Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry, 25–30 July, 2000, Debrecen, Hungary

ON K-CONTACT η-EINSTEIN MANIFOLDS

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Abstract. The object of the present paper is to study a K-contact η-Einstein manifold satisfying certain conditions on the curvature tensor.

1. Introduction

Let (M^n, g) be a contact Riemannian manifold with contact from η, associated vector field ξ, $(1,1)$-tensor field φ and associated Riemannian metric g. If ξ is a killing vector field, then M^n is called a K-contact Riemannian manifold [1], [2]. A K-contact Riemannian manifold is called Sasakian [1], if the relation

$$(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X \tag{1.1}$$

holds, where ∇ denotes the operator of covariant differentiation with respect of g.

Recently, M. C. Chaki and M. Tarafdar [3] studied a Sasakian manifold M^n $(n > 3)$ satisfying the relation $R(X,Y) \,.\, C = 0$, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold and C is the Weyl conformal curvature tensor of type $(1,3)$. Generalizing the result of Chaki and Tarafdar, N. Guha and U. C. De [4] proved that if a K-contact manifold with charcteristic vector field ξ belonging to the k-nullity distribution satisfies the condition $R(\xi, X) \,.\, C = 0$, then $C(\xi, X)Y = 0$ for any vector fields X, Y. In Section 3 of the present paper we prove, without assuming that ξ belongs to the K-nullity distribution, that a K-contact η-Einstein manifold (M^n, g) $(n > 3)$ satisfying the condition $R(X, \xi) \,.\, C = 0$ is a space of constant curvature.

In [5] S. Tanno studied a K-contact manifold satisfying the condition $R(X, \xi) \,.\, S = 0$, where S is the Ricci tensor of type $(0, 2)$. But the condition $R(X, \xi) \,.\, S = 0$ does not imply the condition $S(X, \xi) \,.\, R = 0$. In Section 4 we prove that if a K-contact manifold M^n still satisfies the relation $S(X, \xi) \,.\, R = 0$ than it is an η-Einstein manifold.

Supported by OTKA 32058.

2. Preliminaries

In a K-contact Riemannian manifold the following relations hold: [1], [2], [6]

$$\text{(2.1)} \qquad a)\quad \varphi\xi = 0, \qquad b)\quad \eta(\xi) = 1 \qquad c)\quad g(X,\xi) = \eta(X)$$

$$\text{(2.2)} \qquad \varphi^2 X = -X + \eta(X)\xi$$

$$\text{(2.3)} \qquad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

$$\text{(2.4)} \qquad \nabla_X \xi = -\varphi X$$

$$\text{(2.5)} \qquad g(R(\xi,X)Y),\xi) = \eta(R(\xi,X)Y) = g(X,Y) - \eta(X)\eta(Y)$$

$$\text{(2.6)} \qquad R(\xi,X)\xi = -X + \eta(X)\xi$$

$$\text{(2.7)} \qquad S(X,\xi) = (n-1)\eta(X)$$

$$\text{(2.8)} \qquad (\nabla_X \varphi)(Y) = R(\xi,X)Y$$

for any vector fields X, Y.

A K-contact manifold M^n is said to be η-Einstein if its Ricci tensor S is of the form $S = ag + b\eta \otimes \eta$, where a, b are smooth functions on M.

3. K-contact η-Einstein manifolds satisfying $R(X,\xi)\,.\,C = 0$

Let us consider a K-contact η-Einstein manifold $M^n(n > 3)$ satisfying the relation

$$\text{(3.1)} \qquad R(X,\xi)\,.\,C = 0.$$

In this case we have

$$\text{(3.2)} \qquad S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y).$$

Putting $X = Y = \xi$ in (3.2) and then using (2.7) and (2.1) b, we get

$$\text{(3.3)} \qquad n - 1 = a + b.$$

Also (3.2) implies that

$$\text{(3.4)} \qquad r = an + b.$$

From (3.3) and (3.4) we have

$$\text{(3.5)} \qquad a = \frac{r}{n-1} - 1, \qquad b = n - \frac{r}{n-1}.$$

Again from (3.2) we obtain

$$\text{(3.6)} \qquad QX = \left(\frac{r}{n-1} - 1\right) X + \left(n - \frac{r}{n-1}\right)\eta(X)\xi,$$

where Q denotes the Ricci operator, i.e. $g(QX,Y) = S(X,Y)$.

By definition the Weyl conformal curvature tensor C is given by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\big[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\big] + \frac{r}{(n-1)(n-2)}\big[g(Y,Z)X - g(X,Z)Y\big]. \tag{3.7}$$

Using (3.2) and (3.6) in (3.7), we get

$$C(X,Y)Z = R(X,Y)Z + \left[\frac{2}{(n-1)} - \frac{r}{(n-2)}\right]\big[g(Y,Z)X - g(X,Z)Y\big] - \left[\frac{n}{n-1} - \frac{r}{(n-1)(n-2)}\right]\big[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\big]. \tag{3.8}$$

Now (3.1) gives us by definition

$$R(X,\xi)C(U,V)W - C(R(X,\xi)U,V)W - C(U,R(X,\xi)V)W - C(U,V)R(X,\xi)W = 0, \quad \text{for all} \quad X,U,V,W. \tag{3.9}$$

Substitution of U and W by ξ in (3.9) yields

$$R(X,\xi)C(\xi,V)\xi - C(R(X,\xi)\xi,V)\xi - C(\xi,R(X,\xi)V)\xi - C(\xi,V)R(X,\xi)\xi = 0. \tag{3.10}$$

From (3.8) we get by virtue of (2.1) (b), (2.1) (c) and (2.6),

$$C(\xi,V)\xi = 0, \quad \text{for any vector field } V. \tag{3.11}$$

Hence by virtue of (3.11) we have

$$R(X,\xi)C(\xi,V)\xi = 0. \tag{3.12}$$

Again in view of (2.6) we get

$$C(R(X,\xi)\xi,V)\xi = C(X,V)\xi - \eta(X)C(\xi,V)\xi$$

which implies by means of (3.11) that

$$C(R(X,\xi)\xi,V) = C(X,V)\xi. \tag{3.13}$$

From (3.8) we obtain

$$C(X,V)\xi = R(X,V)\xi - \eta(V)X + \eta(X)V \tag{3.14}$$

for any vector fields X and V. By virtue of (3.13) and (3.14) we have

$$C(R(X,\xi)\xi,V)\xi = R(X,V)\xi - \eta(V)X + \eta(X)V, \tag{3.15}$$

and by virtue of (3.11) we get

$$C(\xi,R(X,\xi)V)\xi = 0. \tag{3.16}$$

Finally using (2.6), we have

$$C(\xi,V)R(X,\xi)\xi = C(\xi,V)X - \eta(X)C(\xi,V)\xi,$$

from which it follows by means of (3.11) and (3.14) that

$$C(\xi,V)R(X,\xi)\xi = R(\xi,V)X - g(X,V)\xi + \eta(X)V. \tag{3.17}$$

Applying (3.12), (3.15), (3.16) and (3.17) in (3.10) we obtain

$$R(X,V)\xi + R(\xi,V)X - g(X,V)\xi - \eta(V)X + 2\eta(X)V = 0. \tag{3.18}$$

Interchanging X and V in (3.18) we have

$$R(V,X)\xi + R(\xi,X)V - g(X,V)\xi - \eta(X)V + 2\eta(V)X = 0. \tag{3.19}$$

Subtracting (3.19) from (3.18) and then using Bianchi's first identity, we get

$$R(X,V)\xi = \eta(V)X - \eta(X)V,$$

from which it follows that

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X. \tag{3.20}$$

In view of (1.1), (2.8) and (3.20), we obtain that the manifold is Sasakian and hence by the result of Chaki and Tarafdar [3], the manifold is a space of constant curvature 1. Thus we have the following

Theorem 1. *A K-contact η-Einstein manifold (M^n,g) $(n>3)$ satisfying the condition $R(X,\xi)\,.\,C=0$ is a space of constant curvature 1.*

A contact Riemannian manifold satisfying the condition $R(X,\xi)\,.\,C=0$ have been studied by C. Baikoussis and T. Koufogiorgos [7].

4. K-contact manifolds satisfying the condition $S(X,\xi)\,.\,R=0$

We consider a K-contact Riemannian manifold M^n satisfying the condition

$$(S(X,\xi)\,.\,R(U,V)W = 0. \tag{4.1}$$

Now by definition we have

$$\begin{aligned}(S(X,\,.\,\xi)\,.\,R(U,V)W = ((X\wedge_s\xi)\,.\,R)(U,V)W &= (X\wedge_s\xi)R(U,V)W\\ &+ R((X,\wedge_S\xi)U,V)W + R((U,(X\wedge_s\xi)V)W\\ &+ R(U,V)(X\wedge_s\xi)W,\end{aligned} \tag{4.2}$$

where the endomorphism $X\wedge_s Y$ is defined by

$$(X\wedge_s y)Z = S(Y,Z)X - S(X,Z)Y. \tag{4.3}$$

Using the definition of (4.3) in (4.2), we get by virtue of (2.7)

$$\begin{aligned}S(X,\xi)\,.\,R(U,V)W = (n-1)\big[&\eta(R(U,V)W)X + \eta(U)R(X,V)W\\ &+ \eta(V)R(U,X)W + \eta(W)R(U,V)X\big]\\ &- S(X,R(U,V)W)\xi - S(X,U)R(\xi,V)W\\ &- S(X,V)R(U,\xi)W - S(X,W)R(U,V)\xi\end{aligned} \tag{4.4}$$

and by virtue of (4.1) and (4.4) we have

$$\begin{aligned}(n-1)\big[\eta(R(U,V)W)X+\eta(U)R(X,V)W+\eta(V)R(U,X)W\\ +\eta(W)R(U,V)W)\big]-S(X,R(U,V)W)\xi-S(X,U)R(\xi,V)W\\ -S(X,V)R(U,\xi)W-S(X,W)R(U,V)\xi=0.\end{aligned}\tag{4.5}$$

Taking the inner product on both sides of (4.5) by ξ, we obtain

$$\begin{aligned}(n-1)\big[\eta(R(U,V)W)\eta(X)+\eta(U)\eta(R(X,V)W\\ +\eta(V)\eta(R(U,X)W)+\eta(W)\eta(R(U,V)X)\big]\\ -S(X,R(U,V)W)-S(X,U)\eta(R(\xi,V)W)\\ -S(X,V)\eta(R(U,\xi)W)-S(X,W)\eta(R(U,V)\xi)=0.\end{aligned}\tag{4.6}$$

Putting $U=W=\xi$ in (4.6) and using (2.5)–(2.7) we get

$$S(X,V)=-(n-1)g(X,V)+2(n-1)\eta(X)\eta(V)$$

which means that the manifold is η-Einstein.

Thus we have the following

Theorem 2. *A K-contact Riemannian manifold (M^n,g) satisfying the condition $S(X,\xi)\,.\,R=0$ is an η-Einstein manifold.*

From Theorem 1 and Theorem 2 we immediately have:

Theorem 3. *A K-contact Riemannians manifold (M^n,g) stisfying the conditions $S(X,\xi)R=0$ and $R(X,\xi)C=0$ is a space of constant curvature 1.*

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