}$. Some of its properties are given in exercises 1.14–1.17.

— Exercise *1.14* For any $\alpha \in \mathbb{R}$ and $t > 0$ the linear operator $\exp(-tA)$ maps $\mathcal{F}_\alpha$ into $\bigcap_{\sigma \geq 0} \mathcal{F}_\sigma$ and possesses the property $\|e^{-tA}u\|_\alpha \leq$ $\leq e^{-t\lambda_1}\|u\|_\alpha$.

— Exercise *1.15* The following semigroup property holds:

$$\exp(-t_1 A)\cdot\exp(-t_2 A) = \exp(-(t_1+t_2)A), \qquad t_1,\ t_2 \geq 0 .$$

— Exercise *1.16* For any $u \in \mathcal{F}_\sigma$ and $\sigma \in \mathbb{R}$ the following equation is valid:

$$\lim_{t \to \tau} \|e^{-tA}u - e^{-\tau A}u\|_\sigma = 0 . \tag{1.8}$$

— Exercise *1.17* For any $\sigma \in \mathbb{R}$ the exponential operator e^{-tA} defines a dissipative compact dynamical system $(\mathcal{F}_\sigma, e^{-tA})$. What can you say about its global attractor?

Let us introduce the following notations. Let $C(a, b; \mathcal{F}_\alpha)$ be the space of strongly continuous functions on the segment $[a, b]$ with the values in $\mathcal{F}_\alpha$, i.e. they are continuous with respect to the norm $\|\cdot\|_\alpha = \|A^\alpha\cdot\|$. In particular, Exercise 1.16 means that $e^{-tA}u \in C(\mathbb{R}_+, \mathcal{F}_\alpha)$ if $u \in \mathcal{F}_\alpha$. By $C^1(a, b; \mathcal{F}_\alpha)$ we denote the subspace of $C(a, b; \mathcal{F}_\alpha)$ that consists of the functions $f(t)$ which possess strong (in $\mathcal{F}_\alpha$) derivatives $f'(t)$ lying in $C(a, b; \mathcal{F}_\alpha)$. The space $C^k(a, b; \mathcal{F}_\alpha)$ is defined similarly for any natural k. We remind that the strong derivative (in $\mathcal{F}_\alpha$) of a function $f(t)$ at a point $t = t_0$ is defined as an element $v \in \mathcal{F}_\alpha$ such that

$$\lim_{h \to 0}\left\|\frac{1}{h}(f(t_0+h) - f(t_0)) - v\right\|_\alpha = 0 .$$

— Exercise *1.18* Let $u_0 \in \mathcal{F}_\sigma$ for some σ. Show that

$$e^{-tA}u_0 \in C^k(\delta, +\infty; \mathcal{F}_\alpha)$$

for all $\delta > 0$, $\alpha \in \mathbb{R}$, and $k = 1, 2, \ldots$. Moreover,

$$\frac{d^k}{dt^k} e^{-tA}u_0 = (-A)^k e^{-tA}u_0, \qquad k = 1,\ 2,\ \ldots .$$

Let $L^2(a, b; \mathcal{F}_\alpha)$ be the space of functions on the segment $[a, b]$ with the values in $\mathcal{F}_\alpha$ for which the integral

$$\|u\|^2_{L^2(a,\, b;\ \mathcal{F}_\alpha)} = \int_a^b \|u(t)\|^2_\alpha\, dt$$

exists. Let $L^\infty(a, b; \mathcal{F}_\alpha)$ be the space of essentially bounded functions on $[a, b]$ with the values in $\mathcal{F}_\alpha$ and the norm

$$\|u\|_{L^\infty(a,\, b;\; D(A^\alpha))} = \operatorname*{ess\ sup}_{t\, \in\, [a,\, b]} \|A^\alpha u(t)\| \quad .$$

We consider the Cauchy problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + Ay = f(t), \quad t \in (a,\, b); \qquad y(a) = y_0, \tag{1.9}$$

where $y \in \mathcal{F}_\alpha$ and $f(t) \in L^2(a,\, b;\; F_{\alpha-1/2})$. The ***weak solution*** (in $\mathcal{F}_\alpha$) to this problem on the segment $[a,\, b]$ is defined as a function

$$y(t) \in C(a,\, b;\; \mathcal{F}_\alpha) \cap L^2(a,\, b;\; \mathcal{F}_{\alpha+1/2}) \tag{1.10}$$

such that $\mathrm{d}y/\mathrm{d}t \in L^2(a,\, b;\; F_{\alpha-1/2})$ and equalities (1.9) hold. Here the derivative $y'(t) \equiv \mathrm{d}y/\mathrm{d}t$ is considered in the generalized sense, i.e. it is defined by the equality

$$\int_a^b \varphi(t)y'(t)\mathrm{d}t = -\int_a^b \varphi'(t)y(t)\mathrm{d}t\,, \qquad \varphi \in C_0^\infty(a,\, b),$$

where $C_0^\infty(a,\, b)$ is the space of infinitely differentiable scalar functions on $(a,\, b)$ vanishing near the points a and b.

Exercise *1.19* Show that every weak solution to problem (1.9) possesses the property

$$\|y(t)\|_\alpha^2 + \int_a^t \|y(\tau)\|^2_{\alpha+\frac{1}{2}}\mathrm{d}\tau \;\le\; \|y_0\|_\alpha + \int_a^t \|f(\tau)\|^2_{\alpha-\frac{1}{2}}\mathrm{d}\tau\,. \tag{1.11}$$

(*Hint*: first prove the analogue of formula (1.11) for $y_n(t) = P_n y(t)$, then use Exercise 1.11).

Exercise *1.20* Prove the theorem on the existence and uniqueness of weak solutions to problem (1.9). Show that a weak solution $y(t)$ to this problem can be represented in the form

$$y(t) = e^{-(t-a)A}y_0 + \int_a^t e^{-(t-\tau)A}f(\tau)\mathrm{d}\tau\,. \tag{1.12}$$

Exercise *1.21* Let $y_0 \in \mathcal{F}_\alpha$ and let a function $f(t)$ possess the property

$$\|f(t_1) - f(t_2)\|_\alpha \le C|t_1 - t_2|^\theta$$

for some $0 < \theta \le 1$. Then formula (1.12) gives us a solution to problem (1.9) belonging to the class

$$C([a,\, b];\; \mathcal{F}_\alpha) \cap C^1(]a,\, b];\; \mathcal{F}_\alpha) \cap C(]a,\, b];\; \mathcal{F}_{\alpha+1})\,.$$

Such a solution is said to be strong in $\mathcal{F}_\alpha$.

The following properties of the exponential operator e^{-tA} play an important part in the further considerations.

Lemma 1.1.

Let Q_N be the orthoprojector onto the closure of the span of elements $\{e_k,\ k \geq N+1\}$ in H and let $P_N = I - Q_N$, $N = 0, 1, 2, \ldots$. Then

1) *for all $h \in H$, $\beta \geq 0$ and $t \in \mathbb{R}$ the following inequality holds:*

$$\left\|A^{\beta} P_N e^{-tA} h\right\| \leq \lambda_N^{\beta} e^{\lambda_N |t|} \|h\| ; \tag{1.13}$$

2) *for all $h \in D(A^{\beta})$, $t > 0$ and $\alpha \geq \beta$ the following estimate is valid:*

$$\left\|A^{\alpha} Q_N e^{-tA} h\right\| \leq \left[\left(\frac{\alpha-\beta}{t}\right)^{\alpha-\beta} + \lambda_{N+1}^{\alpha-\beta}\right] e^{-t\lambda_{N+1}} \left\|A^{\beta} h\right\| , \tag{1.14}$$

in the case $\alpha - \beta = 0$ we supose that $0^0 = 0$ in (1.14).

Proof.

Estimate (1.13) follows from the equation

$$\left\|A^{\beta} P_N e^{-tA} h\right\|^2 = \sum_{k=1}^{N} \lambda_k^{2\beta} e^{-2t\lambda_k} (h,\ e_k)^2 .$$

In the proof of (1.14) we similarly have that

$$\left\|A^{\alpha} Q_N e^{-tA} h\right\|^2 \leq \max_{\lambda \geq \lambda_{N+1}} (\lambda^{\alpha-\beta} e^{-t\lambda})^2 \sum_{k=N+1}^{\infty} \lambda_k^{2\beta} (h,\ e_k)^2 .$$

This gives us the inequality

$$\left\|A^{\alpha} Q_N e^{-tA} h\right\| \leq \frac{1}{t^{\alpha-\beta}} \max_{\mu \geq t\lambda_{N+1}} (\mu^{\alpha-\beta} e^{-\mu}) \left\|A^{\beta} h\right\| .$$

Since $\max\{\mu^{\gamma} e^{-\mu};\ \mu \geq 0\}$ is attained when $\mu = \gamma$, we have that

$$\max_{\mu \geq \lambda_{N+1} t} (\mu^{\gamma} e^{-\mu}) = \begin{cases} (\lambda_{N+1} t)^{\gamma} e^{-\lambda_{N+1} t}, & \text{if} \quad \lambda_{N+1} t \geq \gamma ; \\ \gamma^{\gamma} e^{-\gamma}, & \text{if} \quad \lambda_{N+1} t < \gamma . \end{cases}$$

Therefore,

$$\max_{\mu \geq \lambda_{N+1} t} (\mu^{\gamma} e^{-\mu}) \leq (\gamma^{\gamma} + (\lambda_{N+1} t)^{\gamma})\, e^{-\lambda_{N+1} t} .$$

This implies estimate (1.14). Lemma 1.1 is proved.

In particular, we note that it follows from (1.14) that

$$\left\|A^{\alpha} e^{-tA} h\right\| \leq \left[\left(\frac{\alpha-\beta}{t}\right)^{\alpha-\beta} + \lambda_1^{\alpha-\beta}\right] \cdot e^{-t\lambda_1} \left\|A^{\beta} h\right\| , \qquad \alpha \geq \beta . \tag{1.15}$$

Exercise *1.22* Using estimate (1.15) and the equation

$$e^{-tA}u - e^{-sA}u = -\int_s^t A e^{-\tau A} u \, d\tau , \qquad t \ge s , \quad u \in \mathcal{F}_\beta ,$$

prove that

$$\| e^{-tA}u - e^{-sA}u \|_\theta \le C_{\theta,\,\sigma} |t-s|^{\sigma-\theta} \|u\|_\sigma ; \quad t, s > 0 , \quad (1.16)$$

provided $\theta < \sigma \le 1 + \theta$, where a constant $C_{\theta,\,\sigma}$ does not depend on t and s (cf. Exercise 1.16).

Exercise *1.23* Show that

$$\| A^\alpha e^{-tA} \| \le \left(\frac{\alpha}{t}\right)^\alpha e^{-\alpha} , \qquad t > 0 , \quad \alpha > 0 . \qquad (1.17)$$

Lemma 1.2.

Let $f(t) \in L^\infty(\mathbb{R}, \mathcal{F}_{\alpha-\gamma})$ for $0 \le \gamma < 1$. Then there exists a unique solution $v(t) \in C(\mathbb{R}, \mathcal{F}_\alpha)$ to the nonhomogeneous equation

$$\frac{dv}{dt} + Av = f(t) , \quad t \in \mathbb{R} , \qquad (1.18)$$

that is bounded in $\mathcal{F}_\alpha$ on the whole axis. This solution can be represented in the form

$$v(t) = \int_{-\infty}^t e^{-(t-\tau)A} f(\tau) d\tau . \qquad (1.19)$$

We understand the solution to equation (1.18) on the whole axis as a function $v(t) \in$ $\in C(\mathbb{R}, \mathcal{F}_\alpha)$ such that for any $a < b$ the function $v(t)$ is a weak solution (in $\mathcal{F}_{\alpha-1/2}$) to problem (1.9) on the segment $[a, b]$ with $y_0 = v(a)$.

Proof.

If there exist two bounded solutions to problem (1.18), then their difference $w(t)$ is a solution to the homogeneous equation. Therefore, $w(t) =$ $= \exp\{-(t-t_0)A\} w(t_0)$ for $t \ge t_0$ and for any t_0. Hence,

$$\| A^\alpha w(t) \| \le e^{-(t-t_0)\lambda_1} \| A^\alpha w(t_0) \| \le C e^{-(t-t_0)\lambda_1} .$$

If we tend $t_0 \to -\infty$ here, then we obtain that $w(t) = 0$. Thus, the bounded solution to problem (1.18) is unique. Let us prove that the function $v(t)$ defined by formula (1.19) is the required solution. Equation (1.15) implies that

$$\| A^\alpha e^{-(t-\tau)A} \| \le \left[\left(\frac{\gamma}{t-\tau}\right)^\gamma + \lambda_1^\gamma \right] e^{-(t-\tau)\lambda_1} \operatorname*{ess\,sup}_{\tau \in \mathbb{R}} \| A^{\alpha-\gamma} f(\tau) \|$$

for $t > \tau$ and $0 < \gamma < 1$. Therefore, integral (1.19) exists and it can be uniformly estimated with respect to t as follows:

$$\|A^{\alpha} v(t)\| \le \frac{1+k}{\lambda_1^{1-\gamma}} \operatorname*{ess\,sup}_{\tau \in \mathbb{R}} \|A^{\alpha-\gamma} f(\tau)\|,$$

where $k = 0$ for $\gamma = 0$ and

$$k = \gamma^{\gamma} \int_0^{\infty} s^{-\gamma} e^{-s} \mathrm{d}s \quad \text{for} \quad 0 < \gamma < 1 .$$

The continuity of the function $v(t)$ in $\mathcal{F}_{\alpha}$ follows from the following equation that can be easily verified:

$$v(t) = e^{-(t-t_0)A} v(t_0) + \int_{t_0}^{t} e^{-(t-\tau)A} g(\tau) \mathrm{d}\tau .$$

This also implies (see Exercise 1.18) that $v(t)$ is a solution to equation (1.18). Lemma 1.2 is proved.

§ 2 Semilinear Parabolic Equations in Hilbert Space

In this section we prove theorems on the existence and uniqueness of solutions to an evolutionary differential equation in a separable Hilbert space H of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = B(u, t), \qquad u\big|_{t=s} = u_0, \tag{2.1}$$

where A is a positive operator with discrete spectrum and $B(\cdot, \cdot)$ is a nonlinear continuous mapping from $D(A^{\theta}) \times \mathbb{R}$ into H, $0 \le \theta < 1$, possessing the property

$$\left\|B(u_1, t) - B(u_2, t)\right\| \le M(\rho)\left\|A^{\theta}(u_1 - u_2)\right\| \tag{2.2}$$

for all u_1 and u_2 from the domain $\mathcal{F}_{\theta} = D(A^{\theta})$ of the operator A^{θ} and such that $\left\|A^{\theta} u_j\right\| \le \rho$. Here $M(\rho)$ is a nondecreasing function of the parameter ρ that does not depend on t and $\|\cdot\|$ is a norm in the space H.

A function $u(t)$ is said to be a ***mild solution*** (in $\mathcal{F}_{\theta}$) to problem (2.1) on the half-interval $[s, s+T)$ if it lies in $C(s, s+T'; \mathcal{F}_{\theta})$ for every $T' < T$ and for all $t \in [s, s+T)$ satisfies the integral equation

$$u(t) = e^{-(t-s)A} u_0 + \int_s^t e^{-(t-\tau)A} B(u(\tau), \tau) \mathrm{d}\tau . \tag{2.3}$$

The fixed point method enables us to prove the following assertion on the local existence of mild solutions.

Theorem 2.1.

Let $u_0 \in \mathcal{F}_\theta$***. Then there exists*** T^* ***depending on*** θ ***and*** $\|u_0\|_\theta$ ***such that problem*** (2.1) ***possesses a unique mild solution on the half-interval*** $[s,\ s+T^*)$***. Moreover, either*** $T^* = \infty$ ***or the solution cannot be continued in*** $\mathcal{F}_\theta$ ***up to the moment*** $t = s + T^*$***.***

Proof.

On the space $C_{s,\ \theta} \equiv C(s,\ s+T;\ \mathcal{F}_\theta)$ we define the mapping

$$G[u](t) = e^{-(t-s)A}u_0 + \int_s^t e^{-(t-\tau)A}B(u(\tau),\ \tau)\mathrm{d}\tau .$$

Let us prove that $G[u](t) \in C(s,\ s+T;\ \mathcal{F}_\theta)$ for any $T > 0$. Assume that $t_1,\ t_2 \in \in [s,\ s+T]$ and $t_1 < t_2$. It is evident that

$$G[u](t_2) = e^{-(t_2-t_1)A}G[u](t_1) + \int_{t_1}^{t_2} e^{-(t_2-\tau)A}B(u(\tau),\ \tau)\mathrm{d}\tau . \qquad (2.4)$$

By virtue of (1.8) we have that if $t_2 \to t_1$, then

$$\left\|G[u](t_1) - e^{-(t_2-t_1)A}G[u](t_1)\right\|_\theta \to 0 .$$

Therefore, it is sufficient to estimate the second term in (2.4). Equation (1.15) implies that

$$\left\|\int_{t_1}^{t_2} e^{-(t_2-\tau)A}B(u(\tau),\ \tau)\mathrm{d}\tau\right\|_\theta \le$$

$$\le \int_{t_1}^{t_2}\left[\left(\frac{\theta}{t_2-\tau}\right)^\theta + \lambda_1^\theta\right]\mathrm{d}\tau \cdot \max_{\tau\ \in\ [s,\ s+T]}\|B(u(\tau),\ \tau)\| \le$$

$$\le |t_2-t_1|^{1-\theta}\left\{\frac{\theta^\theta}{1-\theta} + \lambda_1^\theta|t_2-t_1|^\theta\right\}\max_{\tau\ \in\ [s,\ s+T]}\|B(u(\tau),\ \tau)\| \qquad (2.5)$$

(if $\theta = 0$, then the coefficient in the braces should be taken to be equal to 1). Thus, G maps $C_{s,\ \theta} = C(s,\ s+T;\ \mathcal{F}_\theta)$ into itself. Let $v_0(t) = e^{-(t-s)A}u_0$. In $C_{s,\ \theta}$ we consider a ball of the form

$$U = \left\{u(t) \in C_{s,\ \theta}\colon\quad |u - v_0|_{C_{s,\ \theta}} \equiv \max_{[s,\ s+T]}\|u(t) - v_0(t)\|_\theta \le 1\right\} .$$

Let us show that for T small enough the operator G maps U into itself and is contractive. Since $|u|_{C_{s,\ \theta}} \le 1 + \|u_0\|_\theta$ for $u \in U$, equation (2.2) gives

$$\max_{\tau \in [s,\, s+T]} \|B(u(\tau),\, \tau)\| \le \max_{\tau \in [s,\, s+T]} \|B(0,\, \tau)\| +$$

$$+\ (1+\|u_0\|_\theta) M(1+\|u_0\|_\theta) \equiv C(T_0,\, \|u_0\|_\theta)$$

for all $T \le T_0$, where T_0 is a fixed number. Therefore, with the help of (2.5) we find that

$$|G[u]-v_0|_{C_{s,\,\theta}} \le T^{1-\theta} \cdot C_1(T_0,\, \theta,\, \|u_0\|_\theta).$$

Similarly we have

$$|G[u]-G[v]|_{C_{s,\,\theta}} \le T^{1-\theta} \cdot C_2(T_0,\, \theta) M(1+\|u\|_\theta)|u-v|_{C_{s,\,\theta}}$$

for $u,\ v \in U$. Consequently, if we choose T_1 such that

$$T_1^{1-\theta} C_1(T_0,\, \theta,\, \|u_0\|_\theta) \le 1 \quad \text{and} \quad T_1^{1-\theta} C_2(T_0,\, \theta) M(1+\|u_0\|_\theta) < 1\,,$$

we obtain that G is a contractive mapping of U into itself. Therefore, G possesses a unique fixed point in $U \subset C_{s,\,\theta}$. Thus, we have constructed a solution on the segment $[s,\ s+T_1]$. Taking $s+T_1$ as an initial moment, we can construct a solution on the segment $[s+T_1,\ s+T_1+T_2]$ with the initial condition $u_0 = u(s+T_1)$. If we continue our reasoning, then we can construct a solution on some maximal half-interval $[s,\ s+T^*)$. Moreover, it is possible that $T^* = \infty$. **Theorem 2.1 is proved**.

— Exercise *2.1* Let $u_0 \in \mathcal{F}_\theta$ and let $T^* = T^*(\theta,\, u_0)$ be such that $[s,\, s+T^*)$ is the maximal half-interval of the existence of the mild solution $u(t)$ to problem (2.1). Then we have either $T^* < \infty$ and $\overline{\lim}_{t \to s+T^*} \|u(t)\|_\theta = \infty$, or $T^* = \infty$.

— Exercise *2.2* Using equations (1.16) and (2.5), prove that for any mild solution $u(t)$ to problem (2.1) on $[s,\ s+T^*)$ the estimate

$$\|u(t)-u(\tau)\|_\alpha \le C(\theta,\, \alpha,\, T)|t-\tau|^{\theta-\alpha}, \quad t,\ \tau \in [s,\ s+T], \tag{2.6}$$

is valid, provided $u_0 \in \mathcal{F}_\theta$, $0 \le \alpha \le \theta$, and $T \le T^*$.

— Exercise *2.3* Let $u_0 \in \mathcal{F}_\theta$ and let $u(t)$ be a mild solution to problem (2.1) on the half-interval $[s,\ s+T^*)$. Then

$$u(t) \in C(s,\ s+T,\ \mathcal{F}_\theta) \cap C(s+\delta,\ s+T,\ \mathcal{F}_{1-\sigma}) \cap$$

$$\cap\ C^1(s+\delta,\ s+T,\ \mathcal{F}_{-\sigma})$$

for any $\sigma > 0$, $0 < \delta < T$, and $T < T^*$. Moreover, equations (2.1) are valid if they are understood as the equalities in $\mathcal{F}_{-\sigma}$ and $\mathcal{F}_\theta$, respectively.

It is frequently convenient to use the Galerkin method in the study of properties of mild solutions to the problem of the type (2.1). Let P_m be the orthoprojector in H

onto the span of elements $\{e_1, e_2, \ldots, e_m\}$. ***Galerkin approximate solution of the order*** m with respect to the basis $\{e_k\}$ is defined as a continuously differentiable function

$$u_m(t) = \sum_{k=1}^{m} g_k(t)\, e_k \tag{2.7}$$

with the values in the finite-dimensional space $P_m H$ that satisfies the equations

$$\frac{d}{dt} u_m(t) + A u_m(t) = P_m B(u_m(t),\, t)\,, \qquad t > s\,, \quad u_m\big|_{t=s} = P_m u_0\,. \tag{2.8}$$

It is clear that (2.8) can be rewritten as a system of ordinary differential equations for the functions $g_k(t)$.

— Exercise *2.4* Show that problem (2.8) is equivalent to the problem of finding a continuous function $u_m(t)$ with the values in $P_m H$ that satisfies the integral equation

$$u_m(t) = e^{-(t-s)A} P_m u_0 + \int_s^t e^{-(t-\tau)A} P_m B(u_m(\tau),\, \tau)\, d\tau\,. \tag{2.9}$$

— Exercise *2.5* Using the method of the proof of Theorem 2.1, prove the local solvability of problem (2.9) on a segment $[s,\, s+T]$, where the parameter $T > 0$ can be chosen to be independent of m. Moreover, the following uniform estimate is valid:

$$\max_{[s,\, s+T]} \|u_m(t)\|_\theta < R\,, \qquad m = 1,\, 2,\, 3,\, \ldots\,, \tag{2.10}$$

where $R > 0$ is a constant.

The following assertion on the convergence of approximate functions to exact ones holds.

Theorem 2.2.

Let $u_0 \in \mathcal{F}_\theta$. ***Assume that there exists a sequence of approximate solutions*** $u_m(t)$ ***on a segment*** $[s,\, s+T]$ ***for which estimate*** (2.10) ***is valid. Then there exists a mild solution*** $u(t)$ ***to problem*** (2.1) ***on the segment*** $[s,\, s+T]$ ***and***

$$\max_{[s,\, s+T]} \|u(t) - u_m(t)\|_\theta \le C\left(\|(1-P_m)u_0\|_\theta + \frac{1}{\lambda_{m+1}^{1-\theta}} \right), \tag{2.11}$$

where $C = C(\theta,\, R,\, T)$ ***is a positive constant independent of*** s.

Proof.

Let $n > m$. We use (2.9), (1.14) and (1.17) to find that for $\theta > 0$ we have

$$\|u_n(t) - u_m(t)\|_\theta \le \|(P_n - P_m)u_0\|_\theta +$$

$$+\int_s^t \left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{m+1}^\theta\right] e^{-\lambda_{m+1}(t-\tau)} \|B(u_n(\tau),\ \tau)\| \mathrm{d}\tau +$$

$$+ e^{-\theta}\int_s^t \left(\frac{\theta}{t-\tau}\right)^\theta \|B(u_n(\tau),\ \tau) - B(u_m(\tau),\ \tau)\| \mathrm{d}\tau \ .$$

Therefore, equations (2.2) and (2.10) give us that

$$\|u_n(t) - u_m(t)\|_\theta \le \|(P_n - P_m)u_0\|_\theta +$$

$$+\left(\max_{[s,\ s+T]} \|B(0,\ \tau)\| + M(R)R\right) J_m(t,\ s) +$$

$$+ M(R)\theta^\theta e^{-\theta} \int_s^t (t-\tau)^{-\theta} \|u_n(\tau) - u_m(\tau)\|_\theta \mathrm{d}\tau \ , \tag{2.12}$$

where

$$J_m(t,\ s) = \int_s^t \left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{m+1}^\theta\right] e^{-\lambda_{m+1}(t-\tau)} \mathrm{d}\tau \,.$$

It is evident that $J_m(t,\ s) \le J_m(t,\ -\infty)$. By changing the variable in the integral $\xi = \lambda_{m+1}(t-\tau)$, we obtain

$$J_m(t,\ -\infty) = \lambda_{m+1}^{-1+\theta} \left(1 + \theta^\theta \int_0^\infty \xi^{-\theta} e^{-\xi} \mathrm{d}\xi\right) \equiv \lambda_{m+1}^{-1+\theta}(1+k) \,.$$

Thus, equation (2.12) implies

$$\|u_n(t) - u_m(t)\|_\theta \le \|(P_n - P_m)u_0\|_\theta + \frac{a_1(\theta,\ R)}{\lambda_{m+1}^{1-\theta}} +$$

$$+ a_2(\theta,\ R) \int_s^t (t-\tau)^{-\theta} \|u_n(\tau) - u_m(\tau)\|_\theta \mathrm{d}\tau \ .$$

Hence, if we use Lemma 2.1 which is given below, we can find that

$$\|u_n(t) - u_m(t)\|_\theta \le C\left(\|(P_n - P_m)u_0\|_\theta + \frac{1}{\lambda_{m+1}^{1-\theta}}\right) \tag{2.13}$$

for all $t \in [s, s+T]$, where $C > 0$ is a constant depending on θ, R, and T. It is also evident that estimate (2.13) remains true for $\theta = 0$. It means that the sequence of approximate solutions $\{u_m(t)\}$ is a Cauchy sequence in the space $C(s, s+T; \mathcal{F}_\theta)$. Therefore, there exists an element $u(t) \in C(s, s+T; \mathcal{F}_\theta)$ such that equation (2.11) holds. Estimates (2.10) and (2.11) enable us to pass to the limit in (2.9) and to obtain equation (2.3) for $u(t)$. **Theorem 2.2 is proved**.

— Exercise *2.6* Show that if the hypotheses of Theorem 2.2 hold, then the estimate

$$\max_{[s,\, s+T]} \|u(t) - u_m(t)\|_\alpha \le \frac{a_1}{\lambda_{m+1}^{\theta-\alpha}} \|(1-P_m)u_0\|_\theta + \frac{a_2}{\lambda_{m+1}^{1-\alpha}}$$

is valid with $0 \le \alpha \le \theta$. Here a_1 and a_2 are constants independent of θ, R, and T.

The following assertion provides a simple sufficient condition of the global solvability of problem (2.1).

Theorem 2.3.

Assume that the constant $M(\rho)$ ***in*** (2.2) ***does not depend on*** ρ, ***i.e. the mapping*** $B(u, t)$ ***satisfies the global Lipschitz condition***

$$\|B(u_1, t) - B(u_2, t)\| \le M\|u_1 - u_2\|_\theta \tag{2.14}$$

for all $u_j \in \mathcal{F}_\theta$ ***with some constant*** $M > 0$. ***Then problem*** (2.1) ***has a unique mild solution on the half-interval*** $[s, +\infty)$, ***provided*** $u_0 \in \mathcal{F}_\theta$. ***Moreover, for any two solutions*** u_1 ***and*** u_2 ***the estimate***

$$\|u_1(t) - u_2(t)\|_\theta \le a_1 e^{a_2(t-s)} \|u_1(s) - u_2(s)\|_\theta\,, \qquad t \ge s\,, \tag{2.15}$$

holds, where a_1 ***and*** a_2 ***are constants that depend on*** θ, λ_1, ***and*** M ***only.***

The proof of this theorem is based on the following lemma (see the book by Henry [3], Chapter 7).

Lemma 2.1.

Assume that $\varphi(t)$ *is a continuous nonnegative function on the interval* $(0, T)$ *such that*

$$\varphi(t) \le c_0 t^{-\gamma_0} + c_1 \int_0^t (t-\tau)^{-\gamma_1} \varphi(\tau)d\tau, \qquad t \in (0, T), \tag{2.16}$$

where $c_0, c_1 \ge 0$ *and* $0 \le \gamma_0, \gamma_1 < 1$. *Then there exists a constant* $K = K(\gamma_1, c_1, T)$ *such that*

$$\varphi(t) \le \frac{c_0}{1-\gamma_0} t^{-\gamma_0} K(\gamma_1, c_1, T)\,. \tag{2.17}$$

Proof of Theorem 2.3.

Let $u(t)$ be a solution to problem (2.1) on the maximal half-interval of its existence $[s,\ s+T)$. Assume that $T<\infty$. Condition (2.14) gives us that

$$\|B(u,\ t)\| \ \le\ \|B(0,\ t)\| + M\|u\|_\theta \ \le\ M_0(T) + M\|u\|_\theta$$

for all $u\in\mathcal{F}_\theta$ and $t\in[0,\ T]$. Therefore, from (2.3) and (1.15) we find that

$$\|u(t)\|_\theta \ \le\ \|u_0\|_\theta + \int_s^t\Big[\Big(\frac{\theta}{t-\tau}\Big)^\theta + \lambda_1^\theta\Big](M_0(T)+M\|u(\tau)\|_\theta)d\tau\ .$$

Hence, for $t\in[s,\ s+T]$ we have that

$$\|u(t)\|_\theta \ \le\ C_0(\|u_0\|_\theta,\ T,\ \theta) + C_1(T,\ \theta)\int_s^t(t-\tau)^{-\theta}\|u(\tau)\|_\theta d\tau\,.$$

Therefore, Lemma 2.1 implies that the value $\|u(t)\|_\theta$ is bounded on $[s,\ s+T)$ which is impossible (see Exercise 2.1). Thus, the solution exists for any half-interval $[s,\ s+T)$. For the proof of estimate (2.15) we note that, as above, inequalities (2.3) and (1.15) for the function $w(t)=u_1(t)-u_2(t)$ give us that

$$\|w(t)\|_\theta \ \le\ \|w(s)\|_\theta + \int_s^t\Big[\Big(\frac{\theta}{t-\tau}\Big)^\theta + \lambda_1^\theta\Big]M\|w(\tau)\|_\theta d\tau\,.$$

If we apply Lemma 2.1, we find that

$$\|w(t)\|_\theta \le C(\theta,\ \lambda_1,\ M)\|w(s)\| \quad \text{for} \quad s\le t\le s+1\,.$$

Therefore, the estimate

$$\|w(t)\|_\theta \ \le\ C^{n+1}\|w(s)\|_\theta \ \le\ C\exp\{(t-s)\ln C\}\|w(s)\|_\theta$$

holds for $t=s+n+\sigma$, where n is natural and $0\le\sigma\le 1$. Thus, **Theorem 2.3 is proved**.

Exercise *2.7* Using estimate (1.14), prove that if the hypotheses of Theorem 2.3 hold, then the inequality

$$\|Q_N(u_1(t)-u_2(t))\|_\theta \ \le$$

$$\le\ \Big\{e^{-\lambda_{N+1}(t-s)} + \frac{a_3}{\lambda_{N+1}^{1-\theta}}\,e^{a_2(t-s)}\Big\}\|u_1(s)-u_2(s)\|_\theta \qquad (2.18)$$

is valid for any two solutions $u_1(t)$ and $u_2(t)$. Here $Q_N=I-P_N$ and P_N is the orthoprojector onto the span of $\{e_1,\ \dots\ e_N\}$, the number a_2 is the same as in (2.15) and a_3 depends on λ_1, θ, and M.

Let us consider one more case in which we can guarantee the global solvability of problem (2.1). Assume that condition (2.2) holds for $\theta=1/2$ and

$$B(u) = -B_0(u) + B_1(u,\ t), \tag{2.19}$$

where $B_1(u,\ t)$ satisfies the global Lipschitz condition (2.14) with $\theta = 1/2$ and $B_0(u)$ is a potential operator on the space $V = \mathcal{F}_{1/2}$. This means that there exists a Frechét differentiable functional $F(u)$ on V such that $B_0(u) = F'(u)$, i.e.

$$\lim_{\|v\|_{1/2} \to 0} \frac{1}{\|v\|_{1/2}} \left| F(u+v) - F(u) - (B_0(u),\ v) \right| = 0 .$$

Theorem 2.4.

Let (2.2) ***be valid with*** $\theta = 1/2$ ***and let decomposition*** (2.19) ***take place. Assume that the functional*** $F(u)$ ***is bounded below on*** $V = \mathcal{F}_{1/2}$. ***Then problem*** (2.1) ***has a unique mild solution*** $u(t) \in C([s,\ s+T];\ D(A^{1/2}))$ ***on an arbitrary segment*** $[s,\ s+T]$.

Proof.

Let $u_m(t)$ be an approximate solution to problem (2.1) on a segment $[s,\ s+T]$, where T does not depend on m (see Exercise 2.5):

$$\frac{d}{dt} u_m(t) + A u_m(t) = P_m B(u_m(t),\ t), \qquad u_m(s) = P_m u_0 . \tag{2.20}$$

Multiplying (2.20) by $\dot{u}_m(t) = \frac{d}{dt} u_m(t)$ scalarwise in the space H, we find that

$$\|\dot{u}_m\|^2 + \frac{d}{dt}\left\{ \frac{1}{2} \|A^{1/2} u_m\|^2 + F(u_m) \right\} = (B_1(u_m,\ t),\ \dot{u}_m) \le$$

$$\le \frac{1}{2} \|\dot{u}_m\|^2 + \|B_1(0,\ t)\|^2 + M^2 \|A^{1/2} u_m\|^2 .$$

Since $F(u)$ is bounded below, we obtain that

$$\frac{d}{dt} W(u_m(t)) \ \le\ a W(u_m(t)) + b + \|B_1(0,\ t)\|^2$$

with constants a and b independent of m, where

$$W(u) = \frac{1}{2} \|A^{1/2} u\|^2 + F(u) \ .$$

Therefore, Gronwall's lemma gives us that

$$W(u_m(t)) \le \left(W(u_m(s)) + \frac{b}{a} \right) e^{a(t-s)} + 2 \int_s^t e^{a(t-\tau)} \|B_1(0,\ \tau)\|^2 d\tau$$

for all t in the segment $[s,\ s+T]$ of the existence of approximate solutions. Firstly, this estimate enables us to prove the global existence of approximate solutions (cf. Exercise 2.1). Secondly, by virtue of the continuity of the functional W on $V = \mathcal{F}_{1/2}$ Theorem 2.2 enables us to pass to the limit $m \to \infty$ on an arbitrary segment $[s,\ s+T]$ and prove the global solvability of limit problem (2.1). **Theorem 2.4 is proved**.

— Exercise 2.8 Let $u(t)$ be a mild solution to problem (2.1) such that $\|u(t)\|_\theta \le C_T$ for $s \le t \le s+T$. Use Lemma 2.1 to prove that

$$\|u(t)\|_\alpha \le C_T t^{\theta-\alpha}\|u_0\|_\theta, \qquad t \in [s, s+T] \tag{2.21}$$

for all $\theta \le \alpha < 1$, where $C_T = C_T(\alpha, \theta)$ is a positive constant.

— Exercise 2.9 Let $u(t)$ and $v(t)$ be solutions to problem (2.1) with the initial conditions $u_0, v_0 \in \mathcal{F}_\theta$ and such that $\|u(t)\|_\theta + \|v(t)\|_\theta \le C_T$ for t lying in a segment $[s, s+T]$. Then

$$\|u(t)-v(t)\|_\alpha \le C_T(\theta, \alpha)t^{\theta-\alpha}\|u_0-v_0\|_\theta,$$
$$t \in [s, s+T], \qquad \theta \le \alpha < 1.$$

Thus, if $B(u, t) \equiv B(u)$ and the hypotheses of Theorem 2.3 or 2.4 hold, then equation (2.1) generates a dynamical system $(\mathcal{F}_\theta, S_t)$ with the evolutionary operator S_t which is defined by the equality $S_t u_0 = u(t)$, where $u(t)$ is the solution to problem (2.1). The semigroup property of S_t follows from the assertion on the uniqueness of solution.

— Exercise 2.10 Show that Theorem 2.4 holds even if we replace the assumption of semiboundedness of $F(u)$ by the condition $F(u) \ge -\alpha\|A^{1/2}u\|^2 - \beta$ for some $\alpha < 1/2$ and $\beta > 0$.

§ 3 Examples

Here we consider several examples of an application of theorems of Section 2. Our presentation is brief here and is organized in several cycles of exercises. More detailed considerations as well as other examples can be found in the books by Henry, Babin and Vishik, and Temam from the list of references to Chapter 2 (see also Sections 6 and 7 of this chapter).

We first remind some definitions and notations. Let Ω be a domain in $\mathbb{R}^d$ $(d \ge 1)$. The Sobolev space $H^m(\Omega)$ of the order m $(m=0, 1, 2, \dots)$ is defined by the formula

$$H^m(\Omega) = \{f \in L^2(\Omega): \ D^j f \in L^2(\Omega), \ |j| \le m\},$$

where $j = (j_1, j_2, \dots, j_d)$, $j_k = 0, 1, 2, \dots$, $|j| = j_1 + \dots + j_d$,

$$D^j f = \partial_{x_1}^{j_1}\partial_{x_2}^{j_2}\dots\partial_{x_d}^{j_d} f(x), \qquad x = (x_1, \dots, x_d).$$

The space $H^m(\Omega)$ is a separable Hilbert space with the inner product

$$(u, v)_m = \sum_{|j| \le m} \int_{\Omega} D^j u(x)\, D^j v(x) \mathrm{d}x \ .$$

Below we also use the space $H_0^m(\Omega)$ which is constructed as the closure in $H^m(\Omega)$ of the set $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support. For more detailed information we refer the reader to the handbooks on the theory of Sobolev spaces.

Example *3.1* (nonlinear heat equation)

$$\begin{cases} \partial_t u = \nu \Delta u + f(t, x, u, \nabla u), \quad x \in \Omega, \ t > 0, \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0 . \end{cases} \tag{3.1}$$

Here Ω is a bounded domain in $\mathbb{R}^d$, Δ is the Laplace operator, and ν is a positive constant. Assume that $f(t, x, u, p)$ is a continuous function of its variables which satisfies the Lipschitz condition

$$|f(t, x, u, p) - f(t, x, u, q)| \le K\left(|u_1 - u_2|^2 + \sum_{j=1}^{d} |p_j - q_j|^2\right)^{1/2} \tag{3.2}$$

with an absolute constant K. It is clear that the operator $B(u, t)$ defined by the formula

$$B(u, t)(x) = f(t, x, u(x), \nabla u(x)),$$

can be estimated as follows:

$$\|B(u_1, t) - B(u_2, t)\| \le K\left(\|u_1 - u_2\|^2 + \sum_j \|\partial_{x_j}(u_1 - u_2)\|^2\right)^{1/2}. \tag{3.3}$$

Here and below $\|\cdot\|$ is the norm in the space $H = L^2(\Omega)$. It is well-known that the operator $A = -\Delta$ with the Dirichlet boundary condition on $\partial\Omega$ is a positive operator with discrete spectrum. Its domain is $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, $D(A^{1/2}) = H_0^1(\Omega)$. We also note that

$$\|A^{1/2} u\|^2 = (Au, u) = \sum_{j=1}^{d} \int_{\Omega} \left(\frac{\partial u}{\partial x_j}\right)^2 \mathrm{d}x \ , \qquad u \in D(A^{1/2}) = H_0^1(\Omega).$$

Therefore, equation (3.3) for $B(u, t)$ gives us the estimate

$$\|B(u_1, t) - B(u_2, t)\| \le K\left(\frac{1}{\lambda_1} + 1\right)^{1/2} \|A^{1/2} u\|,$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary condition on $\partial\Omega$. Therefore, we can apply Theorem 2.3 with $\theta = 1/2$ to problem (3.1). This theorem guarantees the existence and uniqueness of a mild solution to problem (3.1) in the space $C(\mathbb{R}_+, H_0^1(\Omega))$.

Exercise 3.1 Assume that $f(t, x, u, p) \equiv f(t, x, u)$ is a continuous function of its arguments satisfying the global Lipschitz condition with respect to the variable u. Prove the global theorem on the existence of mild solutions to problem (3.1) in the space $H = L^2(\Omega)$.

Example 3.2

Let us consider problem (3.1) in the case of one spatial variable:

$$\begin{cases} \partial_t u = \nu \partial_x^2 u - g(x, u) + f(t, x, u, \partial_x u), & t > 0,\ x \in (0, 1) \\ u|_{x=0} = u|_{x=1} = 0, \quad u|_{t=0} = u_0 . \end{cases} \tag{3.4}$$

Assume that $g(x, u)$ is a continuously differentiable function with respect to the variable u and $|g'_u(x, u)| \le h(u)$, where $h(u)$ is a function bounded on every compact set of the real axis. We also assume that the function $f(t, x, u, p)$ is continuous and possesses property (3.2). For any element $u \in D(A^{1/2}) = H_0^1(0, 1)$ the following estimates hold:

$$\max_{[0, 1]} |u(x)| \le \|u'\|_{L^2(0,1)} \quad \text{and} \quad \|u\|_{L^2(0, 1)} \le \|u'\|_{L^2(0, 1)},$$

where $u' = \partial_x u$. Therefore, it is easy to find that the inequality

$$\|B(u_1, t) - B(u_2, t)\| \le M(\rho)\|A^{1/2}(u_1 - u_2)\| \tag{3.5}$$

is valid for

$$B(u, t) = -g(x, u(x)) + f(t, x, u(x), u'(x)),$$

provided $\|A^{1/2}u_j\| \equiv \|u'_j\| \le \rho$. Here $\|\cdot\|$ is the norm in the space $H = L^2(0, 1)$ and $M(\rho) = \sup\{h(\xi):\ |\xi| < \rho\} + K\sqrt{2}$. Equation (3.5) and Theorem 2.1 guarantee the local solvability of problem (3.4) in the space $H_0^1(\Omega)$. Moreover, if the function

$$\mathcal{F}(x, y) = \int_0^y g(x, \xi)\mathrm{d}\xi$$

is bounded below, then we can use Theorem 2.4 to obtain the assertion on the existence of mild solutions to problem (3.4) on an arbitrary segment $[0, T]$.

It should be noted that the reasoning in Example 3.2 is also valid for several spatial variables. However, in order to ensure the fulfilment of the estimate of the form (3.5) one should impose additional conditions on the growth of the function $h(u)$. For example, we can require that the equation

$$h(u) \le C_1 + C_2|u|^p$$

be fulfilled, where $p \le 2/(d-2)$ if $d > 2$ and p is an arbitrary number if $d = 2$. In this case the inequality of the form (3.5) follows from the Hölder inequality and the continuity of the embedding of the space $H^1(\Omega)$ into $L^q(\Omega)$, where $q = 2d/(d-2)$ if $d = \dim \Omega > 2$ and q is an arbitrary number if $d = 2$, $q \geq 1$.

The results shown in Examples 3.1 and 3.2 also hold for the systems of parabolic equations. For example, a system of reaction-diffusion equations

$$\begin{cases} \partial_t u = \nu \Delta u + f(t, u, \nabla u), \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x), \end{cases} \tag{3.6}$$

can be considered in a smooth bounded domain $\Omega \subset \mathbb{R}^d$. Here $u = (u_1, u_2, \ldots, u_m)$ and $f(t, u, \xi)$ is a continuous function from $\mathbb{R}_+ \times \mathbb{R}^{(m+1)d}$ into $\mathbb{R}^m$ such that

$$|f(t, u, \xi) - f(t, v, \eta)| \leq M(|u-v| + |\xi - \eta|), \tag{3.7}$$

where M is a constant, $u, v \in \mathbb{R}^m$, $\xi, \eta \in \mathbb{R}^{md}$ and n is an outer normal to $\partial\Omega$.

Exercise *3.2* Prove the global theorem on the existence and uniqueness of mild solutions to problem (3.6) in $\mathcal{F}_{1/2} = [H^1(\Omega)]^m \equiv H^1(\Omega) \times \ldots \times H^1(\Omega)$.

Example *3.3* (nonlocal Burgers equation).

$$\begin{cases} u_t - \nu u_{xx} + (\omega, u) u_x = f(t), \quad 0 < x < l, \ t > 0 \\ u|_{x=0} = u|_{x=l} = 0, \quad u|_{t=0} = u_0. \end{cases} \tag{3.8}$$

Here $f(t)$ is a continuous function with the values in $L^2(0, l)$,

$$\omega \in L^2(0, l), \qquad (\omega, u) = \int_0^l \omega(x) u(x, t) \mathrm{d}x$$

and ν is a positive parameter. Exercises 3.3–3.6 below answer the question on the solvability of problem (3.8).

Exercise *3.3* Prove the local existence of mild solutions to problem (3.8) in the space $\mathcal{F}_{1/2} = H_0^1(0, l)$. *Hint*:

$$A = -\nu \partial_{xx}^2, \quad D(A) = H^2(0, l) \cap H_0^1(0, l),$$

$$B(u) = -(\omega, u) u_x + f(t).$$

Exercise *3.4* Consider the Galerkin approximate solutions to problem (3.8)

$$u_m(t) \equiv u_m(t, x) = \sqrt{\frac{2}{l}} \cdot \sum_{k=1}^m g_k(t) \sin \frac{\pi k}{l} x.$$

Write out the system of ordinary differential equations to determine $\{g_k(t)\}$. Prove that this system is locally solvable.

— Exercise *3.5* Prove that the equations

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|^2 + \nu\|\partial_x u_m(t)\|^2 = (f(t),\ u_m(t)) \tag{3.9}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u_m(t)\|^2 + \nu\|\partial^2_{xx} u_m(t)\|^2 \le$$
$$\le \frac{2}{\nu}\left(|(\omega,\ u)|^2\|\partial_x u_m(t)\|^2 + \|f(t)\|^2\right) \tag{3.10}$$

are valid for any interval of the existence of the approximate solution $u_m(t)$. Here $\|\cdot\|$ is the norm in $L^2(0,\ l)$.

— Exercise *3.6* Use equations (3.9) and (3.10) to prove the global existence of the Galerkin approximate solutions to problem (3.8) and to obtain the uniform estimate of the form

$$\|\partial_x u_m(t)\| \le C(\|\partial_x u_0\|,\ T), \qquad t \in [0,\ T] \tag{3.11}$$

for any $T > 0$.

Thus, Theorem 2.2 guarantees the global existence and uniqueness of weak solutions to problem (3.8) in $\mathscr{F}_{1/2} = H^1_0(0,\ l)$.

— Example *3.4* (Cahn-Hilliard equation).

$$\begin{cases} u_t + \nu\partial_x^4 u - \partial_x^2(u^3 + a u^2 + b u) = 0, \quad x \in (0,\ l), \quad t > 0, \\ \partial_x u\big|_{x=0} = \partial_x^3 u\big|_{x=0} = 0, \quad \partial_x u\big|_{x=l} = \partial_x^3 u\big|_{x=l} = 0, \\ u\big|_{t=0} = u_0(x), \end{cases} \tag{3.12}$$

where $\nu > 0$, a, and $b \in \mathbb{R}$ are constants. The result of the cycle of Exercises 3.7–3.10 is a theorem on the existence and uniqueness of solutions to problem (3.12).

— Exercise *3.7* Prove that the estimate

$$\|\partial_x^2(u\cdot v)\|^2 \le C(\|u\|^2 + \|\partial_x^2 u\|^2)(\|v\|^2 + \|\partial_x^2 v\|^2)$$

is valid for any two functions $u(x)$ and $v(x)$ smooth on $[0,\ l]$. Use this estimate to ascertain that problem (3.12) is locally uniquely solvable in the space

$$V = \left\{u \in H^2(0,\ l) :\ u_x\big|_{x=0} = u_x\big|_{x=l} = 0\right\}$$

(*Hint*: $\nu\partial_x^4 + 1 \mapsto A$, $V = D(A^{1/2})$).

Let us consider the Galerkin approximate solution $u_m(t)$ to problem (3.12):

$$u_m(t) \equiv u_m(t,\ x) = \frac{1}{\sqrt{l}}\, g_0(t) + \sqrt{\frac{2}{l}} \cdot \sum_{k=1}^{m} g_k(t) \cos\frac{\pi k}{l} x .$$

Exercise *3.8* Prove that the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} W(u_m(t)) + \int_0^l \Big(\partial_x [\nu\, \partial_x^2 u_m(t,\ x) - p\,(u_m(t,\ x))]\Big)^2 \mathrm{d}x = 0 \quad (3.13)$$

holds for any interval of the existence of approximate solutions $\{u_m(t)\}$. Here $p\,(u) = u^3 + a\,u^2 + b$ and

$$W(u) = \int_0^l \Big(\frac{\nu}{2} u_x^2 + \frac{1}{4} u^4 + \frac{a}{3} u^3 + \frac{b}{2} u^2\Big) \mathrm{d}x . \qquad (3.14)$$

In particular, equation (3.13) implies that approximate solutions exist for any segment $[0,\ T]$. For $u_0 \in V$ they can be estimated as follows:

$$\|\partial_x u_m(t)\| + \max_{x\,\in\,[0,\ l]} |u_m(t,\ x)| \ \le\ C_T\,, \qquad t \in [0,\ T]\,, \qquad (3.15)$$

where the number C_T does not depend on m.

Exercise *3.9* Using (3.15) show that the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\partial_x^2 u_m(t)\big\|^2 + \big\|\partial_x^4 u_m(t)\big\|^2 \ \le\ C_T(1 + \big\|\partial_x^2 u_m(t)\big\|^2)\,, \quad t \in (0,\ T]$$

holds for any interval $(0, T)$ and for any approximate solution $u_m(t)$ to problem (3.12). (*Hint*: first prove that $\|u'\|^2_{L^4(0,\,l)} \le$ $\le C \max\limits_{[0,\ l]} |u(x)| \cdot \|u'\|_{L^2(0,\ l)}$).

Exercise *3.10* Using Theorem 2.2 and the result of the previous exercise, prove the global theorem on the existence and uniqueness of weak solutions to the Cahn-Hilliard equation (3.12) in the space V (see Exercise 3.7).

Example *3.5* (abstract form of two-dimensional system of Navier-Stokes equations)

In a separable Hilbert space H we consider the evolutionary equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + b(u,\ u) = f(t)\,, \qquad u\big|_{t=0} = u_0\,, \qquad (3.16)$$

where A is a positive operator with discrete spectrum, ν is a positive parameter and $b(u, v)$ is a bilinear mapping from $D(A^{1/2}) \times D(A^{1/2})$ into $\mathcal{F}_{-1/2} = D(A^{-1/2})$ possessing the property

$$(b(u, v), v) = 0 \quad \text{for all} \quad u, v \in D(A^{1/2}) \tag{3.17}$$

and such that for all $u, v \in D(A)$ the estimates

$$\|b(u, v)\| \le C_\delta \|A^{1/2-\delta} u\| \cdot \|A^{1/2+\delta} v\|, \qquad 0 < \delta < \frac{1}{2} \tag{3.18}$$

and

$$\|A^\beta b(u, v)\| \le C_{\delta, \beta} \|A^{(1/2)+\delta} u\| \cdot \|A^{1/2+\beta} v\|, \qquad 0 \le \beta \le \frac{1}{2}, \ 0 < \delta \le \frac{1}{2} \tag{3.19}$$

hold. We also assume that $f(t)$ is a continuous function with the values in H.

Exercise *3.11* Prove that Theorem 2.1 on the local solvability with $\theta = 1/2 + \beta$ is applicable to problem (3.16), where β is a number from the interval $(0, 1/2)$.

Let $\{e_k\}$ be the basis of eigenvectors of the operator A, let $0 < \lambda_1 \le \lambda_2 \le \ldots$ be the corresponding eigenvalues and let P_m be the orthoprojector onto the span of $\{e_1, \ldots, e_m\}$. We consider the Galerkin approximations of problem (3.16):

$$\begin{cases} \dfrac{d}{dt} u_m(t) + \nu A u_m(t) + P_m b(u_m(t), u_m(t)) = P_m f(t), \\ u_m(0) = P_m u_0 . \end{cases} \tag{3.20}$$

Exercise *3.12* Show that the estimates

$$\|u_m(t)\|^2 + \nu \int_0^t \|A^{1/2} u_m(\tau)\|^2 d\tau \le \|u_0\|^2 + \frac{1}{\lambda_1 \nu} \int_0^t \|f(\tau)\|^2 d\tau \tag{3.21}$$

and

$$\|u_m(t)\|^2 \le \|u_m(0)\|^2 e^{-\lambda_1 \nu t} + \frac{1}{\lambda_1}\left(\frac{F}{\nu}\right)^2 (1 - e^{-\lambda_1 \nu t}) \tag{3.22}$$

are valid for an arbitrary interval of the existence of solutions to problem (3.20). Here $F = \sup_{t \ge 0} \|A^{-1/2} f(t)\|$. Using these estimates, prove the global solvability of problem (3.20).

Exercise *3.13* Show that the estimate

$$\frac{d}{dt}\|A^{1/2} u_m(t)\|^2 + \nu \|A u_m(t)\|^2 \le \le \frac{2}{\nu}(\|b(u_m(t), u_m(t))\|^2 + \|f(t)\|^2) \tag{3.23}$$

holds for a solution to problem (3.20).

Using the interpolation inequality (see Exercise 1.13) and estimate (3.18), it is easy to find that

$$\|b(u,\ u)\|^2 \ \le\ C\|u\|\cdot\|A^{1/2}u\|^2\|Au\| \ \le\ \left(\frac{C}{\nu}\right)^2\|u\|^2\|A^{1/2}u\|^4+\frac{\nu^2}{4}\|Au\|^2 .$$

Therefore, equation (3.23) implies that

$$\frac{d}{dt}\left\|A^{1/2}u_m(t)\right\|^2 \ \le\ \sigma_m(t)\left\|A^{1/2}u_m(t)\right\|^2+\frac{2}{\nu}\|f(t)\|^2 , \tag{3.24}$$

where $\sigma_m(t)=(2C^2/\nu^3)\|u_m(t)\|^2\cdot\left\|A^{1/2}u_m(t)\right\|^2$. Hence, Gronwall's inequality gives us that

$$\left\|A^{1/2}u_m(t)\right\|^2 \ \le\ \left\|A^{1/2}u_0\right\|^2\exp\left\{\int_0^t\sigma_m(\tau)d\tau\right\}+\frac{2}{\nu}\int_0^t\|f(\tau)\|^2\exp\left\{\int_0^t\sigma_m(\xi)d\xi\right\}d\tau .$$

It follows from equation (3.21) that the value $\int_0^t\sigma_m(\tau)d\tau$ is uniformly bounded with respect to m on an arbitrary segment $[0,\ T]$. Consequently, the uniform estimates

$$\left\|A^{1/2}u_m(t)\right\| \le C_T , \qquad t\in[0,\ T] , \quad u_0\in D(A^{1/2}) \tag{3.25}$$

are valid for any $T>0$ and $m=1,\ 2,\ \dots$.

Exercise *3.14* Using equations (3.23) and (3.25), prove that if $u_0\in$ $\in D(A^{1/2})$, then

$$\frac{d}{dt}\left\|A^{1/2}u_m(t)\right\|^2+\frac{\nu}{2}\left\|Au_m(t)\right\|^2 \ \le\ C_T , \quad t\in[0,\ T]$$

for any $T>0$.

Exercise *3.15* Let $0<\beta<\frac{1}{2}$ and let $\sup\limits_{[0,\ T]}\left\|A^{\beta}f(t)\right\|^2\le C_T$. Prove that

$$\frac{d}{dt}\left\|A^{1/2+\beta}u_m(t)\right\|^2+\nu\left\|A^{1+\beta}u_m(t)\right\|^2 \ \le$$

$$\le\ C_1\left\|A^{\beta}b(u_m(t),\ u_m(t))\right\|^2+C_T \ .$$

Exercise *3.16* Use the results of Exercises 3.14 and 3.15 and inequality (3.19) to prove the global existence of mild solutions to problem (3.16) in the space $D(A^{1/2+\beta})$, provided that $u_0\in D(A^{1/2+\beta})$ and $f(t)$ is a continuous function with the values in $D(A^{\beta})$. Here β is an arbitrary number from the interval $(0,\ 1/2)$.

§ 4 Existence Conditions and Properties of Global Attractor

In this section we study the asymptotic properties of the dynamical system generated by the autonomous equation

$$\frac{du}{dt} + Au = B(u), \qquad u\big|_{t=0} = u_0, \tag{4.1}$$

where, as before, A is a positive operator with discrete spectrum and $B(u)$ is a nonlinear mapping from $\mathcal{F}_\theta$ into H such that

$$\|B(u_1) - B(u_2)\| \le M(\rho)\|A^\theta(u_1 - u_2)\| \tag{4.2}$$

for all $u_1, u_2 \in \mathcal{F}_\theta = D(A^\theta)$ possessing the property $\|A^\theta u_j\| \le \rho$, $0 \le \theta < 1$. We assume that problem (4.1) has a unique mild (in $\mathcal{F}_\theta$) solution on $\mathbb{R}_+$ for any $u_0 \in \mathcal{F}_\theta$. The theorems that guarantee the fulfilment of this requirement and also some examples are given in Sections 2 and 3 of this chapter. It should be also noted that in this section we use some results of Chapter 1 for the proof of main assertions. Further triple numeration is used in the references to the assertions and formulae of Chapter 1 (first digit is the chapter number).

Let $(\mathcal{F}_\theta, S_t)$ be a dynamical system with the evolutionary operator S_t defined by the formula $S_t u_0 = u(t)$, where $u(t)$ is a mild solution to problem (4.1). As shown in Chapter 1, for the system $(\mathcal{F}_\theta, S_t)$ to possess a compact global attractor, it should be dissipative. It turns out that the condition of dissipativity is not only necessary, but also sufficient in the class of systems considered.

Lemma 4.1.

Let $(\mathcal{F}_\theta, S_t)$ be dissipative and let $B_0 = \{u: \|A^\theta u\| \le R_0\}$ be its absorbing ball. Then the set $B_\alpha = \{u: \|A^\alpha u\| \le R_\alpha\}$ is absorbing for all $\alpha \in (\theta, 1)$, where

$$R_\alpha = (\alpha - \theta)^{\alpha - \theta} R_0 + \frac{\alpha^\alpha}{1-\alpha} \sup\{\|B(u)\|: u \in \mathcal{F}_\theta, \|A^\theta u\| \le R_0\}. \tag{4.3}$$

Proof.

Using equation (2.3) and estimate (1.17), we have

$$\|A^\alpha u(t+1)\| \le (\alpha - \theta)^{\alpha - \theta}\|A^\theta u(t)\| + \int_t^{t+1} \left(\frac{\alpha}{t+1-\tau}\right)^\alpha \|B(u(t))\| dt,$$

where $u(t) = S_t u_0$. Let B be a bounded set in $\mathcal{F}_\theta$. Then $\|S_t y\|_\theta \le R_0$ for all $t \ge t_B$ and $y \in B$. Therefore, the estimate

$$\|B(u(t))\| \le \sup\{\|B(u)\|: u \in B_0\}, \quad t \ge t_B$$

holds for $u(t) = S_t y$. Hence, $\|A^\alpha u(t+1)\| \le R_\alpha$ for all $t \ge t_B$. This implies the assertion of the lemma.

Theorem 4.1.

Assume that the dynamical system $(\mathcal{F}_\theta,\ S_t)$ *generated by problem* (4.1) *is dissipative. Then it is compact and possesses a connected compact global attractor* $\mathcal{A}$. *This attractor is a bounded set in* $\mathcal{F}_\alpha$ *for* $\theta \le \alpha < 1$ *and has a finite fractal dimension.*

Proof.

By virtue of Theorem 1.1 the space $\mathcal{F}_\alpha$ is compactly embedded into $\mathcal{F}_\theta$ for $\alpha > \theta$. Therefore, Lemma 4.1 implies that the dynamical system $(\mathcal{F}_\theta,\ S_t)$ is compact. Hence, Theorem 1.5.1 gives us that $(\mathcal{F}_\theta,\ S_t)$ possesses a connected compact global attractor $\mathcal{A}$. Evidently, $\|A^\alpha u\| \le R_\alpha$ for all $u \in \mathcal{A}$, where R_α is defined by equality (4.3). Thus, we should only establish the finite dimensionality of the attractor. Let us apply the method used in the proof of Theorem 1.9.1 and based on Theorem 1.8.1. Let $S_t u_1$ and $S_t u_2$ be semitrajectories of the dynamical system $(\mathcal{F}_\theta,\ S_t)$ such that $\|S_t u_j\|_\theta \le R$ for all $t \ge 0$, $j = 1,\ 2$. Then equations (2.3) and (1.17) give us that

$$\|S_t u_1 - S_t u_2\|_\theta \ \le\ \|u_1 - u_2\|_\theta + M(R)\theta^\theta \int_0^t (t-\tau)^{-\theta} \|S_t u_1 - S_t u_2\|_\theta \mathrm{d}t\ .$$

Using Lemma 2.1, we find that for all $u_j \in \mathcal{F}_\theta$ such that $\|S_t u_j\|_\theta \le R$ the following estimate holds:

$$\|S_t u_1 - S_t u_2\|_\theta \le C\|u_1 - u_2\|_\theta \quad \text{for} \quad 0 \le t \le 1\,,$$

where the constant C depends on θ and R. Therefore,

$$\|S_t u_1 - S_t u_2\|_\theta \ \le\ C\|S_\tau u_1 - S_\tau u_2\|_\theta \quad \text{for} \quad 0 \le \tau \le t \le \tau + 1$$

for the considered u_1 and u_2. Consequently,

$$\|S_t u_1 - S_t u_2\|_\theta \ \le\ C\|S_{[t]} u_1 - S_{[t]} u_2\|_\theta \ \le\ C^{[t]+1}\|u_1 - u_2\|_\theta\,,$$

where $[t]$ is an integer part of the number t. Thus, the estimate of the form

$$\|S_t u_1 - S_t u_2\|_\theta \le C e^{at}\|u_1 - u_2\|_\theta \tag{4.4}$$

is valid, where the constants C and a depend on θ and R. Similarly, Lemma 1.1 gives

$$\|Q_N(S_t u_1 - S_t u_2)\|_\theta \ \le\ e^{-\lambda_{N+1} t}\|u_1 - u_2\|_\theta +$$

$$+ M(R)\int_0^t \left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{N+1}^\theta\right] e^{-\lambda_{N+1}(t-\tau)} \|S_\tau u_1 - S_\tau u_2\|_\theta \mathrm{d}\tau \tag{4.5}$$

for all $u_1,\ u_2 \in \mathcal{F}_\theta$ such that $\|S_t u_j\|_\theta \le R$, where $Q_N = I - P_N$ and P_N is the orthoprojector onto the first N eigenvectors of the operator A. If we substitute equation (4.4) in the right-hand side of inequality (4.5), then we obtain that

$$\|Q_N(S_t u_1 - S_t u_2)\|_\theta \le (e^{-\lambda_{N+1} t} + CM(R)\, e^{at} J_N(t))\|u_1 - u_2\|_\theta ,$$

where

$$J_N(t) = \int_0^t \left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{N+1}^\theta\right] e^{-\lambda_{N+1}(t-\tau)} d\tau .$$

After the change of variable $\xi = \lambda_{N+1}(t-\tau)$ it is easy to find that $J_N(t) \le$ $\le C_0 \lambda_{N+1}^{-1+\theta}$. Therefore,

$$\|Q_N(S_t u_1 - S_t u_2)\|_\theta \le \left(e^{-\lambda_{N+1} t} + \frac{C(R,\ \theta)}{\lambda_{N+1}^{1-\theta}}\, e^{at}\right)\|u_1 - u_2\|_\theta \tag{4.6}$$

for all $u_1,\ u_2 \in \mathcal{F}_\theta$ such that $\|S_t u_j\|_\theta \le R$. Hence, there exist $t_0 > 0$ and N such that

$$\|Q_N(S_{t_0} u_1 - S_{t_0} u_2)\|_\theta \le \delta \|u_1 - u_2\|_\theta , \qquad 0 < \delta < 1 ,$$

for all $u_1,\ u_2 \in \mathcal{F}_\theta$ such that $\|S_t u_j\|_\theta \le R$. However, the attractor $\mathcal{A}$ lies in the absorbing ball B_0 . Therefore, this estimate and inequality (4.4) mean that the hypotheses of Theorem 1.8.1 hold for the mapping $V = S_{t_0}$. Hence, the attractor $\mathcal{A}$ has a finite fractal dimension. It can be estimated with the help of the parameters in inequalities (4.4) and (4.6). Thus, **Theorem 4.1 is proved**.

Equations (4.4) and (4.6) which are valid for any $R > 0$ enable us to prove the existence of a fractal exponential attractor of the dynamical system $(\mathcal{F}_\theta,\ S_t)$ in the same way as in Section 9 of Chapter 1.

Theorem 4.2.

Assume that the dynamical system $(\mathcal{F}_\theta,\ S_t)$ ***generated by problem*** (4.1) ***is dissipative. Then it possesses a fractal exponential attractor (inertial set).***

Proof.

It is sufficient to verify that the hypotheses of Theorem 1.9.2 (see (1.9.12)–(1.9.14)) hold for $(\mathcal{F}_\theta,\ S_t)$. Let us show that

$$K = \bigcup_{t \ge t_0} S_t B_\alpha , \qquad B_\alpha = \{u:\ \|A^\alpha u\| \le R_\alpha\}$$

can be taken for the compact K in (1.9.12)–(1.9.14). Here α is a number from the interval $(\theta,\ 1)$ and R_α is defined by formula (4.3). We choose the parameter $t_0 = t_0(B_\alpha)$ such that $S_t K \subset B_\alpha$ for $t \ge t_0$. Since K is a bounded invariant set, equation (4.4) is valid for any $u_1,\ u_2 \in K$ with some constants C and a . It is also easy to verify that K is a compact. Indeed, let $\{k_n\} \subset K$. Then $k_n = S_{t_n} y_n$, therewith we can assume that $k_n \to w$, $y_n \to y \in B_\alpha$ and either $t_n \to t_* < \infty$ or $t_n \to \infty$. In the first case with the help of (4.4) we have

$$\left\|S_{t_n} y_n - S_{t_*} y\right\|_\theta \le C e^{a t_n} \|y_n - y\|_\theta + \left\|S_{t_n} y - S_{t_*} y\right\|_\theta .$$

Therefore, $k_n = S_{t_n} y_n \to S_{t_*} y \in K$. In the second case

$$w = \lim_{n \to \infty} S_{t_n} y_n \in \omega(B_\alpha) \subset S_{t_0}(B_\alpha) \subset K .$$

Here $\omega(B_\alpha)$ is the omega-limit set for the semitrajectories emanating from B_α. Thus, K is a compact invariant absorbing set. In particular this means that condition (1.9.12) is fulfilled, therewith we can take any number for $\gamma > 0$. Conditions (1.9.13) and (1.9.14) follow from equations (4.4) and (4.6). Consequently, it is sufficient to apply Theorem 1.9.2 **to conclude the proof of Theorem 4.2**.

Thus, the dissipativity of the dynamical system $(\mathcal{F}_\theta,\ S_t)$ generated by problem (4.1) guarantees the existence of a finite-dimensional global attractor and an inertial set. Under some additional conditions concerning $B(u)$ the requirement of dissipativity can be slightly weakened. We give the following definition. Let $\alpha \le \theta$. The dynamical system $(\mathcal{F}_\theta,\ S_t)$ is said to be $\mathcal{F}_\alpha$-***dissipative*** if there exists $R^*_\alpha > 0$ such that for any set B bounded in $\mathcal{F}_\alpha$ there exists $t_0 = t_0(B)$ such that

$$\left\|A^\alpha S_t y\right\| \equiv \left\|S_t y\right\|_\alpha \le R^*_\alpha \quad \text{for all} \quad y \in B \cap \mathcal{F}_\theta \ \text{ and } \ t \ge t_0 .$$

Lemma 4.2.

Assume that $B(u)$ satisfies the global Lipschitz condition

$$\|B(u_1) - B(u_2)\| \le M \left\|A^\theta (u_1 - u_2)\right\| . \tag{4.7}$$

Let the dynamical system $(\mathcal{F}_\theta,\ S_t)$ generated by mild solutions to problem (4.1) be $\mathcal{F}_\alpha$-dissipative for some $\alpha \in (\theta - 1,\ \theta]$. Then $(\mathcal{F}_\theta,\ S_t)$ is a compact dynamical system, i.e. it possesses an absorbing set which is compact in $\mathcal{F}_\theta$.

Proof.

By virtue of Lemma 4.1 it is sufficient to verify that the system $(\mathcal{F}_\theta,\ S_t)$ is dissipative (i.e. $\mathcal{F}_\theta$-dissipative). If we use expression (2.3) and equation (1.17), then we obtain

$$\|A^\theta u(t+s)\| \le \left(\frac{\theta - \alpha}{s}\right)^{\theta - \alpha} \|A^\alpha u(t)\| + \int_t^{t+s} \left(\frac{\theta}{t + s - \tau}\right)^\theta \|B(u(\tau))\| \, d\tau$$

for positive t and s. Here $u(t) = S_t u_0$. Since $\|B(u)\| \le \|B(0)\| + M\|A^\theta u\|$, we have the estimate

$$\|A^\theta u(t+s)\| \le \left(\frac{1}{s}\right)^{\theta - \alpha} \left\{ (\theta - \alpha)^{\theta - \alpha} \|A^\alpha u(t)\| + \|B(0)\| \frac{\theta^\theta}{1 - \theta} \right\} +$$

$$+M\int_0^s\left(\frac{\theta}{s-\tau}\right)^{\theta}\|A^{\theta}u(t+\tau)\|\mathrm{d}\tau$$

for $0 \le s \le 1$. Hence, we can apply Lemma 2.1 to obtain

$$\|A^{\theta}u(t+s)\| \le C(\theta,\ \alpha,\ M)(1+\|A^{\alpha}u(t)\|)\,s^{\alpha-\theta}, \quad 0 \le s \le 1 .$$

Therefore, if $\|S_t u_0\|_{\alpha} \le R^*_{\alpha}$ for $t \ge t_0(B)$ and $u_0 \in B \cap \mathcal{F}_{\theta}$, then the latter inequality gives us that

$$\|S_t u_0\|_{\theta} \le C(\theta,\ \alpha,\ M)(1+R^*_{\alpha}) \quad \text{for} \quad t \ge 1+t_0(B),$$

i.e. $(\mathcal{F}_{\theta},\ S_t)$ is a dissipative system. Lemma 4.2 is proved.

Exercise *4.1* Show that the assertion of Lemma 4.2 holds if instead of (4.7) we suppose that $B(u)=B_1(u)+B_2(u)$, where $B_1(u)$ possesses property (4.7) and $B_2(u)$ is such that

$$\sup\{B_2(u):\ \|A^{\alpha}u\| \le R^*_{\alpha}\} < \infty .$$

The following assertion contains a sufficient condition of dissipativity of the dynamical system generated by problem (4.1).

Theorem 4.3.

Assume that condition (4.2) ***is fulfilled with*** $\theta=1/2$ ***and*** $B(u)=-F'(u)$ ***is a potential operator from*** $\mathcal{F}_{1/2}$ ***into*** H ***(the prime stands for the Frechét derivative). Let***

$$F(u) \ge -\alpha, \qquad (F'(u),\ u)-\beta F(u) \ge -\gamma\|A^{1/2}u\|^2-\delta \tag{4.8}$$

for all $u \in \mathcal{F}_{1/2}$, ***where*** $\alpha,\ \beta,\ \gamma,$ ***and*** δ ***are real parameters, therewith*** $\beta > 0$ ***and*** $\gamma < 1$. ***Then the dynamical system is dissipative in*** $\mathcal{F}_{1/2}$.

Proof.

In view of Theorem 2.4 conditions (4.8) guarantee the existence of the evolutionary operator S_t. Let us verify the dissipativity. As in the proof of Theorem 2.4 we consider the Galerkin approximations $\{u_m(t)\}$. It is evident that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|^2+\left\|A^{1/2}u_m(t)\right\|^2+(F'(u_m),\ u_m)=0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|A^{1/2}u_m(t)\right\|^2+F(u_m(t))\right)+\left\|\dot{u}_m(t)\right\|^2=0 .$$

If we add these equations and use (4.8), then we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\|u_m\|^2+\frac{1}{2}\left\|A^{1/2}u_m\right\|^2+F(u_m)\right\}+(1-\gamma)\left\|A^{1/2}u_m\right\|^2+\beta F(u_m) \le \delta .$$

Let

$$V(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|A^{1/2}u\|^2 + F(u) + \alpha\,.$$

Therefore, it is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(u_m(t)) + \omega V(u_m(t)) \;\le\; C$$

with some positive constants ω and C that do not depend on m. This (cf. Exercise 1.4.1) easily implies the dissipativity of the system $(\mathcal{F}_{1/2},\ S_t)$, moreover,

$$\frac{1}{2}\left\|A^{1/2}S_t u\right\|^2 \;\le\; V(u_0)\,e^{-\omega t} + \frac{C}{\omega}(1-e^{-\omega t})\,. \tag{4.9}$$

Theorem 4.3 is proved.

Exercise *4.2* Show that the assertion of Theorem 4.3 holds if (4.2) is fulfilled with $\theta = 1/2$ and

$$B(u) = -F'(u) + B_0(u)\,,$$

where $F(u)$ possesses properties (4.8) and $B_0(u)$ satisfies the estimate $\left\|B_0(u)\right\| \le C_\varepsilon + \varepsilon\left\|A^{1/2}u\right\|$ for $\varepsilon > 0$ small enough.

Let us look at the examples of Section 3 again. We assume that the function

$$f(t,\ x,\ u,\ p) \equiv f(x,\ u,\ p) \tag{4.10}$$

in Example 3.1 possesses property (3.2) and

$$\int\limits_{\Omega} f(x,\ u(x),\ \nabla u(x))\,u(x)\,\mathrm{d}x \;\le\; (\lambda_1 - \delta)\int\limits_{\Omega} u^2(x)\,\mathrm{d}x + C \tag{4.11}$$

for all $u \in H_0^1(\Omega)$, where λ_1 is the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary condition on $\partial\Omega$. Here δ and C are positive constants.

Exercise *4.3* Using the Galerkin approximations of problem (3.1) and Lemma 4.2, prove that the dynamical system generated by equation (3.1) is dissipative in $H_0^1(\Omega)$ under conditions (3.2), (4.10), and (4.11).

Therefore, if conditions (3.2), (4.10) and (4.11) are fulfilled, then the dynamical system $(H_0^1(\Omega),\ S_t)$ generated by a mild (in $H_0^1(\Omega)$) solution to problem (3.1) possesses both a finite-dimensional global attractor and an inertial set.

Exercise *4.4* Prove that equation (4.11) holds if

$$f(x,\ u,\ \nabla u) \;=\; f_0(x,\ u) + \sum_{j=1}^{d} a_j\frac{\partial u}{\partial x_j}\,,$$

where a_j are real constants and the function $f_0(x,\ u)$ possesses the property

$$f(x,\ u)u \le (\lambda_1 - \delta)u^2 + C, \quad u \in R$$

for any $x \in \Omega$.

— Exercise *4.5* Consider the dynamical system generated by problem (3.4). In addition to the hypotheses of Example 3.2 we assume that $f(t,\ x,\ u,\ \partial_x u) \equiv 0$ and the function $g(x,\ u)$ possesses the properties

$$\int_0^y g(x,\ \xi)d\xi \ge -\alpha, \qquad y g(x,\ y) - \beta\int_0^y g(x,\ \xi)d\xi \ge -\gamma$$

for some positive α, β, and γ. Then the dynamical system generated by (3.4) is dissipative in $H_0^1(0,\ 1)$.

— Exercise *4.6* Find the analogue of the result of Exercise 4.5 for the system of reaction-diffusion equations (3.6).

— Exercise *4.7* Using equations (3.9) and (3.10) prove that the dynamical system generated by the nonlocal Burgers equation (3.8) with $f(t) \equiv$ $\equiv f \in L^2(0,\ l)$ is dissipative in $H_0^1(\Omega)$.

Let us consider a dynamical system $(V,\ S_t)$ generated by the Cahn-Hilliard equation (3.12). We remind that

$$V = \left\{ u \in H^2(0,\ l):\ u_x\Big|_{x=0} = u_x\Big|_{x=l} = 0 \right\}.$$

— Exercise *4.8* Let the function $W(u)$ be defined by equality (3.14). Show that for any positive R and α the set

$$X_{\alpha,\ R} = \left\{ u \in V:\ W(u) \le R,\ \left|\int_0^l u(x)dx\right| \le \alpha \right\} \tag{4.12}$$

is a closed invariant subset in V for the dynamical system $(V,\ S_t)$ generated by problem (3.12).

— Exercise *4.9* Prove that the dynamical system $(X_{\alpha,\ R},\ S_t)$ generated by the Cahn-Hilliard equation on the set $X_{\alpha,\ R}$ defined by (4.12) is dissipative (*Hint*: cf. Exercise 3.9).

In conclusion of this section let us establish the dissipativity of the dynamical system $(D(A^{1/2+\beta}),\ S_t)$ generated by the abstract form of the two-dimensional Navier-Stokes system (see Example 3.5) under the assumption that $f(t) \equiv f \in D(A^\beta)$. We consider the dynamical system $(P_m H,\ S_t^m)$ generated by the Galerkin approximations (see (3.20)) of problem (3.16).

— Exercise *4.10* Using (3.24) prove that

$$t\left\|A^{1/2}S_t^m u_0\right\|^2 \le \left(c_0 + \int_0^1 \left\|A^{1/2}S_\tau^m u_0\right\|^2 d\tau\right) \times$$

$$\times \exp\left\{c_1 \int_0^1 \left(\left\|S_\tau^m u_0\right\|^2 \left\|A^{1/2}S_\tau^m u_0\right\|\right)^2 d\tau\right\} \tag{4.13}$$

for all $0 < t \le 1$, where c_0 and c_1 are constants independent of m.

— Exercise *4.11* With the help of (3.21), (3.22) and (4.13) verify the property of dissipativity of the system $(D(A^{1/2+\beta}), S_t)$ in the space $D(A^{1/2})$. Deduce its dissipativity (*Hint*: see Exercise 3.14–3.16).

Thus, the dynamical system generated by the two-dimensional Navier-Stokes equations possesses both a finite-dimensional compact global attractor and an inertial set.

§ 5 *Systems with Lyapunov Function*

In this section we consider problem (4.1) on the assumption that condition (4.2) holds with $\theta = 1/2$ and $B(u)$ is a potential operator, i.e. there exists a functional $F(u)$ on $\mathcal{F}_{1/2} = D(A^{1/2})$ such that its Frechét derivative $F'(u)$ possesses the property

$$B(u) = -F'(u), \qquad u \in D(A^{1/2}). \tag{5.1}$$

Below we also assume that the conditions

$$F(u) \ge -\alpha, \quad (F'(u), u) - \beta F(u) \ge -\gamma\|A^{1/2}u\|^2 - \delta \tag{5.2}$$

are fulfilled for all $u \in D(A^{1/2})$, where $\alpha, \delta \in \mathbb{R}$, $\beta > 0$, and $\gamma < 1$. On the one hand, these conditions ensure the existence and uniqueness of mild (in $D(A^{1/2})$) solutions to problem (4.1) (see Theorem 2.4). On the other hand, they guarantee the existence of a finite-dimensional global attractor $\mathcal{A}$ for the dynamical system $(\mathcal{F}_{1/2}, S_t)$ generated by problem (4.1) (see Theorem 4.3). Conditions (5.1) and (5.2) enable us to obtain additional information on the structure of attractor.

Theorem 5.1.

Assume that conditions (5.1), (5.2)***, and*** (4.2) ***hold with*** $\theta = 1/2$. ***Then the global attractor*** $\mathcal{A}$ ***of the dynamical system*** $(\mathcal{F}_{1/2}, S_t)$ ***generated by***

problem (4.1) ***is a bounded set in*** $\mathscr{F}_1 = D(A)$ ***and it coincides with the unstable manifold emanating from the set of fixed points of the system, i.e.***

$$\mathscr{A} = M_+(\mathscr{N}), \tag{5.3}$$

where $\mathscr{N} = \{z \in D(A):\ Az = B(z)\}$ ***(for the definition of*** $M_+(\mathscr{N})$ ***see Section*** 6 ***of Chapter*** 1***).***

The proof of the theorem is based on the following lemmata.

Lemma 5.1.

Assume that a semitrajectory $u(t) = S_t u_0$ *possesses the property* $\|A^\alpha u(t)\| \le R_\alpha$ *for all* $t \ge 0$, *where* $1/2 < \alpha < 1$. *Then*

$$\|A^{1/2}(u(t) - u(s))\| \le C(R_\alpha)|t-s|^{\alpha - 1/2} \tag{5.4}$$

for all $t,\ s \ge 0$.

Proof.

For the sake of definiteness we assume that $t > s \ge 0$. Equation (2.3) implies that

$$u(t) - u(s) = (e^{-(t-s)A} - I)\, u(s) + \int_s^t e^{-(t-\tau)A} B(u(\tau)) \mathrm{d}\tau .$$

Since

$$e^{-(t-s)A} - I = -\int_s^t A\, e^{-(t-\tau)A} \mathrm{d}\tau ,$$

equation (1.17) gives us that

$$\|A^{1/2}(u(t) - u(s))\| \le c_0 \int_s^t (t-\tau)^{\alpha - 3/2} \mathrm{d}\tau \|A^\alpha u(s)\| +$$

$$+ c_1 \int_s^t (t-\tau)^{-1/2} \mathrm{d}\tau \max_{\tau \ge 0} \|B(u(\tau))\| .$$

This implies estimate (5.4).

Lemma 5.2.

There exists $R_1 > 0$ *such that the set* $B_1 = \{u:\ \|Au\| \le R_1\}$ *is absorbing for the dynamical system* $(\mathscr{F}_{1/2},\ S_t)$.

Proof.

By virtue of Theorem 4.3 the system $(\mathscr{F}_{1/2},\ S_t)$ is dissipative. Therefore, it follows from Lemma 4.1 that $B_\alpha = \{u:\ \|A^\alpha u\| \le R_\alpha\}$ is an absorbing set,where

R_α is defined by (4.3), $1/2 < \alpha < 1$. Thus, to prove the lemma it is sufficient to consider semitrajectories $u(t)$ possessing the property $\|A^\alpha u(t)\| \le R_\alpha$, $t \ge 0$. Let us present the solution $u(t)$ in the form

$$u(t) = e^{-(t-s)A}u(s) + \int_s^t e^{-(t-\tau)A}\Big[B(u(\tau)) - B(u(t))\Big]d\tau +$$

$$+ A^{-1}(1 - e^{-(t-s)A})B(u(t)) .$$

Using Lemma 5.1 we find that

$$\|Au(t)\| \le C_0(t-s)^{-1/2}R_{1/2} + C_1(R_\alpha)\int_s^t (t-\tau)^{-\frac{3}{2}+\alpha}d\tau + C_2(R_{1/2}) .$$

This implies that

$$\|Au(t+1)\| \le C(R_\alpha), \qquad t \ge 0 ,$$

provided that $\|A^\alpha u(t)\| \le R_\alpha$ for $t \ge 0$. Therefore, the assertion of Lemma 5.2 follows from Lemma 4.1.

Proof of Theorem 5.1.

The boundedness of the attractor $\mathcal{A}$ in $D(A)$ follows from Lemma 5.2. Let us prove (5.3). We consider the Galerkin approximation $u_m(t)$ of solutions to problem (4.1):

$$\frac{du_m(t)}{dt} + Au_m(t) = P_m B(u_m(t)), \qquad u_m(0) = P_m u_0 .$$

Here P_m is the orthoprojector onto the span of $\{e_1, e_2, \dots, e_m\}$. Let

$$V(u) = \frac{1}{2}(Au, u) + F(u), \quad u \in \mathcal{F}_{1/2} = D(A^{1/2}) . \tag{5.5}$$

It is clear that

$$\frac{d}{dt}V(u_m(t)) = (Au_m(t) + F'(u_m(t)), \dot{u}_m(t)) = -\|P_m[Au_m(t) - B(u_m(t))]\|^2 .$$

This implies that

$$V(u_m(t)) - V(u_m(s)) = -\int_s^t \|P_m[Au_m(\tau) - B(u_m(\tau))]\|^2 d\tau \le$$

$$\le -\int_s^t \|P_N(Au_m(\tau) - B(u_m(\tau)))\|^2 d\tau$$

for $t \ge s$ and for any $N \le m$, where P_N is the orthoprojector onto the span of $\{e_1, \dots, e_N\}$. With the help of Theorem 2.2 and due to the continuity of the functional $V(u)$ we can pass to the limit $m \to \infty$ in the latter equation. As a result, we obtain the estimate

$$V(u(t)) + \int_s^t \left\| P_N(A u(\tau) - B(u(\tau))) \right\|^2 d\tau \ \le \ V(u(s)), \qquad t \ge s, \tag{5.6}$$

for any solution $u(t)$ to problem (4.1) and for any $N \ge 1$. If the semitrajectory $u(t) = S_t u_0$ lies in the attractor $\mathscr{A}$, then we can pass to the limit $N \to \infty$ in (5.6) and obtain the equation

$$V(u(t)) + \int_s^t \| Au(\tau) - B(u(\tau)) \|^2 d\tau \ \le \ V(u(s)) \tag{5.7}$$

for any $t \ge s \ge 0$ and $u(t) \in \mathscr{A}$. Equation (5.7) implies that the functional $V(u)$ defined by equality (5.5) is the Lyapunov function of the dynamical system $(\mathscr{F}_{1/2}, S_t)$ on the attractor $\mathscr{A}$. Therefore, Theorem 1.6.1 implies equation (5.3). **Theorem 5.1 is proved**.

— Exercise *5.1* Using (5.6) show that any solution $u(t)$ to problem (4.1) with $u_0 \in \mathscr{F}_{1/2}$ possesses the property

$$\int_0^t \| Au(\tau) \|^2 d\tau < \infty, \qquad t > 0.$$

Prove the validity of inequality (5.7) for any solution $u(t)$ to problem (4.1).

— Exercise *5.2* Using the results of Exercises 5.1 and 1.6.5 show that if the hypotheses of Theorem 5.1 hold, then a global minimal attractor $\mathscr{A}_{\min}$ of the system $(\mathscr{F}_{1/2}, S_t)$ has the form

$$\mathscr{A}_{\min} = \{ w \in D(A) : \ Aw - B(w) = 0 \} .$$

— Exercise *5.3* Prove that the assertions of Theorems 4.3 and 5.1 hold if we consider the equation

$$\frac{du}{dt} + Au = B(u) + h, \qquad u\big|_{t=0} = u_0 \tag{5.8}$$

instead of problem (4.1). Here h is an arbitrary element from H and $B(u)$ possesses properties (5.1), (5.2) and (4.2) with $\theta = 1/2$ (*Hint*: $V_h(u) = V(u) - (h, u)$).

— Exercise *5.4* Let S_t be an evolutionary operator of problem (5.8). Show that for any $R > 0$ there exist numbers $a_R \ge 0$ and $b_R > 0$ such that

$$\left\| A^{1/2}(S_t u_1 - S_t u_2) \right\| \ \le \ b_R e^{a_R t} \left\| A^{1/2}(u_1 - u_2) \right\|,$$

provided $\left\| A^{1/2} u_j \right\| \le R$, $j = 1, 2$ (*Hint*: $V_h(S_t u_j) \le V_h(u_j) \le C_R$).

Theorem 5.1 and the reasonings of Section 6 of Chapter 1 reduce the question on the structure of global attractor to the problem of studying the properties of stationary points of the dynamical system under consideration. Under some additional conditions on the operator $B(u)$ it can be proved in general that the number of fixed points is finite and all of them are hyperbolic. This enables us to use the results of Section 6 of Chapter 1 to specify the attractor structure. For some reasons (they will be clear later) it is convenient to deal with the fixed points of the dynamical system generated by problem (5.8).

Thus we consider the equation

$$L(u) \equiv Au - B(u) = h\,, \qquad u \in D(A)\,, \tag{5.9}$$

where as before A is a positive operator with discrete spectrum, $B(u)$ is a nonlinear mapping possessing properties (5.1), (5.2) and (4.2) with $\theta = 1/2$, and h is an element of H.

Lemma 5.3.

For any $h \in H$ problem (5.9) *is solvable. If M is a bounded set in H, then the set $L^{-1}(M)$ of solutions to equation* (5.9) *is bounded in $D(A)$ for $h \in M$. If M is compact in H, then $L^{-1}(M)$ is compact in $D(A)$.*

Proof.

Let us consider a continuous functional

$$W(u) = \frac{1}{2}(Au,\ u) + F(u) - (h,\ u) \tag{5.10}$$

on $\mathcal{F}_{1/2} = D(A^{1/2})$. Since $F(u) \ge -\alpha$ for all $u \in \mathcal{F}_{1/2}$, the functional $W(u)$ possesses the property

$$\begin{aligned} W(u) \ &\ge \frac{1}{2}\|A^{1/2}u\|^2 - \alpha - \|A^{-1/2}h\|\,\|A^{1/2}u\| \ \ge \\ &\ge \frac{1}{4}\|A^{1/2}u\|^2 - \alpha - \|A^{-1/2}h\|^2\ . \end{aligned} \tag{5.11}$$

In particular, this means that $W(u)$ is bounded below. Let us consider the functional $W(u)$ on the subspace $P_m\mathcal{F}_{1/2}$ (P_m is the orthoprojector onto the span of elements $e_1,\ e_2,\ \dots,\ e_m$, as before). By virtue of (5.11) there exists a minimum point u_m of this functional in $P_m\mathcal{F}_{1/2}$ which obviously satisfies the equation

$$A u_m - P_m B(u_m) = P_m h\,. \tag{5.12}$$

Equation (5.11) also implies that

$$\begin{aligned} \frac{1}{4}\|A^{1/2}u_m\|^2 \ &\le\ W(u_m) + \alpha + \|A^{-1/2}h\|^2 \ \le \\ &\le\ \inf\{W(u):\ u \in P_m H\} + \alpha + \|A^{-1/2}h\|^2\ . \end{aligned}$$

Hence,

$$\left\|A^{1/2}u_m\right\| \le F(0)+\alpha+\|A^{-1/2}h\|^2 \le C(1+\|A^{-1/2}h\|^2)$$

with a constant C independent of m. Thus, equations (5.12) and (4.2) give us that

$$\|Au_m\| \le \|B(u_m)+h\| \le \|B(0)\|+\|h\|+M(\left\|A^{1/2}u_m\right\|)\left\|A^{1/2}u_m\right\| .$$

Therefore, if $\|h\| \le R$, then $\|Au_m\| \le C(R)$. This estimate enables us to extract a weakly convergent (in $D(A)$) subsequence $\{u_{m_k}\}$ and to pass to the limit as $k \to \infty$ in (5.12) with the help of Theorem 1.1. Thus, the solvability of equation (5.9) is proved. It is obvious that every limit (in $D(A)$) point u of the sequence $\{u_m\}$ possesses the property

$$\|Au\| \le C_R \quad \text{if} \quad \|h\| \le R. \tag{5.13}$$

This means that the complete preimage $L^{-1}(M)$ of any bounded set M in H is bounded in $D(A)$. Now we prove that the mapping L is ***proper***, i.e. the preimage $L^{-1}(M)$ is compact for a compact M. Let $\{u_n\}$ be a sequence from $L^{-1}(M)$. Then the sequence $\{L(u_n)\}$ lies in the compact M and therefore there exist an element $h \in M$ and a subsequence $\{n_k\}$ such that $\left\|L(u_{n_k})-h\right\| \to 0$ as $k \to \infty$. By virtue of (5.13) we can also assume that $\{A\,u_{n_k}\}$ is a weakly convergent sequence in $D(A)$. If we use the equation

$$\left\|Au_{n_k}-B(u_{n_k})-h\right\| \to 0, \qquad k \to \infty,$$

Theorem 1.1, and property (4.2) with $\theta = 1/2$, then we can easily prove that the sequence $\{u_{n_k}\}$ strongly converges in $D(A)$ to a solution u to the equation $L(u) = h$. Lemma 5.3 is proved.

Lemma 5.4.

In addition to (5.1), (5.2), *and* (4.2) *with* $\theta = 1/2$ *we assume that the mapping* $B(\cdot)$ *is Frechét differentiable, i.e. for any* $u \in \mathcal{F}_{1/2}$ *there exists a linear bounded operator* $B'(u)$ *from* $\mathcal{F}_{1/2}$ *into* H *such that*

$$\|B(u+v)-B(u)-B'(u)v\| = o(\left\|A^{1/2}v\right\|) \tag{5.14}$$

for every $v \in \mathcal{F}_{1/2}$, *such that* $\left\|A^{1/2}v\right\| \le 1$. *Then the operator* $A-B'(u)$ *is a Fredholm operator for any* $u \in \mathcal{F}_{1/2} = D(A^{1/2})$.

We remind that a densely defined closed linear operator G in H is said to be ***Fredholm*** (of index zero) if

(a) its image is closed; and
(b) $\dim \mathcal{K}er\, G = \dim \mathcal{K}er\, G^* < \infty$.

Proof of Lemma 5.4.

It is clear that the operator $G \equiv A-B'(u)$ has the structure

$$G = A - C A^{1/2} = A^{1/2}(I - A^{-1/2}C)A^{1/2},$$

where C is a bounded operator in H $(C = B'(u)A^{-1/2})$. By virtue of Theorem 1.1 the operator $K = A^{-1/2}C$ is compact. This implies the closedness of the image of G. Moreover, it is obvious that

$$\dim \mathcal{K}er\, G = \dim \mathcal{K}er\,(I - K)$$

and

$$\dim \mathcal{K}er\, G^* = \dim \mathcal{K}er\,(I - K^*).$$

Therefore, the Fredholm alternative for the compact operator gives us that

$$\dim \mathcal{K}er\, G^* = \dim \mathcal{K}er\, G < \infty .$$

Let us introduce the notion of a ***regular value*** of the operator L seen as an element $h \in H$ possessing the property that for every $u \in L^{-1}h \equiv \{v:\ L(v) = h\}$ the operator $L'(u) = A - B'(u)$ is invertible on H. Lemmata 5.3 and 5.4 enable us to use the Sard-Smale theorem (for the statement and the proof see, e.g., the book by A. V. Babin and M. I. Vishik [1]) and state that the set $\mathcal{R}$ of regular values of the mapping $L(u) = Au - B(u)$ is an open everywhere dense set in H. The following assertion is valid for regular values of the operator L.

Lemma 5.5.

Let h be a regular value of the operator L. Then the set of solutions to equation (5.9) *is finite.*

Proof.

By virtue of Lemma 5.3 the set $\mathcal{N} = \{v:\ L(v) = h\}$ is compact. Since $h \in \mathcal{R}$, the operator $L'(u) = A - B'(u)$ is invertible on H for $u \in \mathcal{N}$. It is also evident that $L'(u)$ has a domain $D(A)$. Therefore, by virtue of the uniform boundedness principle $AL'(u)^{-1}$ is a bounded operator for $u \in \mathcal{N}$. Hence, it follows from (5.14) that

$$\|A(v - w)\| \le \|AL'(v)^{-1}\| \cdot \|L'(v)(v - w)\| =$$

$$= \|AL'(v)^{-1}\| \, \|B(v) - B(w) - B'(v)(v - w)\| = o\left(\|A^{1/2}(v - w)\|\right)$$

for any v and w in $\mathcal{N}$. This implies that for every $v \in \mathcal{N}$ there exists a vicinity that does not contain other points of the set $\mathcal{N}$. Therefore, the compact set $\mathcal{N}$ has no condensation points. Hence, $\mathcal{N}$ consists of a finite number of elements. Lemma 5.5 is proved.

In order to prove the hyperbolicity (for the definition see Section 6 of Chapter 1) of fixed points we should first consider linearization of problem (5.8) at these points. Assume that the hypotheses of Lemma 5.4 hold and $v_0 \in D(A)$ is a stationary solution. We consider the problem

$$\frac{du}{dt} + (A - B'(v_0))u = 0\,, \qquad u\big|_{t=0} = u_0\,. \tag{5.15}$$

Its solution can be regarded as a continuous function in $\mathcal{F}_{1/2} = D(A^{1/2})$ which satisfies the equation

$$u(t) = e^{-tA}\,u_0 + \int\limits_0^t e^{-(t-\tau)A} B'(v_0)u(\tau)\mathrm{d}\tau\,.$$

If $u_0 \in D(A^{1/2})$, then we can apply Theorem 2.3 on the existence and uniqueness of solution. Let T_t stand for the evolutionary operator of problem (5.15).

— Exercise *5.5* Prove that T_t is a compact operator in every space $\mathcal{F}_\alpha$, $0 \le \alpha < 1$, $t > 0$.

— Exercise *5.6* Prove that for any $\rho > 0$ and $t > 0$ the set of points of the spectrum of the operator T_t that are lying outside the disk $\{\lambda : |\lambda| \le \rho\}$ is finite and the corresponding eigensubspace is finite-dimensional.

— Exercise *5.7* Assume that $B'(v_0)$ is a symmetric operator in H. Prove that the spectrum of the operator T_t is real.

The next assertion contains the conditions wherein the evolutionary operator S_t belongs to the class $C^{1+\alpha}$, $\alpha > 0$, on the set of stationary points.

Theorem 5.2.

Assume that conditions (5.1), (5.2), ***and*** (4.2) ***are fulfilled with*** $\theta = 1/2$. ***Let*** $B(u)$ ***possess a Frechét derivative in*** $\mathcal{F}_{1/2}$ ***such that for any*** $R > 0$

$$\|B(u+v) - B(u) - B'(u)v\| \le C\|A^{1/2}v\|^{1+\alpha}, \qquad \alpha > 0, \tag{5.16}$$

provided that $\|A^{1/2}v\| \le R$, ***where the constant*** $C = C(u, R)$ ***depends on*** u ***and*** R ***only. Then the evolutionary operator*** S_t ***of problem*** (5.8) ***has a Frechét derivative at every stationary point*** v_0. ***Moreover,*** $([S_t(v_0)]', u) = T_t u$, ***where*** T_t ***is the evolutionary operator of linear problem*** (5.15).

Proof.

It is evident that

$$S_t[v_0 + u] - v_0 - T_t u = \int\limits_0^t e^{-(t-\tau)A} \Big\{ B(S_\tau[v_0+u]) - B(v_0) - B'(v_0)T_\tau u \Big\} \mathrm{d}\tau\,.$$

Therefore, using (1.17) and (5.16) we have

$$\|A^{1/2}\psi(t)\| \le \int_0^t (2e[t-\tau])^{-1/2}\Big\{C\Big\|A^{1/2}(S_\tau[v_0+u]-v_0)\Big\|^{1+\alpha}+$$

$$+\|B'(v_0)\|\cdot\|A^{1/2}\psi(\tau)\|\Big\}\mathrm{d}\tau\ ,$$

where $\psi(t)=S_t[v_0+u]-v_0-T_t u$. Using the result of Exercise 5.4 we obtain that

$$\Big\|A^{1/2}(S_t[v_0+u]-v_0)\Big\| \le b_R e^{a_R t}\|A^{1/2}u\| \quad \text{if} \quad \|A^{1/2}u\| \le R\,.$$

Thus,

$$\|A^{1/2}\psi(t)\| \le C_0\sqrt{t}\,e^{(1+\alpha)a_R t}\|A^{1/2}u\|^{1+\alpha}+C_1\int_0^t (t-\tau)^{-1/2}\|A^{1/2}\psi(\tau)\|\mathrm{d}\tau\,.$$

Consequently, Lemma 2.1 gives us the estimate

$$\Big\|A^{1/2}(S_t[v_0+u]-v_0-T_t u)\Big\| \le C_T\|A^{1/2}u\|^{1+\alpha}, \qquad t\in[0,\ T]\,.$$

This implies the assertion of Theorem 5.2.

Exercise *5.8* Assume that the constant C in (5.16) depends on R only, provided that $\|A^{1/2}u\| \le R$ and $\|A^{1/2}v\| \le R$. Prove that $S_t \in C^{1+\alpha}$ for any $0<\alpha<1$.

The reasoning above leads to the following result on the properties of the set of fixed points of problem (5.8).

Theorem 5.3.

Assume that conditions (5.1), (5.2) ***and*** (4.2) ***are fulfilled with*** $\theta=1/2$ ***and the operator*** $B(u)$ ***possesses a Frechét derivative in*** $\mathcal{F}_{1/2}$ ***such that equation*** (5.16) ***holds with the constant*** $C=C(R)$ ***depending only on*** R ***for*** $\|A^{1/2}u\| \le R$ ***and*** $\|A^{1/2}v\| \le R$. ***Then there exists an open dense (in*** H***) set*** $\mathcal{R}$ ***such that for*** $h\in\mathcal{R}$ ***the set of fixed points of the system*** $(\mathcal{F}_{1/2},\ S_t)$ ***generated by problem*** (5.8) ***is finite. If in addition we assume that*** $B'(z)$ ***is a symmetric operator for*** $z\in D(A)$, ***then fixed points are hyperbolic.***

In particular, this theorem means that if $h\in\mathcal{R}$, then the global attractor of the dynamical system generated by equation (5.8) possesses the properties given in Exercises 1.6.9–1.6.12. Moreover, it is possible to apply Theorem 1.6.3 as well as the other results related to finiteness and hyperbolicity of the set of fixed points (see, e.g., the book by A. V. Babin and M. I. Vishik [1]).

E x a m p l e *5.1*

Let us consider a dynamical system generated by the nonlinear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} + g(x,\ u(x)) = h(x), \quad 0 < x < 1, \quad t > 0, \\ u\big|_{x=0} = u\big|_{x=1} = 0, \quad u\big|_{t=0} = u_0(x), \end{cases} \tag{5.17}$$

in $H_0^1(0,\ 1)$. Assume that $g(x,\ u)$ is twice continuously differentiable with respect to its variables and the conditions

$$\int_0^y g(x,\ \xi)\,\mathrm{d}\xi \ \ge -\alpha, \qquad y\,g(x,\ y) - \beta \int_0^y g(x,\ \xi)\,\mathrm{d}\xi \ \ge -\gamma$$

are fulfilled with some positive constants $a,\ b$, and γ.

E x e r c i s e *5.9* Prove that the dynamical system generated by equation (5.17) possesses a global attractor $\mathscr{A} = M_+(\mathscr{N})$, where $\mathscr{N}$ is the set of stationary solutions to problem (5.17).

E x e r c i s e *5.10* Prove that there exists a dense open set $\mathscr{R}$ in $L^2(0,\ 1)$ such that for every $h(x) \in \mathscr{R}$ the set $\mathscr{N}$ of fixed points of the dynamical system generated by problem (5.17) is finite and all the points are hyperbolic.

It should be noted that if a property of a dynamical system holds for the parameters from an open and dense set in the corresponding space, then it is frequently said that this property is a ***generic property***.

However, it should be kept in mind that the generic property is not the one that holds almost always. For example one can build an open and dense set $\mathscr{R}$ in $[0,\ 1]$, the Lebesgue measure of which is arbitrarily small $(\ \le \varepsilon)$. To do that we should take

$$\mathscr{R} = \bigcup_{k=1}^{\infty} \left\{ x \in (0,\ 1):\ |x - r_k| < \varepsilon \cdot 2^{-k-2} \right\},$$

where $\{r_k\}$ is a sequence of all the rational numbers of the segment $[0,\ 1]$. Therefore, it should be remembered that generic properties are quite frequently encountered and stay stable during small perturbations of the properties of a system.

§ 6 Explicitly Solvable Model of Nonlinear Diffusion

In this section we study the asymptotic properties of solutions to the following nonlinear diffusion equation

$$\begin{cases} u_t - \nu u_{xx} + \left(\varkappa \int\limits_0^1 |u(x,\ t)|^2 \mathrm{d}x - \Gamma \right) u + \rho u_x = 0\,, \quad 0 < x < 1\,,\ t > 0\,, \\ u|_{x=0} = u|_{x=1} = 0\,, \quad u|_{t=0} = u_0(x)\,, \end{cases} \tag{6.1}$$

where $\nu > 0$, $\varkappa > 0$, Γ and ρ are parameters. The main feature of this problem is that the asymptotic behaviour of its solutions can be completely described with the help of elementary functions. We do not know whether problem (6.1) is related to any real physical process.

Exercise *6.1* Show that Theorem 2.4 which guarantees the global existence and uniqueness of mild solutions is applicable to problem (6.1) in the Sobolev space $H_0^1(0,\ 1)$.

Exercise *6.2* Write out the system of ordinary differential equations for the functions $\{g_k(t)\}$ that determine the Galerkin approximations

$$u_m(t) = \sqrt{2} \sum_{k=1}^{m} g_k(t) \sin \pi k x \tag{6.2}$$

of the order m of a solution to problem (6.1).

Exercise *6.3* Using the properties of the functions $u_m(t)$ defined by equation (6.2) prove that the mild solution $u(t,\ x)$ possesses the properties

$$\frac{1}{2}\|u(t)\|^2 + \int\limits_0^t (\nu\|\partial_x u(\tau)\|^2 + \varkappa\|u(\tau)\|^4 - \Gamma\|u(\tau)\|^2)\mathrm{d}\tau \ = \ \frac{1}{2}\|u_0\|^2 \tag{6.3}$$

and

$$\|u(t)\|^2 \ \le \ \|u_0\|^2 e^{-2\nu\pi^2 t} + \frac{\Gamma^2}{4\nu\varkappa\pi^2}(1 - e^{-2\nu\pi^2 t})\,. \tag{6.4}$$

Here and below $\|\cdot\|$ is a norm in $L^2(0,\ 1)$.

Exercise *6.4* Using equations (6.3) and (6.4) prove that the dynamical system generated by problem (6.1) in $H_0^1(0,\ 1)$ is dissipative.

Therefore, by virtue of Theorem 4.1 the dynamical system $(H_0^1(0,\ 1),\ S_t)$ generated by equation (6.1) possesses a finite-dimensional global attractor.

Let $u(t) = u(x,\ t) \in C(\mathbb{R}_+;\ H_0^1(0,\ 1))$ be a mild solution to problem (6.1) with the initial condition $u_0(x) \in H_0^1(0,\ 1)$. Then the function $u(x,\ t)$ can be considered as a mild solution to the linear problem

$$\begin{cases} u_t - \nu u_{xx} + b(t)u + \rho u_x = 0, & 0 < x < 1, \quad t > 0\ , \\ u\big|_{x=0} = u\big|_{x=1} = 0, & u\big|_{t=0} = u_0(x)\ , \end{cases} \tag{6.5}$$

where $b(t)$ is a scalar continuous function defined by the formula

$$b(t) = \varkappa \int_0^1 |u(x,\ t)|^2 \mathrm{d}x - \Gamma\ .$$

We consider the function

$$v(x,\ t) = u(x,\ t) \exp\left\{ \int_0^t b(\tau)\, \mathrm{d}\tau + \frac{\rho^2}{4\nu} t - \frac{\rho}{2\nu} x \right\}. \tag{6.6}$$

Then it is easy to check that $v(x,\ t)$ is a mild solution to the heat equation

$$\begin{cases} v_t = \nu v_{xx}, \quad x \in (0,\ 1),\ t > 0\ , \\ v\big|_{x=0} = v\big|_{x=1} = 0, \quad v\big|_{t=0} = u_0(x) \exp\left\{ -\dfrac{\rho}{2\nu} x \right\}. \end{cases} \tag{6.7}$$

The following assertion shows that the asymptotic properties of equation (6.7) completely determine the dynamics of the system generated by problem (6.1).

Lemma 6.1.

Every mild (in $H_0^1(0,\ 1)$) solution $u(x,\ t)$ to problem (6.1) *can be rewritten in the form*

$$u(x,\ t) = \frac{w(x,\ t)}{\left\{ 1 + 2\varkappa \int_0^t \|w(\tau)\|^2\, \mathrm{d}\tau \right\}^{1/2}}\ , \tag{6.8}$$

where $w(x,\ t)$ has the form

$$w(x,\ t) = v(x,\ t) \exp\left\{ \left(\Gamma - \frac{\rho^2}{4\nu} \right) t + \frac{\rho}{2\nu} x \right\} \tag{6.9}$$

and $v(x,\ t)$ is the solution to problem (6.7).

Proof.

Let $w(x,\ t)$ have the form (6.9). Then we obtain from (6.6) that

$$w(x,\ t) = u(x,\ t)\exp\left\{\varkappa\int_0^t \|u(\tau)\|^2 d\tau\right\}. \tag{6.10}$$

Therefore,

$$\|w(x,\ t)\|^2 = \|u(x,\ t)\|^2 \exp\left\{2\varkappa\int_0^t \|u(\tau)\|^2 d\tau\right\} = \frac{1}{2\varkappa}\frac{d}{dt}\exp\left\{2\varkappa\int_0^t \|u(\tau)\|^2 d\tau\right\}.$$

Hence,

$$\exp\left\{2\varkappa\int_0^t \|u(\tau)\|^2 d\tau\right\} = 1 + 2\varkappa\int_0^t \|w(\tau)\|^2 d\tau.$$

This and equation (6.10) imply (6.8). Lemma 6.1 is proved.

Now let us find the fixed points of problem (6.1). They satisfy the equation

$$-\nu u_{xx} + (\varkappa\|u\|^2 - \Gamma)u + \rho u_x = 0, \quad u|_{x=0} = u|_{x=1} = 0.$$

Therefore, $u(x) = w(x)\exp\left\{\frac{\rho}{2\nu}x\right\}$, where $w(x)$ is the solution to the problem

$$-\nu w_{xx} + \left(\varkappa\|u\|^2 - \Gamma + \frac{\rho^2}{4\nu}\right)w = 0, \quad w|_{x=0} = w|_{x=1} = 0.$$

However, this problem has a nontrivial solution $w(x)$ if and only if

$$w(x) = C\sin\pi n x \quad \text{and} \quad \varkappa\|u\|^2 - \Gamma + \frac{\rho^2}{4\nu} + \nu(\pi n)^2 = 0,$$

where n is a natural number. Since $u = w\exp\left\{\frac{\rho}{2\nu}x\right\}$, we obtain the equation

$$\varkappa C^2\int_0^1 e^{\frac{\rho}{\nu}x}\sin^2\pi n x\, dx + \nu(\pi n)^2 - \Gamma + \frac{\rho^2}{4\nu} = 0$$

which can be used to find the constant C. After the integration we have

$$\varkappa C^2\frac{\nu}{\rho}(e^{\rho/\nu} - 1)\frac{2n^2\nu^2\pi^2}{\rho^2 + 4n^2\nu^2\pi^2} = \Gamma - \frac{\rho^2}{4\nu} - \nu(\pi n)^2.$$

The constant C can be found only when the parameter n possesses the property $\Gamma - \rho^2/(4\nu) - \nu(\pi n)^2 > 0$. Thus, we have the following assertion.

Lemma 6.2.

Let $\gamma = \gamma(\Gamma,\ \nu,\ \rho) \equiv \Gamma - \frac{\rho^2}{4\nu}$. *If* $\gamma \le \nu\pi^2$, *then problem* (6.1) *has a unique fixed point* $\bar{u}_0(x) \equiv 0$. *If* $(\pi n)^2 < \gamma \le \pi^2(n+1)^2$ *for some* $n \ge 1$, *then the fixed points of problem* (6.1) *are*

$$\bar{u}_0(x) \equiv 0, \qquad \bar{u}_{\pm k}(x) = \pm\mu_k e^{\frac{\rho}{2\nu}x}\sin\pi k x, \qquad k = 1,\ 2,\ \dots,\ n, \tag{6.11}$$

where

$$\mu_k \ \equiv\ \mu_k(\Gamma,\ \rho,\ \nu) = \frac{1}{4\nu^2\pi k}\sqrt{\frac{2\rho\delta_k(\Gamma,\ \rho,\ \nu)}{\varkappa\left(\exp\left(\frac{\rho}{\nu}\right) - 1\right)}}, \tag{6.12}$$

$$\delta_k(\Gamma,\ \rho,\ \nu) = [4\Gamma\nu - \rho^2 - 4(\pi n\nu)^2]\cdot[\rho^2 + 4(\pi n\nu)^2].$$

Exercise *6.5* Show that every subspace

$$H_N = \mathrm{Lin}\left\{e^{\frac{\rho}{2\nu}x}\sin\pi k x:\quad k = 1,\ 2,\ \dots,\ N\right\} \tag{6.13}$$

is positively invariant for the dynamical system $(H_0^1(0,\ 1),\ S_t)$ generated by problem (6.1).

Theorem 6.1.

Let $\gamma \equiv \Gamma - \frac{\rho^2}{4\nu} < \nu\pi^2$. ***Then for any mild solution*** $u(t)$ ***to problem*** (6.1) ***the estimate***

$$\|\partial_x u(t)\| \ \le\ C(\rho,\ \nu)e^{-(\nu\pi^2 - \gamma)t}\|\partial_x u_0\|, \qquad t \ge 0, \tag{6.14}$$

is valid. If $\gamma \ge \nu\pi^2$ ***and*** $\nu\pi^2 N^2 > \gamma$, ***then the subspace*** H_{N-1} ***defined by formula*** (6.13) ***is exponentially attracting:***

$$\mathrm{dist}_{H_0^1(0,\ 1)}(S_t u_0,\ H_{N-1}) \ \le\ C(\rho,\ \nu)\,e^{-(\nu\pi^2N^2 - \gamma)t}\|\partial_x u_0\|. \tag{6.15}$$

Here S_t ***is the evolutionary operator in*** $H_0^1(0,\ 1)$ ***corresponding to*** (6.1).

Proof.

Let $w(x,\ t)$ be of the form (6.9). Then

$$w_\rho(t) \equiv e^{-\frac{\rho}{2\nu}x}w(t) = \sum_{k=1}^{\infty} e^{-(\nu(\pi k)^2 - \gamma)t}\,C_{k,\ \rho}\,e_k(x), \tag{6.16}$$

where $e_k(x) = \sqrt{2}\sin\pi k x$ and

$$C_{k,\ \rho} = \int_0^1 e^{-\frac{\rho}{2\nu}x}u_0(x)\,e_k(x)\mathrm{d}x.$$

Assume that $\gamma < \nu\pi^2$. Then it is obvious that

$$\|\partial_x w_\rho(t)\|^2 = \sum_{k=1}^{\infty} (\pi k)^2 C_{k,\rho}^2 \exp\{-2(\nu(\pi k)^2 - \gamma)t\} \le$$
$$\le \exp\{-2(\nu\pi^2 - \gamma)t\}\left\|\partial_x\left(e^{-\frac{\rho}{2\nu}x} u_0\right)\right\|^2. \tag{6.17}$$

However, equation (6.8) implies that

$$\left\|\partial_x\left(e^{-\frac{\rho}{2\nu}x} u(t)\right)\right\| \le \|\partial_x w_\rho(t)\|.$$

Therefore, estimate (6.14) follows from (6.17) and from the obvious equality $\|u\| \le 1/\pi\|\partial_x u\|$. When $\gamma \ge \nu\pi^2$ and $N > (1/\pi)\sqrt{\gamma/\nu}$ the function $w(t)$ in (6.9) can be rewritten in the form

$$w(t) = h_N(t) + e^{\frac{\rho}{2\nu}x} w_{\rho,N}(t), \tag{6.18}$$

where

$$h_N(t) = \sqrt{2}\, e^{\frac{\rho}{2\nu}x} \sum_{k=1}^{N-1} e^{-(\nu(\pi k)^2 - \gamma)t} C_{k,\rho} \sin\pi k x \in H_{N-1}$$

and $w_{\rho,N}(t)$ can be estimated (cf. (6.17)) as follows:

$$\|\partial_x w_{\rho,N}(t)\|^2 \le \exp\{-2(\nu(\pi N)^2 - \gamma)t\}\left\|\partial_x\left(e^{-\frac{\rho}{2\nu}x} u_0\right)\right\|^2. \tag{6.19}$$

Thus, it is clear that

$$\operatorname{dist}_{H_0^1(0,\,1)}(S_t u_0, H_{N-1}) \le C(\rho, \nu)\|\partial_x w_{\rho,N}(t)\|.$$

Consequently, estimate (6.15) is valid. **Theorem 6.1 is proved**.

In particular, Theorem 6.1 means that if $\gamma \equiv \Gamma - \rho^2/(4\nu) < \nu\pi^2$, then the global attractor of problem (6.1) consists of a single zero element, whereas if $\gamma \ge \nu\pi^2$, the attractor lies in an exponentially attracting invariant subspace H_{N_0}, where $N_0 = [(1/\pi)\sqrt{\gamma/\nu}]$ and $[\cdot]$ is a sign of the integer part of a number.

The following assertion shows that the global minimal attractor of problem (6.1) consists of fixed points of the system. It also provides a description of the corresponding basins of attraction.

Theorem 6.2.

Let $\gamma \equiv \Gamma - \dfrac{\rho^2}{4\nu} > \nu\pi^2$. ***Assume that the number*** $\pi^{-1}\sqrt{\gamma/\nu}$ ***is not integer. Suppose that*** N_0 ***is the greatest integer such that*** $\nu(\pi N_0)^2 < \gamma$. ***Let***

$$C_j(u_0) = \sqrt{2}\int_0^1 e^{-\frac{\rho}{2\nu}x}\, u_0(x)\sin\pi j x\,\mathrm{d}x$$

and let $u(t)$ ***be a mild solution to problem*** (6.1).

(a) ***If*** $C_j(u_0) = 0$ ***for all*** $j = 1, 2, \ldots, N_0$, ***then***

$$\|\partial_x u(t)\| \le C(\nu, \rho)\exp\{-(\nu\pi^2(N_0+1)^2 - \gamma)t\}\|\partial_x u_0\|, \qquad t \ge 0. \quad (6.20)$$

(b) ***If*** $C_j(u_0) \ne 0$ ***for some*** j ***between*** 1 ***and*** N_0, ***then there exist positive numbers*** $C = C(\nu, \rho, \gamma; u_0)$ ***and*** $\beta = \beta(\gamma, \nu)$ ***such that***

$$\|\partial_x(u(t) - \overline{u}_{\sigma k})\| \le Ce^{-\beta t}, \qquad t \ge 0, \quad (6.21)$$

where $\overline{u}_{\pm k}(x)$ ***is defined by formula*** (6.11), $\sigma = \operatorname{sign} C_k(u_0)$, ***and*** k ***is the smallest index between*** 1 ***and*** N_0 ***such that*** $C_k(u_0) \ne 0$.

Proof.

In order to prove assertion (a) it is sufficient to note that the value $h_N(t)$ is identically equal to zero in decomposition (6.18) when $N = N_0 + 1$. Therefore, (6.20) follows from (6.19).

Now we prove assertion (b). In this case equation (6.18) can be rewritten in the form

$$w(t) = g_k(t) + h_k(t) + w_{\rho,\, N_0+1}(t), \quad (6.22)$$

where

$$g_k(t) = \sqrt{2}\, e^{\frac{\rho}{2\nu}x}\, e^{-[\nu(\pi k)^2 - \gamma]t}\, C_k(u_0)\sin\pi k x,$$

$$h_k(t) = \sqrt{2}\, e^{\frac{\rho}{2\nu}x}\sum_{j=k+1}^{N_0} e^{-[\nu(\pi j)^2 - \gamma]t}\, C_{j,\,\rho}\sin\pi j x,$$

and $h_k(t) \equiv 0$ if $k = N_0$. Moreover, the estimate

$$\left\|\partial_x w_{\rho,\, N_0+1}(t)\right\| \le C(\rho, \nu)e^{-(\nu\pi^2(N_0+1)^2 - \gamma)t}\|\partial_x u_0\| \quad (6.23)$$

is valid for $w_{\rho,\, N_0+1}(t)$. It is also evident that

$$\|\partial_x h_k(t)\| \le C(\rho, \nu)\, e^{(\gamma - \nu\pi^2(k+1)^2)t}\|\partial_x u_0\|. \quad (6.24)$$

Since

$$\left|\|w(t)\|^2 - \|g_k(t)\|^2\right| \le (\|w(t)\| + \|g_k(t)\|)\,\left|\|w(t)\| - \|g_k(t)\|\right| \le$$

$$\le (2\|g_k(t)\| + \|h_k(t)\| + \|w_{\rho,\, N_0+1}(t)\|)\,(\|h_k(t)\| + \|w_{\rho,\, N_0+1}(t)\|),$$

using (6.23) and (6.24) we obtain

$$\left|\|w(t)\|^2 - \|g_k(t)\|^2\right| \le C\exp\{2[\gamma - \nu\pi^2(k^2 + (k+1)^2)]t\}\|\partial_x u_0\|^2.$$

Integration gives

$$2\varkappa\int_0^t \|g_k(\tau)\|^2 d\tau = 4\varkappa[C_k(u_0)]^2\int_0^1 e^{\frac{\rho}{\nu}x}\sin^2\pi k x\,dx\int_0^t e^{2(\gamma-\nu(\pi k)^2)\tau}d\tau =$$

$$= 2\left(\frac{C_k(u_0)}{\mu_k}\right)^2 (e^{2(\gamma-\nu(\pi k)^2)t}-1)\ ,$$

where μ_k is defined by formula (6.12). Hence,

$$1+2\varkappa\int_0^t \|w(\tau)\|^2 d\tau = 1+2\left(\frac{C_k(u_0)}{\mu_k}\right)^2 e^{2(\gamma-\nu(\pi k)^2)t}+a_k(t)\,, \tag{6.25}$$

where

$$|a_k(t)| \le C\left[1+\exp\left\{2\left(\gamma-\nu\pi^2\frac{k^2+(k+1)^2}{2}\right)t\right\}\right]\|\partial_x u_0\|^2, \qquad k\le N_0-1\,.$$

Let

$$v_k(t) = \frac{g_k(t)}{\left(1+2\varkappa\int_0^t\|w(\tau)\|^2 d\tau\right)^{1/2}}\,.$$

We consider the case when $1\le k\le N_0-1$. Equations (6.23)–(6.25) imply that

$$\|\partial_x(u(t)-v_k(t))\| \le \|\partial_x w_{\rho,\,N_0+1}(t)\| + \frac{\|\partial_x h_k(t)\|}{\left(1+2\varkappa\int_0^t\|w(\tau)\|^2 d\tau\right)^{1/2}} \le$$

$$\le C(\rho,\nu)\|\partial_x u_0\|\left\{e^{-(\nu\pi^2(N_0+1)^2-\gamma)t}+\right.$$

$$\left.+\frac{\exp\{(\gamma-\nu\pi^2(k+1)^2)t\}}{\left(1+2\left(\frac{C_k(u_0)}{\mu_k}\right)^2 e^{2(\gamma-\nu(\pi k)^2)t}+a_k(t)\right)^{1/2}}\right\}\,.$$

Therefore,

$$\|\partial_x(u(t)-v_k(t))\| \le$$

$$\le C(\rho,\nu,u_0)\left\{e^{-(\nu\pi^2(N_0+1)^2-\gamma)t}+e^{-\nu\pi^2((k+1)^2-k^2)t}\,G(t)^{-1/2}\right\}\ ,$$

where

$$G(t) = (1 + a_k(t))\, e^{-2(\gamma - \nu\pi^2 k^2)t} + 2\Big(\frac{C_k(u_0)}{\mu_k}\Big)^2 .$$

It follows from (6.25) that $G(t) > 0$ for all $t \ge 0$ and $G(0) = 1$. Moreover, the above-mentioned estimate for $a_k(t)$ enables us to state that

$$\lim_{t \to \infty} G(t) = 2\Big(\frac{C_k(u_0)}{\mu_k}\Big)^2 > 0 .$$

This implies that there exists a constant $G_{\min} = G_{\min}(\gamma, k, \nu, u_0) > 0$ such that $G(t) \ge G_{\min}$ for all $t \ge 0$. Consequently,

$$\|\partial_x(u(t) - v_k(t))\| \le C(\rho, \nu, \gamma; u_0)\, e^{-\alpha t}, \tag{6.26}$$

where $\alpha = \alpha(k, \nu, \gamma) = \min\{\nu\pi^2(N_0+1)^2 - \gamma,\ \nu\pi^2(2k+1)\}$. Now we consider the value $v_k(t)$. Evidently,

$$v_k(t) = \varphi_k(t, u_0)\bar{u}_{\sigma k}(x),$$

where

$$\varphi_k(t, u_0) = \frac{\sqrt{2}\,|C_k(u_0)|\,\exp\{(\gamma - \nu\pi^2 k^2)t\}}{\mu_k\Big(1 + 2\Big(\frac{C_k(u_0)}{\mu_k}\Big)^2 \exp\{2(\gamma - \nu\pi^2 k^2)t\} + a_k(t)\Big)^{1/2}} .$$

Simple calculations give us that

$$|\varphi_k(t, u_0) - 1| \le C(\rho, \nu, \gamma, u_0)\, e^{-\bar{\beta} t}$$

with the constant $\bar{\beta} = \nu\pi^2(2k+1)$. This and equation (6.26) imply (6.21), provided $k \le N_0 - 1$. We offer the reader to analyse the case when $k = N_0$ on his/her own. **Theorem 6.2 is proved**.

Theorem 6.2 enables us to obtain a complete description of the basins of attraction of each fixed point of the dynamical system $(H_0^1(0, 1), S_t)$ generated by problem (6.1).

Corollary 6.1.

Let $\gamma \equiv \Gamma - \rho^2/(4\nu) > \nu\pi^2$. Assume that the number $\pi^{-1}\sqrt{\gamma/\nu}$ is not integer. Let N_0 be the greatest integer possessing the property $\nu(\pi N_0)^2 < \gamma$. We denote

$$C_j(u) = \sqrt{2}\int_0^1 e^{-\frac{\rho}{2\nu}x}\, u(x) \sin \pi j x \, dx$$

and define the sets

$$\mathcal{D}_k = \{u \in H_0^1(0, 1):\ C_j(u) = 0,\ j = 1, \ldots, k-1,\ C_k(u) > 0\},$$

$$\mathcal{D}_{-k} = \{u \in H_0^1(0, 1):\ C_j(u) = 0,\ j = 1, \ldots, k-1,\ C_k(u) < 0\}$$

for $k = 1, 2, \ldots, N_0$. *We also assume that*

$$\mathscr{D}_0 = \{u \in H_0^1(0, 1):\ C_j(u) = 0,\ j = 1, 2, \ldots, N_0\}.$$

Then for any $l = 0, \pm 1, \pm 2, \ldots, \pm N_0$ *we have that*

$$\lim_{t \to \infty} \|S_t v - \bar{u}_l\|_{H_0^1(0, 1)} = 0 \Leftrightarrow v \in \mathscr{D}_l,$$

where $\{\bar{u}_l\}$ *are the fixed points of problem* (6.1) *which are defined by equalities* (6.11).

The next assertion gives us a complete description of the global attractor of problem (6.1).

Theorem 6.3.

Assume that the hypotheses of Theorem 6.2 ***hold and*** N_0 ***is the same as in Theorem*** 6.2. ***Then the global attractor*** $\mathscr{A}$ ***of the dynamical system*** $(H_0^1(0, 1), S_t)$ ***generated by equation*** (6.1) ***is the closure of the set***

$$\mathrm{A} = \left\{ v(x) = \frac{\sqrt{2} \sum_{k=1}^{N_0} \xi_k\, e^{\frac{\rho}{2\nu}x} \sin \pi k x}{\left(1 + 2\varkappa \sum_{k, j=1}^{N_0} \xi_k \xi_j \frac{\alpha_{kj}}{\nu_k + \nu_j}\right)^{1/2}} \ :\ \xi_j \in \mathbb{R} \right\}, \tag{6.27}$$

where $\nu_k = \gamma - \nu(\pi k)^2$, $k = 1, 2, \ldots, N_0$, ***and***

$$\alpha_{kj} = 2\int_0^1 e^{\frac{\rho}{\nu}x} \sin \pi k x \cdot \sin \pi j x \,\mathrm{d}x, \qquad k, j = 1, 2, \ldots, N_0.$$

Every complete trajectory $\{u(t):\ t \in \mathbb{R}\}$ ***which lies in the attractor and does not coincide with any of the fixed points*** $\bar{u}_k$, $k = 0, \pm 1, \ldots, \pm N_0$, ***has the form***

$$u(t) = \frac{\sqrt{2} \sum_{k=1}^{N_0} \xi_k\, e^{\nu_k t}\, e^{\frac{\rho}{2\nu}x} \sin \pi k x}{\left(1 + 2\varkappa \sum_{k, j=1}^{N_0} \xi_k \xi_j \frac{\alpha_{kj}}{\nu_k + \nu_j}\, e^{(\nu_k + \nu_j)t}\right)^{1/2}}, \tag{6.28}$$

where ξ_k ***are real numbers,*** $k = 1, 2, \ldots, N_0$, $t \in \mathbb{R}$.

— Exercise *6.6* Show that for all $\xi \in \mathbb{R}^{N_0}$ the function

$$a(t, \xi) = 2\varkappa \sum_{k, j=1}^{N_0} \xi_k \xi_j \frac{\alpha_{kj}}{\nu_k + \nu_j} e^{(\nu_k + \nu_j)t}, \qquad t \geq 0 \tag{6.29}$$

is nonnegative and it is monotonely nondecreasing with respect to t (Hint: $a'_t(t, \xi) \geq 0$ and $a(t, \xi) \to 0$ as $t \to -\infty$).

Proof of Theorem 6.3.

Let h belong to the set A given by formula (6.27). Then by virtue of Lemma 6.1 we have

$$S_t h = \frac{w(x,\ t)}{\left(1 + 2\varkappa \int\limits_0^t \|w(\tau)\|^2 \mathrm{d}\tau\right)^{1/2}}, \tag{6.30}$$

where

$$w(t) = \sqrt{\frac{2}{1 + a(0,\ \xi)}}\ e^{\frac{\rho}{2\nu}x} \sum_{k=1}^{N_0} \xi_k e^{\nu_k t} \sin \pi k x$$

and the value $a(t,\ \xi)$ is defined according to (6.29). Simple calculations show that

$$2\varkappa \int\limits_0^t \|w(\tau)\|^2 \mathrm{d}\tau = \frac{a(t,\ \xi) - a(0,\ \xi)}{1 + a(0,\ \xi)}.$$

Therefore, it is easy to see that $S_t h = u(t)$ for $t \geq 0$, where $u(t)$ has the form (6.28). It follows that $S_t \mathrm{A} = \mathrm{A}$ and that a complete trajectory $u(t)$ lying in A has the form (6.28). In particular, this means that $\mathrm{A} \subset \mathcal{A}$. To prove that $\overline{\mathrm{A}} = \mathcal{A}$ it is sufficient to verify using Theorem 6.1 and the reduction principle (see Theorem 1.7.4) that for any element $h \in H_{N_0}$ there exists a semitrajectory $v(t) \subset \mathrm{A}$ such that

$$\lim_{t \to \infty} \|\partial_x (S_t h - v(t))\| = 0$$

uniformly with respect to h from any bounded set in H_{N_0}. To do this, it should be kept in mind that for

$$h = \sqrt{2} \sum_{k=1}^{N_0} \xi_k\, e^{\frac{\rho}{2\nu}x} \sin \pi k x \ \in\ H_{N_0} \tag{6.31}$$

$S_t h$ has the form (6.30) with

$$w(t) = \sqrt{2}\ e^{\frac{\rho}{2\nu}x} \sum_{k=1}^{N_0} \xi_k e^{\nu_k t} \sin \pi k x.$$

Therefore, it is easy to find that

$$S_t h = \frac{w(t)}{(1 + a(t,\ \xi) - a(0,\ \xi))^{1/2}},$$

where $a(t,\ \xi)$ is given by formula (6.29). Hence, if we choose

$$v(t) = \frac{w(t)}{(1 + a(t,\ \xi))^{1/2}} \ \in\ \mathrm{A},$$

we find that

$$S_t h - v(t) = \psi(t,\ h) S_t h\,, \tag{6.32}$$

where

$$\psi(t,\ h) = 1 - \left(1 - \frac{a(0,\ \xi)}{1 + a(t,\ \xi)}\right)^{1/2}.$$

Using the obvious inequality

$$1 - (1-x)^{1/2} \le x\,, \qquad 0 \le x \le 1\,,$$

we obtain that

$$\psi(t,\ h) \le \frac{a(0,\ \xi)}{1 + a(t,\ \xi)}\,,$$

provided that h has the form (6.31). It is evident that

$$a(t,\ \xi) \ \ge\ a(0,\ \xi) + c_0 \int_0^t e^{2\nu_{N_0}\tau} d\tau \cdot \sum_{k=1}^{N_0} \xi_k^2\ .$$

Therefore,

$$\psi(t,\ h) \le C\left(\int_0^t e^{2\nu_{N_0}\tau} d\tau\right)^{-1}.$$

Consequently, the dissipativity property of S_t in $H_0^1(0,\ 1)$ and equations (6.32) give us that

$$\|\partial_x (S_t h - v(t))\| \ \le\ C\left(\int_0^t e^{2\nu_{N_0}\tau} d\tau\right)^{-1}, \qquad t \ge t_B$$

for all $h \in B \subset H_{N_0}$, where B is an arbitrary bounded set. Thus, $\overline{\mathrm{A}} = \mathscr{A}$ and therefore **Theorem 6.3 is proved**.

— Exercise *6.7* Show that the set A coincides with the unstable manifold $M_+(0)$ emanating from zero, provided that the hypotheses of Theorem 6.3 hold.

— Exercise *6.8* Show that the set $\mathrm{A} = M_+(0)$ from Theorem 6.3 can be described as follows:

$$\mathrm{A} = \left\{ v = \sqrt{2} \sum_{k=1}^{N_0} \eta_k\, e^{\frac{\rho}{2\nu}x} \sin \pi k x : \right.$$

$$\left. \left(2\varkappa \sum_{k,\ j=1}^{N_0} \eta_k \eta_j \frac{\alpha_{kj}}{\nu_k + \nu_j} < 1\,, \quad \eta \in \mathbb{R}^{N_0} \right) \right\}\ .$$

Therewith, the global attractor $\mathcal{A}$ has the form

$$\mathcal{A} = \left\{ v = \sqrt{2} \sum_{k=1}^{N_0} \eta_k \, e^{\frac{\rho}{2\nu}x} \sin \pi k x : \; 2\varkappa \sum_{k,\, j=1}^{N_0} \eta_k \, \eta_j \, \frac{\alpha_{kj}}{\nu_k + \nu_j} \le 1 \right\}.$$

Exercise *6.9* Prove that the boundary

$$\partial \mathrm{A} = \left\{ v = \sqrt{2} \sum_{k=1}^{N_0} \eta_k \, e^{\frac{\rho}{2\nu}x} \sin \pi k x : \right.$$

$$\left. \left(2\varkappa \sum_{k,\, j=1}^{N_0} \eta_k \, \eta_j \, \frac{\alpha_{kj}}{\nu_k + \nu_j} = 1 \,, \quad \eta \in \mathbb{R}^{N_0} \right) \right\}$$

of the set A is a strictly invariant set.

Exercise *6.10* Show that any trajectory γ lying in $\partial \mathrm{A}$ has the form $\{u(t),\; t \in \mathbb{R}\}$, where

$$u(t) = \frac{\sum_{k=1}^{N_0} \eta_k \, e^{\nu_k t} \, e^{\frac{\rho}{2\nu}x} \sin \pi k x}{\sqrt{\varkappa} \left(\sum_{k,\, j=1}^{N_0} \eta_k \, \eta_j \, \frac{\alpha_{kj}}{\nu_k + \nu_j} \, e^{(\nu_k + \nu_j) t} \right)^{1/2}}, \qquad \eta \in \mathbb{R}^{N_0}.$$

Exercise *6.11* Using the result of Exercise 6.10 find the unstable manifold $M_+(\overline{u}_k)$ emanating from the fixed point $\overline{u}_k$, $k = \pm 1, \pm 2, \ldots, \pm N_0$.

Exercise *6.12* Find out for which pairs of fixed points $\{\overline{u}_k, \overline{u}_j\}$, $k, j = \pm 1, \pm 2, \ldots, \pm N_0$, there exists a heteroclinic trajectory connecting them, i.e. a complete trajectory $\{u_{kj}(t) :\; t \in \mathbb{R}\}$ such that

$$\overline{u}_k = \lim_{t \to -\infty} u_{kj}(t), \qquad \overline{u}_j = \lim_{t \to +\infty} u_{kj}(t).$$

Exercise *6.13* Display graphically the global attractor $\mathcal{A}$ on the plane generated by the vectors $e_1 = e^{[\rho/(2\nu)]x} \sin \pi x$ and $e_2 = e^{[\rho/(2\nu)]x} \times \sin 2\pi x$ for $N_0 = 2$.

Exercise 6.14 Study the structure of the global attractor of the dynamical system generated by the equation

$$\begin{cases} u_t - \nu u_{xx} + \left(\varkappa \int\limits_0^1 (|u(x,t)|^2 + |v(x,t)|^2)\,\mathrm{d}x - \Gamma \right) u - \alpha v = 0, \quad 0 < x < 1, \; t > 0, \\ v_t - \nu v_{xx} + \left(\varkappa \int\limits_0^1 (|u(x,t)|^2 + |v(x,t)|^2)\,\mathrm{d}x - \Gamma \right) v + \alpha u = 0, \quad 0 < x < 1, \; t > 0, \\ u|_{x=0} = u|_{x=1} = v|_{x=0} = v|_{x=1} = 0 \end{cases}$$

in $H_0^1(0,\ 1) \times H_0^1(0,\ 1)$, where ν, $\varkappa$, Γ, α are positive parameters.

§ 7 *Simplified Model of Appearance of Turbulence in Fluid*

In 1948 German mathematician E. Hopf suggested (see the references in [3]) to consider the following system of equations in order to illustrate one of the possible scenarios of the turbulence appearance in fluids:

$$\begin{cases} u_t = \mu u_{xx} - v * v - w * w - u * 1, & (7.1) \\ v_t = \mu v_{xx} + v * u + v * a + w * b, & (7.2) \\ w_t = \mu w_{xx} + w * u - v * b + w * a, & (7.3) \end{cases}$$

where the unknown functions u, v, and w are even and 2π-periodic with respect to x and

$$(f * g)(x) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(x-y)\, g(y)\,\mathrm{d}y.$$

Here $a(x)$ and $b(x)$ are even 2π-periodic functions and μ is a positive constant. We also set the initial conditions

$$u|_{t=0} = u_0(x), \quad v|_{t=0} = v_0(x), \quad w|_{t=0} = w_0(x). \tag{7.4}$$

As in the previous section the asymptotic behaviour of solutions to problem (7.1)–(7.4) can be explicitly described.

Let us introduce the necessary functional spaces. Let

$$H = \{ f \in L^2_{\mathrm{loc}}(\mathbb{R}) \colon\ f(x) = f(-x) = f(x + 2\pi) \}.$$

Evidently H is a separable Hilbert space with the inner product and the norm defined by the formulae:

$$(f,\ g) = \int_0^{2\pi} f(x)\, g(x)\, \mathrm{d}x\,, \qquad \|f\|^2 = (f,\ f)$$

There is a natural orthonormal basis

$$\left\{ \frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}} \cos x,\ \frac{1}{\sqrt{\pi}} \cos 2x,\ \ldots \right\}$$

in this space. The coefficients $C_n(f)$ of decomposition of the function $f \in H$ with respect to this basis have the form

$$C_0(f) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)\, \mathrm{d}x\,, \qquad C_n(f) = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \cos nx\, \mathrm{d}x\,.$$

— Exercise *7.1* Let $f,\ g \in H$. Then $f * g \in H$ and

$$\|f * g\| \le \frac{1}{\sqrt{2\pi}} \|f\| \cdot \|g\|\,. \tag{7.5}$$

The Fourier coefficients C_n of the functions $f * g$, f, and g obey the equations

$$C_0(f * g) = \frac{1}{\sqrt{2\pi}} C_0(f)\, C_0(g)\,, \qquad C_n(f * g) = \frac{1}{2\sqrt{\pi}} C_n(f)\, C_n(g)\,. \tag{7.6}$$

— Exercise *7.2* Let p_m be the orthoprojector onto the span of elements $\{\cos kx,\ k = 0,\ 1,\ \ldots,\ m\}$ in H. Show that

$$p_m(f * g) = (p_m f) * g = f * (p_m g)\,. \tag{7.7}$$

Let us consider the Hilbert space $\mathbb{H} = H^3 \equiv H \times H \times H$ with the norm $\|(u;\ v;\ w)\|_{\mathbb{H}} = (\|u\|^2 + \|v\|^2 + \|w\|^2)^{1/2}$ as the phase space of problem (7.1)–(7.4). We define an operator A by the formula

$$A(u,\ v,\ w) = (u - \mu u_{xx};\ v - \mu v_{xx};\ w - \mu w_{xx})\,, \quad (u;\ v;\ w) \in D(A)$$

on the domain

$$D(A) = [H^2_{\mathrm{loc}}(\mathbb{R})]^3 \cap \mathbb{H}\,,$$

where $H^2_{\mathrm{loc}}(\mathbb{R})$ is the second order Sobolev space.

— Exercise *7.3* Prove that A is a positive operator with discrete spectrum. Its eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ have the form:

$$\lambda_{3k} = \lambda_{3k+1} = \lambda_{3k+2} = 1 + \mu k^2\,, \qquad k = 0,\ 1,\ 2,\ \ldots\,,$$

while the corresponding eigenelements are defined by the formulae

$$\left.\begin{aligned} e_0 = \frac{1}{\sqrt{2\pi}}(1;\ 0;\ 0), \quad e_1 = \frac{1}{\sqrt{2\pi}}(0;\ 1;\ 0), \quad e_2 = \frac{1}{\sqrt{2\pi}}(0;\ 0;\ 1), \\ e_{3k} = \frac{1}{\sqrt{\pi}}(\cos kx;\ 0;\ 0), \quad e_{3k+1} = \frac{1}{\sqrt{\pi}}(0;\ \cos kx;\ 0), \\ e_{3k+2} = \frac{1}{\sqrt{\pi}}(0;\ 0;\ \cos kx), \end{aligned}\right\} \tag{7.8}$$

where $k = 1,\ 2,\ \dots$

Let

$$\begin{aligned} b_1(u,\ v,\ w) &= -v*v - w*w - u*1, \\ b_2(u,\ v,\ w) &= v*u + v*a + w*b, \\ b_3(u,\ v,\ w) &= w*u - v*b + w*a. \end{aligned}$$

Equation (7.5) implies that $b_j(u,\ v,\ w) \in H$, provided a, b, u, v, and w are the elements of the space H, $j = 1,\ 2,\ 3$. Therefore, the formula

$$B(u,\ v,\ w) = (b_1(u,\ v,\ w) + u;\ b_2(u,\ v,\ w) + v;\ b_3(u,\ v,\ w) + w)$$

gives a continuous mapping of the space $\mathbb{H}$ into itself.

— Exercise *7.4* Prove that

$$\|B(y_1) - B(y_2)\|_{\mathbb{H}} \le C\Big(1 + \|a\| + \|b\| + \|y_1\|_{\mathbb{H}} + \|y_2\|_{\mathbb{H}}\Big)\|y_1 - y_2\|_{\mathbb{H}},$$

where $y_j = (u_j;\ v_j;\ w_j) \in \mathbb{H}$, $j = 1,\ 2$.

Thus, if $a,\ b \in H$, then problem (7.1)–(7.4) can be rewritten in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} + Ay = B(y), \qquad y\big|_{t=0} = y_0,$$

where A and B satisfy the hypotheses of Theorem 2.1. Therefore, the Cauchy problem (7.1)–(7.4) has a unique mild solution $y(t) = (u(t),\ v(t),\ w(t))$ in the space $\mathbb{H}$ on a segment $[0,\ T]$, provided that $a,\ b \in H$. In order to prove the global existence theorem we consider the Galerkin approximations of problem (7.1)–(7.4). The Galerkin approximate solution $y_m(t)$ of the order $3m$ with respect to basis (7.8) can be presented in the form

$$y_m(t) = \Big(u^{(m)}(t);\ v^{(m)}(t);\ w^{(m)}(t)\Big) = \sum_{k=0}^{m-1}\Big(u_k(t);\ v_k(t);\ w_k(t)\Big)\cos kx, \tag{7.9}$$

where $u_k(t)$, $v_k(t)$, and $w_k(t)$ are scalar functions. By virtue of equations (7.7) it is easy to check that the functions $u^{(m)}(t)$, $v^{(m)}(t)$, and $w^{(m)}(t)$ satisfy equations (7.1)–(7.3) and the initial conditions

$$u^{(m)}(0) = p_m u_0, \qquad v^{(m)}(0) = p_m v_0, \qquad w^{(m)}(0) = p_m w_0. \tag{7.10}$$

Thus, approximate solutions exist, locally at least. However, if we use (7.1)–(7.3) we can easily find that

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|u^{(m)}(t)\|^2+\|v^{(m)}(t)\|^2+\|w^{(m)}(t)\|^2\right\}+$$

$$+\ \mu\left\{\left\|u_x^{(m)}\right\|^2+\left\|v_x^{(m)}\right\|^2+\left\|w_x^{(m)}\right\|^2\right\}=$$

$$=\ -(u^{(m)}*1,\ u^{(m)})+(v^{(m)}*a,\ v^{(m)})+(w^{(m)}*a,\ w^{(m)})\ .$$

Therefore, inequality (7.5) leads to the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y_m(t)\|_{\mathbb{H}}^2\ \le\ C(1+\|a\|)\|y_m(t)\|_{\mathbb{H}}^2\ .$$

This implies the global existence of approximate solutions $y_m(t)$ (see Exercise 2.1). Therefore, Theorem 2.2 guarantees the existence of a mild solution to problem (7.1)–(7.4) in the space $\mathbb{H}=H^3$ on the time interval of any length T. Moreover, the mild solution $y(t)=(u(t);\ v(t);\ w(t))$ possesses the property

$$\max_{[0,\,T]}\|y(t)-y_m(t)\|\to 0\,,\qquad m\to\infty$$

for any segment $[0,\,T]$. Approximate solution $y_m(t)$ has the structure (7.9).

— Exercise *7.5* Show that the scalar functions $\{u_k(t);\ v_k(t);\ w_k(t)\}$ involved in (7.9) are solutions to the system of equations:

$$\begin{cases} \dot{u}_k+\mu k^2u_k\ =\ -v_k^2-w_k^2-u_0\delta_{k0}\ , & (7.11)\\ \dot{v}_k+\mu k^2v_k\ =\ v_ku_k+v_ka_k+w_kb_k\ , & (7.12)\\ \dot{w}_k+\mu k^2w_k\ =\ w_ku_k-v_kb_k+w_ka_k\ . & (7.13)\end{cases}$$

Here $k=1,\ 2,\ \dots$, $\delta_{k0}=0$ for $k\neq 0$, $\delta_{00}=1$, the numbers $a_k=$ $=C_k(a)$ and $b_k=C_k(b)$ are the Fourier coefficients of the functions $a(x)$ and $b(x)$.

Thus, equations (7.1)–(7.3) generate a dynamical system $(\mathbb{H},\ S_t)$ with the evolutionary operator S_t defined by the formula

$$S_t\,y_0=(u(t);\ v(t);\ w(t))\,,$$

where $(u(t);\ v(t);\ w(t))$ is a mild solution (in $\mathbb{H}$) to the Cauchy problem (7.1)–(7.4), $y_0=(u_0,\ v_0,\ w_0)$. An interesting property of this system is given in the following exercise.

— Exercise *7.6* Let $\mathcal{L}_k$ be the span of elements $\{e_{3k},\ e_{3k+1},\ e_{3k+2}\}$, where $k=0,\ 1,\ 2,\ \dots$ and $\{e_n\}$ are defined by equations (7.8). Then the subspace $\mathcal{L}_k$ of the phase space $\mathbb{H}$ is positively invariant with respect to S_t $(S_t\,\mathcal{L}_k\subset\mathcal{L}_k)$.

Therefore, the phase space $\mathbb{H}$ of the dynamical system $(\mathbb{H}, S_t)$ falls into the orthogonal sum

$$\mathbb{H} = \sum_{k=0}^{\infty} \oplus \mathcal{L}_k$$

of invariant subspaces. Evidently, the dynamics of the system $(\mathbb{H}, S_t)$ in the subspace $\mathcal{L}_k$ is completely determined by the system of three ordinary differential equations (7.11)–(7.13).

Lemma 7.1.

Assume that $a, b \in H$ *and*

$$C_0 = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} a(x)\,\mathrm{d}x < 0\,. \tag{7.14}$$

Then the dynamical system $(\mathbb{H}, S_t)$ *is dissipative.*

Proof.

Let us fix N and then consider the initial conditions $y_0 = (u_0, v_0, w_0)$ from the subspace

$$\mathbb{H}_N = \sum_{k=0}^{N} \oplus \mathcal{L}_k = \sum_{k=0}^{N} \oplus \operatorname{Lin}\{e_{3k}, e_{3k+1}, e_{3k+2}\},$$

where $\{e_n\}$ are defined by equations (7.8). It is clear that $\mathbb{H}_N$ is positively invariant and the trajectory

$$y(t) = (u(t);\ v(t);\ w(t)) = \sum_{k=0}^{N} (u_k(t);\ v_k(t);\ w_k(t)) \cos kx$$

of the system is a function satisfying (7.1)–(7.4) in the classical sense. Let p_m be the orthoprojector in H onto the span of elements $\{\cos kx:\ n = 0, 1, \ldots, m\}$. We introduce a new variable $\tilde{u}(t) = u(t) + \alpha^m$ instead of the function $u(t)$. Here $\alpha^m = (p_m - p_0)a$. Equations (7.1)–(7.3) can be rewritten in the form

$$\begin{cases} \tilde{u}_t - \mu \tilde{u}_{xx} = -v * v - w * w + (-\tilde{u} + \alpha^m) * 1 - \mu \alpha^m_{xx}\,, & (7.15) \\ v_t - \mu v_{xx} = v * \tilde{u} + v * (q_m a) + w * b + v * (p_0 a)\,, & (7.16) \\ w_t - \mu w_{xx} = w * \tilde{u} - v * b + w * (q_m a) + w * (p_0 a)\,, & (7.17) \end{cases}$$

where $q_m = 1 - p_m$. The properties of the convolution operation (see Exercises 7.1 and 7.2) enable us to show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\tilde{u}\|^2 + \|v\|^2 + \|w\|^2) + \mu(\|\tilde{u}_x\|^2 + \|v_x\|^2 + \|w_x\|^2) =$$
$$= ((-\tilde{u} + \alpha^m) * 1,\ \tilde{u}) + \mu(\alpha_x^m,\ \tilde{u}_x) + (v * (q_m a),\ v) +$$
$$+ (w * (q_m a),\ w) + (v * (p_0 a),\ v) + (w * (p_0 a),\ w)\ . \tag{7.18}$$

It is clear that

$$((-\tilde{u} + \alpha^m) * 1,\ \tilde{u}) = -[C_0(\tilde{u})]^2, \qquad (v * (p_0 a),\ v) = \frac{1}{\sqrt{2\pi}} C_0(a)[C_0(v)]^2,$$

where $C_0(f)$ is the zeroth Fourier coefficient of the function $f(x) \in H$. Moreover, the estimate

$$\|q_m v\| \le \|v_x\|, \qquad m \ge 1,$$

holds. We choose $m \ge 1$ such that $(1/\sqrt{2\pi})\|q_m a\| \le \mu/2$. Then equation (7.18) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}\|^2 + \|v\|^2 + \|w\|^2\right) + \mu\left(\|\tilde{u}_x\|^2 + \|v_x\|^2 + \|w_x\|^2\right) +$$
$$+ 2|C_0(\tilde{u})|^2 + \sqrt{\frac{2}{\pi}}|C_0(a)|\left(|C_0(v)|^2 + |C_0(w)|^2\right) \le \mu\|\alpha_x^m\|^2.$$

If we use the inequality

$$\|f\|^2 \le |C_0(f)|^2 + \|f_x\|^2,$$

then we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\tilde{u}\|^2 + \|v\|^2 + \|w\|^2) + \nu(\|u\|^2 + \|v\|^2 + \|w\|^2) \le \mu\|\alpha_x^m\|^2,$$

where $\nu = \min(\mu,\ 2,\ \sqrt{2/\pi}|C_0(a)|)$. Consequently, the estimate

$$\|\tilde{y}(t)\|_{\mathbb{H}}^2 \le \|\tilde{y}(0)\|_{\mathbb{H}}^2 e^{-\nu t} + \frac{\mu}{\nu}(1 - e^{-\nu t})\|\alpha_x^m\|^2 \tag{7.19}$$

is valid for $\tilde{y}(t) = (\tilde{u}(t),\ v(t),\ w(t))$, provided $\|q_m a\| \le \mu\sqrt{\pi/2}$ and $\tilde{y}_0 = y_0 + (\alpha^m,\ 0,\ 0)$ where $y_0 \in \mathbb{H}_N$. By passing to the limit we can extend inequality (7.19) over all the elements $y_0 \in \mathbb{H}$. Thus, the system $(\mathbb{H},\ S_t)$ possesses an absorbing set

$$\mathcal{B}_m = \{(u,\ v,\ w):\ \|u + (p_m - p_0)a\|^2 + \|v\|^2 + \|w\|^2 \le R_m^2\}, \tag{7.20}$$

where m is such that $\|a - p_m a\| \le \mu\sqrt{\pi/2}$ and

$$R_m^2 = \mu\|[(p_m - p_0)a]_x\|^2\left(\min\left(\mu,\ 2,\ \sqrt{\frac{2}{\pi}}|C_0(a)|\right)\right)^{-1} + 1.$$

Exercise 7.7 Show that the ball $\mathcal{B}_m$ defined by equality (7.20) is positively invariant.

Exercise *7.8* Consider the restriction of the dynamical system $(\mathbb{H},\ S_t)$ to the subspace

$$\widetilde{\mathbb{H}} = \left\{ h(x) = (u(x);\ v(x);\ w(x)) \in \mathbb{H},\quad \int_0^{2\pi} h(x)\mathrm{d}x = 0 \right\} = \sum_{n\ge 1} \mathcal{L}_n .$$

Show that $(\widetilde{\mathbb{H}},\ S_t)$ is dissipative not depending on the validity of condition (7.14).

Lemma 7.2.

Assume that the hypotheses of Lemma 7.1 *hold and let* p_m *be the orthoprojector onto the span of elements* $\{\cos kx:\ k = 0,\ 1,\ \dots,\ m\}$ *in* H, $q_m = 1 - p_m$. *Then the estimate*

$$\|y_m(t)\|_{\mathbb{H}}^2 \le \|y_0\|_{\mathbb{H}}^2 \exp\left\{-2\left((m+1)^2\mu - \frac{\|q_m a\|}{\sqrt{2\pi}}\right)t\right\} \tag{7.21}$$

holds for all m *such that* $\|q_m a\| < \mu\sqrt{2\pi}(m+1)^2$. *Here* $y_m(t) = (q_m u(t);\ q_m v(t);\ q_m w(t))$ *and* $(u(t);\ v(t);\ w(t))$ *is the mild solution to problem* (7.1)–(7.4) *with the initial condition* $y_0 = (u_0;\ v_0;\ w_0)$.

Proof.

As in the proof of Lemma 7.1 we assume that $y_0 = (u_0;\ v_0;\ w_0) \in \mathbb{H}_N$ for some N. If we apply the projector q_m to equalities (7.15)–(7.17), then we get the equations

$$\begin{cases} u_t^m - \mu u_{xx}^m = -v^m * v^m - w^m * w^m , \\ v_t^m - \mu v_{xx}^m = v^m * u^m + w^m * b + v^m * q_m a , \\ w_t^m - \mu w_{xx}^m = w^m * u^m - v^m * b + w^m * q_m a , \end{cases}$$

where $u^m = q_m u$, $v^m = q_m v$ and $w^m = q_m w$. Therefore, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u^m\|^2 + \|v^m\|^2 + \|w^m\|^2\right) + \mu\left(\|u_x^m\|^2 + \|v_x^m\|^2 + \|w_x^m\|^2\right) \le$$

$$\le \frac{1}{\sqrt{2\pi}}\|q_m a\|\left(\|v^m\|^2 + \|w^m\|^2\right) \tag{7.22}$$

as in the proof of the previous lemma. It is easy to check that $\|(q_m h)_x\|^2 \ge (m+1)^2\|h\|^2$ for every $h \in p_N H$. Therefore, inequality (7.22) implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_m(t)\|_{\mathbb{H}}^2 + \left(\mu(m+1)^2 - \frac{1}{\sqrt{2\pi}}\|q_m a\|\right)\|y_m(t)\|_{\mathbb{H}}^2 \le 0 .$$

Hence, equation (7.21) is valid. Lemma 7.2 is proved.

Lemmata 7.1 and 7.2 enable us to prove the following assertion on the existence of the global attractor.

Theorem 7.1.

Let $a, b \in H$ ***and let condition*** (7.14) ***hold. Then the dynamical system*** $(\mathbb{H}, S_t)$ ***generated by the mild solutions to problem*** (7.1)–(7.4) ***possesses a global attractor*** $\mathcal{A}_\mu$. ***This attractor is a compact connected set. It lies in the finite-dimensional subspace***

$$\mathbb{H}_N = \sum_{k=0}^{N} \oplus \operatorname{Lin}\{e_{3k},\ e_{3k+1},\ e_{3k+1}\},$$

where the vectors $\{e_n\}$ ***are defined by equalities*** (7.8) ***and the parameter*** N ***is defined as the smallest number possessing the property*** $\|q_N a\| < \mu\sqrt{2\pi}(N+1)^2$. ***Here*** q_N ***is the orthoprojector onto the subspace generated by the elements*** $\{\cos nx :\ n \ge N+1\}$ ***in*** H.

To prove the theorem it is sufficient to note that the dynamical system is compact (see Lemma 4.1). Therefore, we can use Theorem 1.5.1. In particular, it should be noted that belonging of the attractor $\mathcal{A}_\mu$ to the subspace $\mathcal{H}_N$ means that $\dim_f \mathcal{A}_\mu \le 3(N+1)$, where N is an arbitrary number possessing the property $\|q_N a\| < \mu\sqrt{2\pi}(N+1)^2$. Below we describe the structure of the attractor and evaluate its dimension exactly.

According to Lemma 7.2 the subspace $\mathbb{H}_N$ is a uniformly exponentially attracting and positively invariant set. Therefore, by virtue of Theorem 1.7.4 it is sufficient to study the structure of the global attractor of the finite-dimensional dynamical system $(\mathbb{H}_N, S_t)$. To do that it is sufficient to study the qualitative behaviour of the trajectory in each invariant subspace $\mathcal{L}_k$, $0 \le k \le N$ (see Exercise 7.6). This behaviour is completely described by equations (7.11)–(7.13) which get transformed into system (1.6.4)–(1.6.6) studied before if we take $\mu = \mu k^2 + \delta_{k0}$, $\nu = \mu k^2 - a_k$, and $\beta = b_k$. Therefore, the results contained in Section 6 of Chapter 1 lead us to the following conclusion.

Theorem 7.2.

Let the hypotheses of Theorem 7.1 ***hold. Then the global minimal attractor*** $\mathcal{A}_{\min}^{(\mu)}$ ***of the dynamical system*** $(\mathbb{H}, S_t)$ ***generated by mild solutions to problem*** (7.1)–(7.4) ***has the form*** $\mathcal{A}_{\min}^{(\mu)} = \{0\} \cup \Sigma_\mu$, ***where***

$$\Sigma_\mu = \bigcup_{k \in J_\mu(a)} \ \bigcup_{0 \le \varphi < 2\pi} \left\{ (u_k;\ r_k \cos\varphi;\ r_k \sin\varphi) \frac{\cos kx}{\sqrt{\pi}} \right\}.$$

Here $u_k = \mu k^2 - a_k$, $r_k = k(a_k\mu - \mu^2 k^2)^{1/2}$, ***the values*** $a_k = C_k(a)$ ***are the***

Fourier coefficients of the function $a(x)$***, and the number*** k ***ranges over the set of indices*** $J_\mu(a)$ ***such that*** $0 < \mu k^2 < a_k$***. Topologically*** Σ_μ ***is a torus (i.e. a cross product of circumferences) of the dimension*** $\mathrm{Card}\, J_\mu(a)$***. The global attractor*** $\mathcal{A}_\mu$ ***of the system*** $(\mathbb{H}, S_t)$ ***can be obtained from*** $\mathcal{A}^{(\mu)}_{\min}$ ***by attaching the unstable manifold*** $M_+(0)$ ***emanating from the zero element of the space*** $\mathbb{H}$***. Moreover,*** $\dim \mathcal{A}_\mu = 2\,\mathrm{Card}\, J_\mu(a)$.

It should be noted that appearance of a limit invariant torus of high dimension posesssing the structure described in Theorem 7.2 is usually assosiated with the Landau-Hopf picture of turbulence appearance in fluids. Assume that the parameter μ gradually decreases. Then for some fixed choice of the function $a(x)$ the following picture is sequentially observed. If μ is large enough, then there exists only one attracting fixed point in the system. While μ decreases and passes some critical value μ_1, this fixed point loses its stability and an attracting limit cycle arises in the system. A subsequent decrease of μ leads to the appearance of a two-dimensional torus. It exists for some interval of values of μ: $\mu_3 < \mu < \mu_2$ $(< \mu_1)$. Then tori of higher dimensions arise sequentially. Therefore, the character of asymptotic behaviour of typical trajectories becomes more complicated as μ decreases. According to the Landau-Hopf scenario, movement along an infinite-dimensional torus corresponds to the turbulence.

§ 8 On Retarded Semilinear Parabolic Equations

In this section we show how the above-mentioned ideas can be used in the study of the asymptotic properties of dynamical systems generated by the retarded perturbations of problem (2.1). It should be noted that systems corresponding to ordinary retarded differential equations are quite well-studied (see, e. g., the book by J. Hale [4]). However, there are only occasional journal publications on the retarded partial differential equations. The exposition in this section is quite brief. We give the reader an opportunity to restore the missing details independently.

As before, let A be a positive operator with discrete spectrum in a separable Hilbert space H and let $C(a, b, \mathcal{F}_\theta)$ be the space of strongly continuous functions on the segment $[a, b]$ with the values in $\mathcal{F}_\theta = D(A^\theta)$, $\theta \ge 0$. Further we also use the notation $C_\theta = C(-r, 0; \mathcal{F}_\theta)$, where $r > 0$ is a fixed number (with the meaning of the delay time). It is clear that C_θ is a Banach space with the norm

$$|v|_{C_\theta} \equiv \max\left\{\|A^\theta v(\sigma)\| :\ \sigma \in [-r;\ 0]\right\} .$$

Let B be a (nonlinear) mapping of the space C_θ into H possessing the property

$$\|B(v_1)-B(v_2)\| \le M(R)|v_1-v_2|_{C_\theta}, \qquad 0 \le \theta < 1, \tag{8.1}$$

for any $v_1, v_2 \in C_\theta$ such that $|v_j|_{C_\theta} \le R$, where $R>0$ is an arbitrary number and $M(R)$ is a nondecreasing function. In the space H we consider a differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}+Au = B(u_t), \qquad t \ge t_0, \tag{8.2}$$

where u_t denotes the element from C_θ determined with the help of the function $u(t)$ by the equality

$$u_t(\sigma) = u(t+\sigma), \qquad \sigma \in [-r,\ 0].$$

We equip equation (8.2) with the initial condition

$$u_{t_0}(\sigma) = u(t_0+\sigma) = v(\sigma), \qquad \sigma \in [-r,\ 0], \tag{8.3}$$

where v is an element from C_θ.

The simplest example of problem (8.2) and (8.3) is the Cauchy problem for the nonlinear retarded diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f_1(u(t,\ x)) - f_2(u(t-r,\ x)), & x \in \Omega,\ t > t_0, \\ u|_{\partial\Omega} = 0, \quad u(\sigma)|_{\sigma \in [t_0-r,\ t_0]} = v(\sigma). \end{cases} \tag{8.4}$$

Here r is a positive parameter, $f_1(u)$ and $f_2(u)$ are the given scalar functions. As in the non-retarded case (see Section 2), we give the following definition.

A function $u(t) \in C(t_0-r,\ t_0+T;\ \mathcal{F}_\theta)$ is called a ***mild (in $\mathcal{F}_\theta$) solution*** to problem (8.2) and (8.3) on the half-interval $[t_0,\ t_0+T)$ if (8.3) holds and $u(t)$ satisfies the integral equation

$$u(t) = e^{-(t-t_0)A}v(0) + \int_{t_0}^{t} e^{-(t-t_0)A}B(u_\tau)\mathrm{d}\tau. \tag{8.5}$$

The following analogue of Theorem 2.1 on the local solvability of problem (8.2) and (8.3) holds.

Theorem 8.1.

Assume that (8.1) ***holds. Then for any initial condition*** $v \in C_\theta$ ***there exists*** $T > 0$ ***such that problem*** (8.2) ***and*** (8.3) ***has a unique mild solution on the half-interval*** $[t_0,\ t_0+T)$.

Proof.

As in Section 2, we use the fixed point method. For the sake of simplicity we consider the case $t_0 = 0$ (for arbitrary $t_0 \in \mathbb{R}$ the reasoning is similar). In the space $C_\theta(0,\ T) \equiv C(0,\ T;\ \mathcal{F}_\theta)$ we consider a ball

$$\mathcal{B}_\rho = \{w \in C_\theta(0, T) : \ |w - \hat{v}|_{C_\theta(0, T)} \le \rho\},$$

where $\hat{v}(t) = \exp\{-At\}v(0)$ and $v(\sigma) \in C_\theta$ is the initial condition for problem (8.2) and (8.3). We also use the notation

$$|w|_{C_\theta(0, T)} \equiv \max\{\|A^\theta w(t)\| : \ t \in [0, T]\}.$$

We consider the mapping K from $C_\theta(0, T)$ into itself defined by the formula

$$[Kw](t) = \hat{v}(t) + \int_0^t e^{-(t-\tau)A} B(w_\tau) d\tau .$$

Here we assume that $w(\sigma) = v(\sigma)$ for $\sigma \in [-r, 0)$. Using the estimate (see Exercise 1.23)

$$\|A^\theta e^{-sA}\| \le \left(\frac{\theta}{es}\right)^\theta, \qquad s > 0, \tag{8.6}$$

(for $\theta = 0$ we suppose $\theta^\theta = 1$) we have that

$$\|A^\theta(Kw_1(t) - Kw_2(t))\| \le \int_0^t \left(\frac{\theta}{e(t-\tau)}\right)^\theta \|B(w_{1,\tau}) - B(w_{2,\tau})\| d\tau . \tag{8.7}$$

If $w \in \mathcal{B}_\rho$, then

$$\max_{[0, T]} \|A^\theta w(t)\| \le \rho + \|A^\theta v(0)\| .$$

This easily implies that

$$|w_\tau|_{C_\theta} \le \rho + |v|_{C_\theta}, \qquad \tau \in [0, T],$$

where, as above, $w_\tau \in C_\theta$ is defined by the formula

$$w_\tau(\sigma) = w(\tau + \sigma), \qquad \sigma \in [-r, 0].$$

Hence, estimate (8.1) for $w_j \in \mathcal{B}_\rho$ gives us that

$$\|B(w_{1,\tau}) - B(w_{2,\tau})\| \le M(\rho + |v|_{C_\theta})|w_{1,\tau} - w_{2,\tau}|_{C_\theta}.$$

Since $w_1(\sigma) = w_2(\sigma)$ for $\sigma \in [-r, 0)$, the last estimate can be rewritten in the form

$$\|B(w_{1,\tau}) - B(w_{2,\tau})\| \le M(\rho + |v|_{C_\theta})|w_1 - w_2|_{C_\theta(0, T)}$$

for $\tau \in [0, T]$. Therefore, (8.7) implies that

$$|Kw_1 - Kw_2|_{C_\theta(0, T)} \le \frac{1}{1-\theta}\left(\frac{\theta}{e}\right)^\theta T^{1-\theta} M(\rho + |v|_{C_\theta})|w_1 - w_2|_{C_\theta(0, T)},$$

if $w_j \in \mathcal{B}_\rho$, $j = 1, 2$. Similarly, we have that

$$|Kw - \hat{v}|_{C_\theta(0, T)} \le \frac{1}{1-\theta}\left(\frac{\theta}{e}\right)^\theta T^{1-\theta} \left\{\|B(0)\| + M(\rho + |v|_{C_\theta})(\rho + |v|_{C_\theta})\right\}$$

for $w \in \mathcal{B}_\rho$. These two inequalities enable us to choose $T = T(\theta, \rho, |v|_{C_\theta}) > 0$ such that K is a contractive mapping of $\mathcal{B}_\rho$ into itself. Consequently, there exists a unique fixed point $w(t) \in C_\theta(0, T)$ of the mapping K. The structure of the operator K implies that $w(+0) = [Kw](+0) = v(0)$. Therefore, the function

$$u(t) = \begin{cases} w(t), & t \in [0, T], \\ v(t), & t \in [-r, 0], \end{cases}$$

lies in $C(-r, T; \mathcal{F}_\theta)$ and is a mild solution to problem (8.2), (8.3) on the segment $[0, T]$. Thus, **Theorem 8.1 is proved**.

In many aspects the theory of retarded equations of the type (8.2) is similar to the corresponding reasonings related to the problem without delay (see (2.1)). The exercises below partially confirm that.

— Exercise *8.1* Prove the assertions similar to the ones in Exercises 2.1–2.5 and in Theorem 2.2.

— Exercise *8.2* Assume that the constant $M(R)$ in (8.1) does not depend on R. Prove that problem (8.2) and (8.3) has a unique mild solution on $[t_0, \infty)$, provided $v(\sigma) \in C_\theta$. Moreover, for any pair of solutions $u_1(t)$ and $u_2(t)$ the estimate

$$\|u_1(t) - u_2(t)\|_\theta \le a_1 e^{a_2(t - t_0)} |v_1 - v_2|_{C_\theta} \tag{8.8}$$

is valid, where $v_j(\sigma)$ is the initial condition for $u_j(t)$ (see (8.3)).

For the sake of simplicity from now on we restrict ourselves to the case when the mapping B has the form

$$B(v) = B_0(v(0)) + B_1(v), \qquad v = v(\sigma) \in C_{1/2}, \tag{8.9}$$

where $B_0(\cdot)$ is a continuous mapping from $\mathcal{F}_{1/2} \equiv D(A^{1/2})$ into H, $B_1(\cdot)$ continuously maps $C_{1/2}$ into H and possesses the property $B_1(0) = 0$. We also assume that $B_0(\cdot)$ is a potential operator, i.e. there exists a continuously Frechét differentiable function $F(u)$ on $\mathcal{F}_{1/2}$ such that $B_0(u) = -F'(u)$. We require that

$$F(u) \ge -\alpha, \qquad (F'(u), u) - \beta F(u) \ge \gamma \|u\|^2 - \delta \tag{8.10}$$

for all $u \in \mathcal{F}_{1/2}$, where α, β, γ, and δ are real parameters, β and γ are positive (cf. Section 2). As to the retarded term $B_1(v)$, we consider the uniform estimate

$$\|B_1(v_1) - B_2(v_2)\| \le M \int_{-r}^{0} \|A^{1/2}(v_1(\sigma) - v_2(\sigma))\| d\sigma \tag{8.11}$$

to be valid. Here M is an absolute constant and $v_j(\sigma) \in C \equiv C_{1/2}$.

Exercise *8.3* Assume that conditions (8.9)–(8.11) hold. Then problem (8.2) and (8.3) has a unique mild solution (in $\mathcal{F}_{1/2}$) on any segment $[t_0,\ t_0+T]$ for every initial condition v from $C = C_{1/2}$.

Therefore, we can define an evolutionary operator S_t acting in the space $C \equiv C_{1/2}$ by the formula

$$(S_t v)(\sigma) = u_t(\sigma) \equiv u(t+\sigma), \qquad \sigma \in [-r,\ 0], \tag{8.12}$$

where $u(t)$ is a mild solution to problem (8.2) and (8.3).

Exercise *8.4* Prove that the operator S_t given by formula (8.12) satisfies the semigroup property: $S_t \circ S_\tau = S_{t+\tau}$, $S_0 = I$, $t,\ \tau \ge 0$ and the pair $(C_{1/2},\ S_t)$ is a dynamical system.

Theorem 8.2.

Let conditions (8.9)–(8.11) ***and*** (8.1) ***with*** $\theta = 1/2$ ***hold. Assume that the parameters in*** (8.10) ***and*** (8.11) ***satisfy the equation***

$$\frac{r^2}{2\gamma}(2+\gamma)M^2 \exp\left\{r \cdot \min(2,\ \gamma,\ \beta)\right\} \le \min(2,\ \gamma,\ \beta)\,.$$

Then the dynamical system $(C,\ S_t)$ ***generated by equality*** (8.12) ***is a dissipative compact system.***

Proof.

We reason in the same way as in the proof of Theorem 4.3. Using the Galerkin approximations it is easy to find that a solution to problem (8.2) and (8.3) satisfies the equalities

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \|A^{1/2}u(t)\|^2 + (F'(u(t)),\ u(t)) \;=\; (B_1(u_t),\ u(t))$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|A^{1/2}u(t)\|^2 + 2F(u(t))) + \|\dot u(t)\|^2 \;=\; (B_1(u_t),\ \dot u(t))\ .$$

If we add these two equations and use (8.10), then we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(u(t)) + \|A^{1/2}u(t)\|^2 + \gamma\|u(t)\|^2 + \|\dot u(t)\|^2 + \beta F(u(t)) \;&\le\\ \le\; \delta + \|B_1(u_t)\|\,(\|u(t)\| + \|\dot u(t)\|)\ ,&\end{aligned} \tag{8.13}$$

where

$$V(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|A^{1/2}u\|^2 + F(u) + \alpha\ . \tag{8.14}$$

Using (8.11) it is easy to find that there exists a constant $D_0 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(u(t)) + \omega_1 V(u(t)) \le D_0 + \frac{1}{2}\omega_2 \int_{t-r}^{t} \|A^{1/2} u(\tau)\|^2 \mathrm{d}\tau ,$$

where

$$\omega_1 = \min(2,\ \gamma,\ \beta), \qquad \omega_2 = \frac{r}{2}\Big(1 + \frac{2}{\gamma}\Big) M^2 .$$

Consequently, the inequality

$$\dot{\psi}(t) + \omega_1 \psi(t) \le D_0 + \omega_2 \int_{t-r}^{t} \psi(\tau)\mathrm{d}\tau$$

is valid for $\psi(t) = V(u(t))$. Therefore, we have

$$\dot{\varphi}(t) \le D_0\, e^{\omega_1 t} + \omega_2\, e^{\omega_1 r} \int_{t-r}^{t} \varphi(\tau)\mathrm{d}\tau$$

for the function

$$\varphi(t) = e^{\omega_1 t}\, \psi(t) = e^{\omega_1 t}\, V(u(t)) .$$

If we integrate this inequality from 0 to t, then we obtain

$$\varphi(t) \le \varphi(0) + \frac{D_0}{\omega_1}(e^{\omega_1 t} - 1) + \omega_2 e^{\omega_1 r}\, r \int_{-r}^{t} \varphi(\tau)\mathrm{d}\tau .$$

Therefore, Gronwall's lemma gives us that

$$V(u(t)) \le (V(u(0)) + C_0 |v|^2_{C_{1/2}})\, e^{-\omega_3 t} + C_1 ,$$

provided that

$$\omega_2 e^{\omega_1 r} r < \omega_1 .$$

Here C_1 and C_2 are positive numbers, $\omega_3 = \omega_1 - \omega_2 e^{\omega_1 r} r > 0$. This implies the dissipativity of the dynamical system $(C,\ S_t)$. In order to prove its compactness we note that the reasoning similar to the one in the proof of Lemma 4.1 enables us to prove the existence of the absorbing set $\mathscr{B}_\theta$ which is a bounded subset of the space $C_\theta = C(-r,\ 0;\ D(A^\theta))$ for $1/2 < \theta < 1$. Using the equality (cf. (8.5))

$$u(t) = e^{-(t-s)A}\, u(s) + \int_{s}^{t} e^{-(t-\tau)A} B(u_\tau)\mathrm{d}\tau$$

for $t \ge s$ large enough, we can show that the equation

$$\|A^{1/2}(u(t) - u(s))\| \le C|t-s|^\beta$$

holds for the solution $u(t)$ in the absorbing set $\mathscr{B}_\theta$. Here the constant C depends on $\mathscr{B}_\theta$ and the parameters of the problem only, $\beta > 0$. This circumstance enables us to prove the existence of a compact absorbing set for the dynamical system $(C,\ S_t)$. **Theorem 8.2 is proved**.

Theorem 8.2 and the results of Chapter 1 enable us to prove the following assertion on the attractor of problem (8.2) and (8.3).

Theorem 8.3.

Assume that the hypotheses of Theorem 8.2 ***hold. Then the dynamical system*** $(C,\ S_t)$ ***possesses a compact connected global attractor*** $\mathscr{A}$ ***which is a bounded set in the space*** $C_\theta = C(-r,\ 0;\ \mathscr{F}_\theta)$ ***for each*** $\theta < 1$.

It should be noted that the finite dimensionality of this attractor can be proved in some cases.

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