Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry, 25–30 July, 2000, Debrecen, Hungary

SYMMETRY ALGEBRA FOR CONTROL SYSTEMS

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Abstract. A description of the full symmetry algebra for a general nonlinear system of ordinary differential equations is given in terms of its general solution and differential constants. The full symmetry algebra of a system is a module over the ring of its differential constants; the module is generated by partial derivatives of the general solution by the independent constants. Special solutions, such as an envelope of a family of solutions, are described naturally in this context. These results are extended to control systems; in such case, differential constants become operators on controls. Examples are provided.

1. Introduction

The study of symmetries of ordinary differential equation (ODE) was initiated by Sophus Lie [1] and has a long history, see [2] for details. The latest results were obtained in [4] and [5].

To find symmetries for a particular equation still remains a hard task. This publication deals, however, with another problem. We give a full description of the symmetry algebra of a system of ODE in a nondegenerate situation using the general solution whose (local) existence is guaranteed by classical theorems. For a linear system of ODEs this result was obtained in [3] and it was recently generalized to the normal form scalar ODEs in [5].

Given a general solution, our description of the symmetry algebra is effective and explicit: the full symmetry algebra of a system is a module over the ring of its differential constants; the module is generated by partial derivatives of the general solution by the independent constants. Special solutions, such as an envelope of a family of solutions, are described naturally in this context. The interconnection between differential invariants, symmetries and a general solution is quite transparent in the case of ODEs and may be used as a model aplicable in other situations.

We give two such applications below. First, we describe the symmetries of a boundary/initial value problem for a one-dimensional wave equation. The second, main application deals with symmetries of a control system. In both cases, differential invariants become nonlocal ones.

1991 *Mathematics Subject Classification.* 58G37, 34H05.

Key words and phrases. Symmetry, control system, general solution.

This work was partially supported by INTAS grant 96-0793.

2. Full symmetry algebra for a general nonlinear ordinary differential equation and a system of equations

2.1. **General solution and differential constants.** We begin with trivialities to introduce notation.

Let $\mathcal{E}$ denote a general scalar ordinary differential equation of nth order:

$$y^{(n)} - F(x, y, y', \dots, y^{(n-1)}) = 0. \tag{1}$$

Its *general solution* (or a *general integral*) is of the form

$$\Phi(x, y, c_1, c_2, \dots, c_n) = 0. \tag{2}$$

When (2) is solved with respect to y, we get

$$y = f(x, c_1, \dots, c_n); \tag{3}$$

almost any solution of (1) is obtained from (3) by a proper choice of constants c_i. (The solution that is not produced by the general one is called a *special solution*. Such solutions are discussed below.)

Remark 1. The existence of a general solution of (1) is by no means guaranteed. Yet if F is continuously differentiable, the classical theorem on a differentiable dependence of a solution of ODE on initial data guaranties an existence of a local form of (2) in a neighborhood of a chosen solution. In this local form the initial datum $y^{(k)}(x_0)$ is taken as a differential constant c_k, $k = 0, \dots, n-1$. Below we deal mostly with a global general solution, but it is always possible to make a correspondent local statement.

Differentiating (3) by x, we obtain the following system of n independent equations

$$\begin{cases} y = f(x, c_1, \dots, c_n), \\ y' = f'(x, c_1, \dots, c_n), \\ \dots\dots\dots\dots\dots\dots\dots \\ y^{(n-1)} = f^{(n-1)}(x, c_1, \dots, c_n) \end{cases} \tag{4}$$

One can obtain an expression (not necessary explicit) for c_i solving (4). Thus

$$c_i = c_i(x, y, y', \dots, y^{(n-1)}), \quad i = 1, \dots, n. \tag{5}$$

In this way, all c_i are differential constants of order less than n. In other words, they are differential operators of order $n-1$, or functions on the jet space $J^{n-1}(\mathbb{R})$.

In the case of a system of m differential equations,

$$\mathbf{y}^{(n)} - \mathbf{F}(x, \mathbf{y}, \mathbf{y}', \dots, \mathbf{y}^{(n-1)}) = 0, \tag{6}$$

where $\mathbf{y} = (y_1, \dots, y_m)$, $\mathbf{F} = (F_1, \dots, F_m)$, the general solution is of the form

$$\mathbf{y} = \mathbf{f}(x, c_1, \dots, c_{mn}). \tag{7}$$

2.2. **Full symmetry algebra.** By definition of a solution, if in the right-hand side of (3) $f(x, y, c_1, \dots, c_n)$ is substituted for y in (1), we obtain the identity

$$f^{(n)} - F(x, f, f', \dots, f^{(n-1)}) \equiv 0. \tag{8}$$

Hence

$$\frac{\partial}{\partial c_i}\left(f^{(n)} - F(x, f, f', \dots, f^{(n-1)})\right) = 0 \tag{9}$$

for all i, or

$$\left(D^n - \sum_{j=1}^{n} \frac{\partial F(x, y, y', \dots, y^{(n-1)})}{\partial y_j} D^j\right)\Bigg|_{y=f(x,y,c_1,\dots,c_n)} f_{c_i} = 0, \tag{10}$$

where $D = d/dx$ is the total derivative with respect to x and f_{c_i} denotes the partial derivatice over c_i.

Recall that

$$\mathcal{L}_{y^{(n)}-F} \overset{\text{def}}{=} D^n - \sum_{j=1}^{n} \frac{\partial F(x, y, y', \dots, y^{(n-1)})}{\partial y_j} D^j \tag{11}$$

is called the *universal linearization* of the operator $y^{(n)} - F$ and that a solution ϕ of the equation

$$\left(\mathcal{L}_{y^{(n)}-F}\right) \varphi\big|_{\mathcal{E}} = 0 \tag{12}$$

is a *symmetry* of $\mathcal{E}$. Thus we have

Theorem 1. *The partial derivatives f_{c_i}, $i = 1, \dots, n$, form a full functionally independent basis of symmetries for equation* (1).

Remark 2. Let φ be a symmetry. Then it defines a flow on a set of solutions by the formula:

$$\frac{\partial y}{\partial \tau} = \varphi|_y, \tag{13}$$

where $y = f(x, y, c_1, \dots, c_n)$. A solution of this equation is a one-parameter family of solutions of (1). By (3), it has the form

$$y = f(x, c_1(\tau), \dots, c_n(\tau)). \tag{14}$$

Hence

$$\varphi|_y = \left(\sum_{i=1}^{n} \frac{\partial c_i}{\partial \tau} f_{c_i}\right)\Bigg|_y \tag{15}$$

for any solution y of equation (1). Therefore,

$$\varphi = \sum_{i=1}^{n} \frac{\partial c_i}{\partial \tau} f_{c_i} \tag{16}$$

holds everywhere on $\mathcal{E}$.

Note that the derivatives $\partial c_i/\partial\tau|_y$ depend on y, that is, on $c_1, \ldots, c_1$, which are functions on $J^{n-1}(\mathbb{R})$ by virtue of (5). Since any choice of arbitrary functions $c_i(\tau)$ define some symmetry by (14), the functions $\partial c_i/\partial\tau|_y$ are also arbitrary.

Thus, we got the general form of a symmetry for equation (1):

$$\varphi = \sum_{i=1}^{n} A_i(c_1, \ldots, c_n) \frac{\partial}{\partial c_i} f(x, y, c_1, \ldots, c_n); \tag{17}$$

here f is a general solution, A_i are arbitrary functions and c_i are functions on $J^{n-1}(\mathbb{R})$ given by system (4).

Formula (17) gives a representation of the algebra of vector fields on $\mathbb{R}^n$ in the full symmetry algebra of (6) by the isomorphism

$$\sum_{i=1}^{n} A_i(c_1, \ldots, c_n) \frac{\partial}{\partial c_i} \longleftrightarrow \sum_{i=1}^{n} A_i(c_1, \ldots, c_n) \frac{\partial}{\partial c_i} f(x, c_1, \ldots, c_n) \tag{18}$$

(on the left-hand side, c_i are coordinates in $\mathbb{R}^n$; on the right-hand side they denote differential invariants (5) of (1) or special functions on $J^{n-1}(\mathbb{R})$).

Theorem 1 give an explicit representation of this correspondence, provided the general solution is known. Yet its existence is guaranteed only locally; hence, the formula (18) is also generally local.

Remark 3. Theorem 1 generalizes easily to the case of a system of differential equations (6). Its full symmetry algebra is isomorphic to the algebra of vector fields on $\mathbb{R}^{mn}$: the representation is given by

$$\sum_{i=1}^{mn} A_i(c_1, \ldots, c_{mn}) \frac{\partial}{\partial c_i} \longleftrightarrow \partial\mathbf{f} \times \mathbf{A},$$

where $\partial\mathbf{f}, \mathbf{A}$ are respectively $m \times mn$ and $mn \times 1$ matrices with matrix elements given by the formulas

$$(\partial\mathbf{f})_{j,i} = \frac{\partial f_j}{\partial c_i}, \quad (\mathbf{A})_i = A_i.$$

Remark 4. A full symmetry algebra is a module over the ring of the equation differential constants. The module is generated by partial derivatives of a general solution by independent constants.

Let us call f_{c_i}, $i = 1, \ldots, n$, *basic symmetries.* They correspond to the flows $y(\tau) = f(x, c_1, \ldots, c_i + \tau, \ldots, c_n)$. Thus, in the case of an explicit general solution (3) basic symmetries are $f_{c_i} = y_{c_i}$. If a general solution of (1) is given in an implicit form (2), then

$$y_{c_i} = -\left(\frac{\partial\Phi}{\partial c_i}\right) \Big/ \left(\frac{\partial\Phi}{\partial y}\right). \tag{19}$$

2.3. **Special and invariant solutions.** *Invariant* solution y of (1) is a solution that satisfies the condition $\varphi(y) = 0$ for some symmetry φ of the form (17). Hence an invariant solution satisfy the system of equations

$$\begin{cases} \mathcal{E}(f) & = y^{(n)} - F(x, y, y', \dots, y^{(n-1)}) = 0, \\ \phi(y) & = \sum_{i=1}^{n} A_i(c_1(y), \dots, c_n(y)) \dfrac{\partial}{\partial c_i} f(x, y, c_1(y), \dots, c_n(y)) = 0. \end{cases} \tag{20}$$

Since c_i are constants on solutions of (1), so are $A_i(c_1(y), \dots, c_n(y))$. Thus (20) is simply

$$\begin{cases} \mathcal{E}(f) & = y^{(n)} - F(x, y, y', \dots, y^{(n-1)}) = 0, \\ \phi(y) & = \sum_{i=1}^{n} A_i f_{c_i}(x, y, c_1, \dots, c_n) = 0 \end{cases} \tag{21}$$

with constant A_i and c_i. The second condition in (21) means that basic symmetries are linearly dependent on an invariant solution. If $\operatorname{rank}\{f_{c_1}, \dots, f_{c_n}\}|_y = n - k$, we introduce the notion of a k-invariant solution.

Consider a simple case of (21),

$$\begin{cases} y^{(n)} - F(x, y, y', \dots, y^{(n-1)}) = 0 \\ f_{c_i} = 0. \end{cases} \tag{22}$$

Its solution is a fixed point of the flow $c_i \to c_i + \tau$. Geometrically, such a solution is an envelope for the family of solutions generated by this flow, see Subsection 2.4.

2.4. **Examples.**

Example 1. Consider the equation

$$y'' + \frac{9}{8}(y')^4 = 0.$$

It is invariant with respect to the translations in both x and y, hence its symmetry algebra is obvious. Its general solution is as follows:

$$\Phi(x, y, c_1, c_2) = (y + c_1)^3 - (x + c_2)^2 = 0,$$

or

$$y = f(x, c_1, c_2) = (x + c_2)^{\frac{2}{3}} - c_1.$$

Therefore, its basic symmetries are $f_{c_1} = -1$, $f_{c_2} = \frac{2}{3}(x + c_2)^{-\frac{1}{3}}$. They depend on the differential constants c_1, c_2 that may be found from the system (4),

$$(y + c_1)^3 = (x + c_2)^2,$$
$$3y'(y + c_1)^2 = 2(x + c_2).$$

It follows that

$$c_1 = \left(\frac{2}{3y'}\right)^2 - y,$$
$$c_2 = \left(\frac{2}{3y'}\right)^3 - x.$$

Now, basic symmetries come to

$$f_{c_1} = -1,$$
$$f_{c_2} = y',$$

which are (not surprisingly) translations in y and x respectively.

So the general symmetry for this equation is of the form (17)

$$\begin{aligned}\varphi &= A_1(c_1, c_2) f_{c_1} + A_2(c_1, c_2) f_{c_2} \\ &= -A_1\left(\left(\frac{2}{3y'}\right)^2 - y, \left(\frac{2}{3y'}\right)^3 - x\right) + A_2\left(\left(\frac{2}{3y'}\right)^2 - y, \left(\frac{2}{3y'}\right)^3 - x\right) y',\end{aligned}$$

where A_1, A_2 are arbitrary functions in two variables.

Invariant solutions must satisfy the system (21)

$$A + y'B = 0,$$
$$y'' + \frac{9}{8}(y')^4 = 0$$

for some constants A, B. It follows that $y' = 0$, so $y = \text{const}$. This is a special solution (in the sense it is not obtained from the general integral). Each special solution is an envelope for the family

$$(y - \text{const})^3 - (x + c_2)^2 = 0$$

for all c_2.

Example 2. Linear equations (cf. [4])

$$y^{(n)} + \sum_{i=0}^{n-1} a_i(x) y^{(i)} = 0.$$

Here the general integral is if the form

$$y = \sum_{i=1}^{n} c_i f_i(x),$$

where $f_i(x)$ are independent solutions, i.e., their Wronskian is nonzero:

$$W = W(f_1, \dots, f_i, \dots, f_n) = \begin{vmatrix} f_1 & \dots & f_i & \dots & f_n \\ f_1' & \dots & f_i' & \dots & f_n' \\ \dots & \dots & \dots & \dots & \dots \\ f_1^{(n-1)} & \dots & f_i^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} \neq 0$$

Independent solutions f_i coincide with basic symmetries in this case: $f_i = f_{c_i}$.

Differential constant c_i is given by the formula

$$c_i(y, y', \dots, y^{(n-1)}) = \frac{W_i}{W},$$

where W_i is obtained from W by changing the entries of the ith column of W for $y, y', \dots, y^{(n-1)}$ in the respective order.

The general form of a symmetry is

$$\varphi = \sum_{i=1}^{n} A_i \left(\frac{W_1}{W}, \dots, \frac{W_i}{W}, \dots, \frac{W_n}{W} \right) f_i(x).$$

Example 3. Linear boundary problem

$$u_{tt} - u_{xx} = 0, \quad u|_{x=0} = u|_{x=\pi} = 0.$$

This example is a rather wide generalization of the previous one. Fourier general solution on $[0, \pi]$ for this string is

$$u = \sum_{n=0}^{\infty} \sin nx (a_n \cos nt + b_n \sin nt),$$

where a_n, b_n are constants, but neither differential nor local: the Fourier coefficient formula states that

$$a_n = \frac{2}{\pi} \int_0^{\pi} u|_{t=0} \sin nx \, dx, \quad b_n = \frac{2}{\pi n} \int_0^{\pi} u_t|_{t=0} \sin nx \, dx \tag{23}$$

A general form of the symmetry is given by

$$\varphi = \sum_{n=0}^{\infty} \sin nx \big(A_n(a_1, b_1, \dots, a_i, b_i, \dots) \cos nt + B_n(a_1, b_1, \dots, a_i, b_i, \dots) \sin nt \big).$$

Here A_n, B_n are arbitrary functions depending on any finite number of a_i, b_j given by (23).

3. Full symmetry algebra for a general control system

3.1. **General solution and differential constants.** Consider a first order control system

$$\mathbf{y}' = \mathbf{F}(x, \mathbf{y}, \mathbf{v}(x)), \tag{24}$$

where $\mathbf{y} \in \mathbb{R}^m$ is an m-vector of unknown functions and $\mathbf{v}(\mathbf{x}) \in \mathbb{R}^k$ in a k-vector of control functions.

With any fixed choice of controls, (24) comes to (6), where $n = 1$. Thus, the general solution of (24) is of the form

$$\mathbf{y} = \mathbf{f}(x, c_1, \dots, c_m, \mathbf{v}(x)), \tag{25}$$

where c_i are constants. From (25) it follows that there exists (at least an implicit) dependence

$$c_i = c_i(x, \mathbf{y}(x), \mathbf{y}'(x), \mathbf{v}(x)), \quad i = 1, \dots, m, \tag{26}$$

of constants c_i on x, $\mathbf{y}(x)$, $\mathbf{y}'(x)$, $\mathbf{v}(x)$. Both $\mathbf{f}$ and c_i are operators on $\mathbf{v}$. Examples below show that these operators may be nonlocal.

3.2. **Full symmetry algebra.** Technically, equation (24) is an equation with two types of dependent variables, that is, with $\mathbf{y}$ and $\mathbf{v}$. Let us put this equation in the form

$$\mathcal{H}(\mathbf{y}, \mathbf{v}) = \mathbf{y}' - \mathbf{F}(x, \mathbf{y}, \mathbf{v}(x)) = 0.$$

The symmetry equation in this case is as follows:

$$(D - \mathbf{F_y})\mathbf{A} - \mathbf{F_v}\mathbf{B}|_{\mathcal{H}=0} = 0, \tag{27}$$

where $(\mathbf{A}, \mathbf{B})$ is a symmetry (if it defines a flow, then $\mathbf{y}_\tau = \mathbf{A}$, $\mathbf{v}_\tau = \mathbf{B}$). Besides, $\mathbf{F_y}$ is an $m \times m$ matrix with the entries $(F_i)_{y_j}$ and $\mathbf{F_v}$ is an $m \times k$ matrix with the entries $(F_i)_{v_j}$.

It is convenient to put (27) in a vector form,

$$(D - \mathbf{F_y}, -\mathbf{F_v}) \cdot \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}\Bigg|_{\mathcal{H}=0} = 0. \tag{28}$$

The left factor in this formula is the linearization of $\mathcal{H}$ denoted by $\mathcal{L}_\mathcal{H}$.

Theorem 2. *Partial derivative vectors*

$$\begin{pmatrix} \mathbf{f_c} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{f_v} \\ \mathrm{I} \end{pmatrix} \tag{29}$$

form a full functionally independent basis of symmetries for equation (24).

Proof. In terms of the general solution, the general form of a flow on the set of solutions of equation (24) is given by the formula

$$\mathbf{y} = \mathbf{f}(x, c_1(\tau), \dots, c_m(\tau), \mathbf{v}(x, \tau)), \tag{30}$$

where τ is a parameter of the flow. Since (30) is a solution for any τ, we have

$$\frac{d}{d\tau}\big(\mathbf{f}'(x, c_1(\tau), \dots, c_m(\tau), \mathbf{v}(x, \tau)) \\ - \mathbf{F}(x, \mathbf{f}(x, c_1(\tau), \dots, c_m(\tau), \mathbf{v}(x, \tau)), \mathbf{v}(x, \tau))\big) = 0.$$

It follows that

$$\begin{aligned}\big(\,(D - \mathbf{F_y})(\mathbf{f_c} \cdot \mathbf{c}_\tau + \mathbf{f_v} \cdot \mathbf{v}_\tau) - \mathbf{F_v}\mathbf{v}_\tau\big)\big|_{\mathcal{H}=0} \\ = (D - \mathbf{F_y}, -\mathbf{F_v}) \cdot \begin{pmatrix} \mathbf{f_c} \cdot \mathbf{c}_\tau + \mathbf{f_v} \cdot \mathbf{v}_\tau \\ \mathbf{v}_\tau \end{pmatrix}\Bigg|_{\mathcal{H}=0} \\ = \mathcal{L}_\mathcal{H} \begin{pmatrix} \mathbf{f_c} \cdot \mathbf{c}_\tau + \mathbf{f_v} \cdot \mathbf{v}_\tau \\ \mathbf{v}_\tau \end{pmatrix}\Bigg|_{\mathcal{H}=0} = 0.\end{aligned} \tag{31}$$

Thus, the general solution of the symmetry equation is (cf. (16))

$$\begin{pmatrix} \mathbf{f_c} \\ 0 \end{pmatrix} \cdot \mathbf{c}_\tau + \begin{pmatrix} \mathbf{f_v} \\ \mathrm{I} \end{pmatrix} \cdot \mathbf{v}_\tau. \tag{32}$$

Here $\mathbf{f_c} = (f_i)_{c_j}$ is an $m \times m$ matrix, $\mathbf{f_v}$ is an $m \times k$ matrix and I is the $k \times k$ identity matrix.

To obtain the general form of the symmetry for equation (24) it remains to notice that

1. $\mathbf{v}_\tau$ is an arbitrary vector-function;
2. for any fixed $\mathbf{v}$, equation (24) coincides with (6), so $c_{i\tau}$ are the components of a vector field on the solution space. Therefore, $c_{i\tau} = \mathcal{A}_i(\mathbf{c}, \mathbf{v})$ are arbitrary functions;
3. c_i are constants on solution of (24) given by (26).

Finally, we can write down the general form of a symmetry for (24):

$$\varphi = \begin{pmatrix} \mathbf{f_c} \\ 0 \end{pmatrix} \cdot \mathcal{A}(\mathbf{c}, \mathbf{v}(x)) + \begin{pmatrix} \mathbf{f_v} \\ \mathrm{I} \end{pmatrix} \cdot \mathbf{u}(x). \tag{33}$$

Here $\mathcal{A}(\mathbf{c}, \mathbf{v}(x))$ and $\mathbf{u}(x)$ are arbitrary proper-sized matrices. $\square$

Remark 5. Generally, the solution (25) and its derivatives as well as expressions of the type $\mathcal{A}(\mathbf{c}, \mathbf{v}(x))$ or $\mathbf{u}(x)$ are operators on $\mathbf{v}(x)$. If they are differential operators of order l, we obtain lth order higher symmetries by formula (33).

Example 4. A linear scalar equation

$$y' = xy + v(x). \tag{34}$$

The general solution in this case is

$$y = e^{\frac{x^2}{2}} \int_{x_0}^{x} e^{-\frac{t^2}{2}} v(t)\, dt + c \cdot e^{\frac{x^2}{2}}.$$

Thus,

$$c = y \cdot e^{\frac{-x^2}{2}} - I(x), \text{ where } I(x) = \int_{x_0}^{x} e^{-\frac{t^2}{2}} v(t)\, dt,$$

is constant on any solution of (34).

Therefore, from (33) it follows that the general form of the symmetry in this example is

$$\varphi = \begin{pmatrix} e^{\frac{x^2}{2}} \\ 0 \end{pmatrix} \cdot \mathcal{A}(y \cdot e^{\frac{-x^2}{2}} - I(x), v(x)) + \begin{pmatrix} e^{\frac{x^2}{2}} \int_{x_0}^{x} e^{-\frac{t^2}{2}} [\,\bullet\,]\, dt \\ 1 \end{pmatrix} \cdot u(x). \tag{35}$$

Here $\mathcal{A}(c, v(x))$ and $u(x)$ are arbitrary operator and function respectively; $f_v = e^{\frac{x^2}{2}} \int_{x_0}^{x} e^{-\frac{t^2}{2}} [\,\bullet\,]\, dt$ is an operator acting on $u(x)$ by the formula

$$\left(e^{\frac{x^2}{2}} \int_{x_0}^{x} e^{-\frac{t^2}{2}} [\,\bullet\,]\, dt \right) u(x) = e^{\frac{x^2}{2}} \int_{x_0}^{x} e^{-\frac{t^2}{2}} u(t)\, dt.$$

This example shows that, since a general solution $f = f(v)$ of a control system is an operator on controls, f_v in formula (33) is a linearization of this operator.

References

[1] S. Lie, *Über die Integration durch bestimmte Integrale von einer Klasse linear partieller Differentialgleichung*, Arch. fur Math. **6** (1881), pp. 328–368

[2] *Symmetries and Conservation Laws for Differential equations of Mathematical Physics*, I. S. Krasil'shchik and A. M. Vinogradov (eds), AMS Translations of Math. Monographs v. **182** (1998), 333 pp.

[3] A. V. Samokhin, *Symmetries of Linear Ordinary Differential Equations*, Amer. Math. Soc. Transl. (2), **167** (1995).

[4] A. V. Samokhin, *Symmetries of Linear and Linearizable Systems of Differential Equations*, Acta Appl. Math., **56** (1999), 253–300.

[5] P. W. Doyle, *Symmetry and Ordinary Differential Constraints*, Int. J. of Nonlinear Mech., **34**, (1999), 1089–1102

[6] A. V. Samokhin, *Symmetries of control systems*, Banach Center Publications, **33** (1996), 337–342

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