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EIGENVALUE ESTIMATES AND NODAL LENGTH OF EIGENFUNCTIONS

ALESSANDRO SAVO

Introduction

This is a survey, without proofs, of the main results in [S]. We refer the reader to that paper for complete proofs and additional facts.

Let M be a compact 2-dimensional Riemannian manifold with metric g, and let ϕ be an eigenfunction of the *Laplace operator* Δ associated to the pair (M, g). The *nodal set* of ϕ is simply its zero set $\phi^{-1}(0)$: our main purpose is to estimate its size from below.

According to a result of S. Y. Cheng[C], the nodal set has the following properties:

• $\phi^{-1}(0)$ is a finite union of C^2-immersed circles, and the set of its *singular points*, that is the set $S = \{x \in M : \phi(x) = |\nabla\phi(x)| = 0\}$, is always finite, so that $\phi^{-1}(0) \setminus S$ is a finite union of smooth arcs. Moreover, the nodal lines through a singular point form an equiangular system.

Hence, with the exception of the finite set S, the nodal set of an eigenfunction is a regular curve. Our main result is a simple, explicit lower bound of its length.

Theorem. *Let M be a two-dimensional compact, smooth Riemannian manifold without boundary, and let ϕ be an eigenfunction associated to the eigenvalue λ. Then the bound:*

$$\text{Length}\,[\phi^{-1}(0)] > \frac{1}{11}\text{Area}\,(M)\sqrt{\lambda} \tag{1}$$

holds if λ is large (more precisely, if it is greater than an explicitable constant depending on the diameter and on a lower bound of the curvature). If the curvature is everywhere non negative, then the bound holds for all eigenvalues.

The geometric dependance on the area in the right-hand side is necessary (just consider a homothety) hence the estimate is geometrically sharp. The numerical constant $1/11$ is not far from being sharp itself; this can be seen by considering the eigenfunctions $\phi(x, y) = \sin mx$, $m \in \mathbf{N}$, on the flat torus $S^1 \times S^1$, which have nodal length equal to $\frac{1}{\pi}\text{Area}\,(M)\sqrt{\lambda}$ (as far as we know, $1/\pi$ could be the optimal constant).

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For the generalization to manifolds with boundary, we refer to [S].

Let us now give a short history of the problem. In 1972, Brüning and Gromes [B-Gr] proved that, if Ω is an open domain in the Euclidean plane, and ϕ is an eigenfunction associated to the eigenvalue λ (for the Dirichlet boundary condition) then

$$\text{Length}\,[\phi^{-1}(0)\cap\Omega] + \frac{1}{2}\text{Length}\,(\partial\Omega) \geq \frac{\text{Area}\,(\Omega)}{2j}\sqrt{\lambda} - \frac{\pi j(k-1)}{2\sqrt{\lambda}},$$

where j is the first zero of the Bessel function J_0, and k is the number of holes of Ω. Note that if λ is large enough then, for the interior nodal length, one has a lower bound of the type:

$$\text{Length}\,[\phi^{-1}(0)\cap\Omega] \geq c_0 \text{Area}\,(\Omega)\sqrt{\lambda}. \tag{2}$$

where c_0 is a numerical constant. Later, Brüning proved in [B] that, if M is a compact 2-manifold without boundary, then, for large eigenvalues λ:

$$\text{Length}\,[\phi^{-1}(0)] \geq C_1\sqrt{\lambda}, \tag{3}$$

with C_1 now depending on the *curvature* and the *injectivity radius* of M.

Our result is a generalization to manifolds of Brüning-Gromes' Euclidean bound (2); in particular, it implies that the constant C_1 in (3) can be taken to be *independent* of the curvature.

In arbitrary dimension, the nodal set $\phi^{-1}(0)$ of an eigenfunction ϕ is almost a regular hypersurface, in the sense that the singular set S has codimension at least 2. If Vol_{n-1} denotes, by a slight abuse of language, Hausdorff $(n-1)-$dimensional measure, then Yau conjectured that

$$C_1\sqrt{\lambda} \leq \text{Vol}_{n-1}[\phi^{-1}(0)] \leq C_2\sqrt{\lambda} \tag{4}$$

for constants C_1 and C_2 depending only on M.In their remarkable work [D-F], Donnelly and Feffermann proved the conjecture for *analytic* manifolds and for large eigenvalues; in other words, the ratio $\text{Vol}_{n-1}[\phi^{-1}(0)]/\sqrt{\lambda}$ does not go to 0 or ∞ as the eigenvalue tends to infinity. However, no explicit expression of C_1 and C_2 is given.

Our method does not extend, at least immediately, to higher dimensions. So, we wonder if inequalilty (1) extends to the arbitrary dimension n: in other words, does there exist a numerical constant c_0 such that

$$\text{Vol}_{n-1}[\phi^{-1}(0)] \geq c_0 \text{Vol}_n(M)\sqrt{\lambda},$$

when λ is sufficiently large (or, for all λ under some curvature assumptions)?

The main theorem.

The main bound (1) in the introduction is a corollary of Theorem 1 below. Fix the eigenfunction ϕ on the Riemann surface M associated to the eigenvalue λ. Let p be a point on the nodal set of ϕ, and define $\nu(p)$, its *order*, to be the number of nodal lines through p (so that p is singular iff $\nu(p) \geq 2$; note that $\nu(p)$ is also the order of vanishing of ϕ at p). Denote by

$$|S| = \sum_{p \in S} (\nu(p) - 1) \tag{5}$$

the *total number of singular points* (counted with their "multiplicities"). Then $|S| \geq 0$, and $|S| = 0$ iff the nodal set of ϕ is smooth.

We assume that the Gaussian curvature K of M is bounded below by $-\alpha^2$, with $\alpha \geq 0$; we let $K^- = \min\{K, 0\}$, $D = \mathrm{Diam}\,(M)$, $c(\alpha, D) = \sqrt{\pi^2 + \frac{1}{4}\alpha^2 D^2}$ and $\psi(\lambda, \alpha, D) = \min\{c(\alpha, D), \sqrt{\pi^2 + \frac{\alpha^2 c(\alpha,D)^2}{4\lambda}}\}$.

Theorem 1. *Let M be a compact oriented Riemann surface without boundary, and ϕ an eigenfunction associated to the eigenvalue λ. Then:*

$$\begin{aligned} \mathrm{Length}\,[\phi^{-1}(0)] \quad &\geq \quad \frac{4\mathrm{Area}\,(M)}{4\pi + \pi^2 \psi(\lambda, \alpha, D)} \sqrt{\lambda} \\ &\cdot \quad \left\{ 1 + \frac{\pi \psi(\lambda, \alpha, D)}{2\mathrm{Area}\,(M)} \left(2\pi |S| - \int_M |K^-| \right) \cdot \frac{1}{\lambda} \right\}. \end{aligned}$$

The bound (1) now follows because $\lim_{\lambda \to \infty} \psi(\lambda, \alpha, D) = \pi$, $|S| \geq 0$ and $\frac{4}{4\pi + \pi^3} > \frac{1}{11}$.

Corollary 2. *If the curvature of M is everywhere non-negative then, for all eigenfunctions ϕ one has:* $\mathrm{Length}\,[\phi^{-1}(0)] \geq \frac{4\mathrm{Area}\,(M)}{4\pi + \pi^3} \sqrt{\lambda}$. *Furthermore,*

$$\mathrm{Length}\,[\phi^{-1}(0)] \geq \frac{2}{4 + \pi^2} \frac{\mathrm{Area}\,(M)}{\mathrm{Diam}\,(M)} \tag{6}$$

Note that (6) follows combining the first bound with the estimate of [L-Y]: $\lambda_1(M) \geq \frac{\pi^2}{4D^2}$, thus giving a purely geometric estimate (it is interesting only for low eigenvalues, in particular the first).

Finally, a few remarks on the singularity set S, which is of independent geometric interest. The following estimate of its cardinality $|S|$ is due to Dong [Do]:

$$|S| \leq \frac{1}{4\pi} \mathrm{Area}\,(M) + \frac{1}{2\pi} \int_M |K^-|. \tag{7}$$

Using Dong's estimate, and the proof of Theorem 1, one also has:

$$|S| \leq \frac{1}{4} \mathrm{Length}\,[\phi^{-1}(0)] \sqrt{\lambda} + \frac{1}{2\pi} \int_M |K^-|. \tag{8}$$

Dong also gives an estimate of the number of singular points inside a small disk. Similar estimates were given in [D-F2]; these estimate are then used to establish upper bounds for the nodal length.

Note finally that $|S|$ also counts (with multiplicity) the number of critical points of the eigenfunction which lie on the nodal set. It would be desirable to have a bound for the whole critical set of an eigenfunction.

An eigenvalue estimate, and the outline of the proof

We first observe that, if ϕ is an eigenfunction on the manifold M associated to the eigenvalue λ, and if Ω is a nodal domain of ϕ (that is, a connected component of $M \setminus \phi^{-1}(0)$), then

$$\lambda = \lambda_1(\Omega) \tag{9}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of Ω for the Dirichlet boundary conditions.

The strategy in the proof of Theorem 1 is to give an upper bound of $\lambda_1(\Omega)$ by the square of the boundary volume, and then sum over all nodal domains. If Ω is a *simply connected* (or doubly connected) domain in the Euclidean plane, such an upper bound is due to Polya [P]:

$$\lambda_1(\Omega) \leq \frac{\pi^2}{4}\frac{\mathrm{Length}\,(\partial\Omega)^2}{\mathrm{Area}\,(\Omega)^2}, \tag{10}$$

We re-write (10) as:

$$\mathrm{Area}\,(\Omega)\sqrt{\lambda_1(\Omega)} \leq \frac{\pi}{2}\mathrm{Length}\,(\partial\Omega). \tag{11}$$

In particular, if ϕ is an eigenfunction on the flat torus, such that all of its nodal domains are simply connected, or doubly connected, we obtain immediately the (sharp) bound:

$$\mathrm{Length}\,[\phi^{-1}(0)] \geq \frac{1}{\pi}\mathrm{Area}\,(M)\sqrt{\lambda} \tag{12}$$

simply by summing (11) over all nodal domains of ϕ. However, the above condition on the the nodal domains may not be met; moreover, we want to consider arbitrary curved manifolds.

So the first step is to generalize Polya's bound to an arbitrary 2-manifold with boundary; this is accomplished by the following Proposition, which is valid in arbitrary dimensions:

Proposition 3. *Let Ω be a domain with piecewise-smooth boundary in an arbitrary manifold M. If $\lambda_1(\Omega)$ denotes the first eigenvalue of Ω for the Dirichlet boundary conditions, then:*

$$\lambda_1(\Omega) \leq \frac{\pi^2}{4}C(\Omega)^2\frac{\mathrm{Vol}_{\,n-1}(\partial\Omega)^2}{\mathrm{Vol}_{\,n}(\Omega)^2}, \tag{13}$$

where $C(\Omega) = \sup_r \left\{\frac{\mathrm{Vol}_{\,n-1}[\rho^{-1}(r)]}{\mathrm{Vol}_{\,n-1}(\partial\Omega)}\right\}$.

Here ρ denotes the distance function from the boundary of Ω, so that $\rho^{-1}(r)$ is the family of *interior parallels*, that is, the hypersurfaces of Ω at constant distance r from the boundary (note that the interior parallels are not in general regular, so by $\mathrm{Vol}_{\,n-1}[\rho^{-1}(r)]$ we really mean the volume of their regular part).

In general, the constant $C(\Omega)$ can be upper bounded in terms of a lower bound of the mean curvature of $\partial\Omega$ and a lower bound of the Ricci curvature of Ω (see

[S]). In some cases, however, the volume of the interior parallels is a non-increasing function of the distance r from the boundary (so that $C(\Omega) = 1$); this happens for example when both the mean curvature of $\partial\Omega$ and the Ricci curvature of the domain are non-negative (or when $\Omega \subseteq \mathbf{R}^2$ is simply connected, or doubly connected, the situation considered by Polya), and one has in these cases:

$$\lambda_1(\Omega) \leq \frac{\pi^2}{4} \frac{\mathrm{Vol}_{n-1}(\partial\Omega)^2}{\mathrm{Vol}_n(\Omega)^2}.$$

An upper bound of $\lambda_1(\Omega)$ in dimension 2. Proposition 3 takes an interesting form in dimension 2; in that case, the constant $C(\Omega)$ can be upper bounded by the *integrals* of the curvatures of the domain (rather then the pointwise bounds of the curvatures needed in higher dimensions). This is an important fact in deriving the nodal length estimates we are looking for. In fact, for the length of the interior parallels one can prove the following inequality, valid for all r:

$$\text{Length}\,[\rho^{-1}(r)] \leq \text{Length}\,(\partial\Omega) + R(\Omega)\max\{\int_\Omega K^+ - 2\pi\chi(\Omega), 0\}. \tag{14}$$

where $R(\Omega)$ is the inner radius of Ω (that is, the radius of the largest disk which fits into Ω), and K^+ is the positive part of the Gaussian curvature. (This bound is rather delicate, and extends the validity of the corresponding bound due to Fiala and Hartmann in special cases (see [Fi], [H])). Eventually one proves

Proposition 4. *Let Ω be a domain with piecewise-smooth boundary in an oriented Riemann surface M. Then:*

$$\sqrt{\lambda_1(\Omega)} \leq \frac{\pi}{2}\frac{\text{Length}\,(\partial\Omega)}{\text{Area}\,(\Omega)} + \frac{\pi}{2}\frac{R(\Omega)}{\text{Area}\,(\Omega)}\max\{\int_\Omega K^+ - 2\pi\chi(\Omega), 0\}.$$

The main steps of the proof of Theorem 1. Fix the eigenfunction ϕ on our 2-manifold M, associated to the eigenvalue λ, and consider the family $\{\Omega_i\}_{i\in I}$ of all nodal domains of ϕ: then $\lambda_1(\Omega_i) = \lambda$ for all i (note that I is a finite index by Courant's nodal domain theorem).

Step 1. Apply Proposition 4 to Ω_i and get, for all $i \in I$:

$$2\text{Area}\,(\Omega_i)\sqrt{\lambda} \leq \pi\text{Length}\,(\partial\Omega_i) + \pi R(\Omega_i)\max\{B(\Omega_i), 0\}. \tag{15}$$

where $B(\Omega_i) = \int_{\Omega_i} K^+ - 2\pi\chi(\Omega_i)$, $K^+ = \max\{K, 0\}$ denotes the positive part of the Gaussian curvature K of M, and $R(\Omega_i)$ is the inner radius of Ω_i.

Step 2. Estimate the inner radius of each nodal domain:

$$R(\Omega_i) \leq \psi(\lambda, \alpha, D)/\sqrt{\lambda} \tag{16}$$

where $\psi(\lambda, \alpha, D)$ has been defined in Theorem 1 (we use here the domain monotonicity of the first eigenvalue, and Cheng's comparison theorem). Note that, since $\psi(\lambda, \alpha, D) \to \pi$ as $\lambda \to \infty$, the nodal set becomes $\pi/\sqrt{\lambda}$–dense in M, as $\lambda \to \infty$.

STEP 3. Sum (15) over all $i \in I$, keeping in mind that summing the boundary lengths of all nodal domains gives twice the nodal length:

$$\text{Area}\,(M)\sqrt{\lambda} \leq \pi\text{Length}\,[\phi^{-1}(0)] + \frac{\pi}{2}\frac{\psi(\lambda,\alpha,D)}{\sqrt{\lambda}}\sum_{i\in I_-} B(\Omega_i). \tag{17}$$

where $I_- = \{i \in I : B(\Omega_i) > 0\}$.

STEP 4. Show that:

$$\sum_{i\in I_-} B(\Omega_i) \leq \frac{\pi\sqrt{\lambda}}{2}\text{Length}\,[\phi^{-1}(0)] + \int_M |K^-| - 2\pi|S|. \tag{18}$$

Taking into account (17) and (18) we get Theorem 1.

Few words about inequality (18). First observe that

$$\sum_{i\in I} B(\Omega_i) = \int_M |K^-| - 2\pi|S| \tag{19}$$

which follows from the addition formula $\sum_{i\in I}\chi(\Omega_i) = \chi(M) + |S|$ and the Gauss-Bonnet theorem. Hence:

$$\sum_{i\in I_-} B(\Omega_i) = \sum_{i\in I_+} |B(\Omega_i)| + \int_M |K^-| - 2\pi|S| \tag{20}$$

the main point is now to show (see Lemma 11 in [S]) that if $i \in I_+$ then:

$$|B(\Omega_i)| \leq \frac{\lambda}{2}\text{Area}\,(\Omega_i) \leq \frac{\pi}{4}\sqrt{\lambda}\text{Length}\,(\partial\Omega_i) \tag{21}$$

and (18) follows by summing (21) over $i \in I_+$. Note that, when M is a flat torus, (21) reads $2\pi \leq \frac{\lambda}{2}\text{Area}\,(\Omega_i)$ which is essentially the Faber-Krahn inequality applied to Ω_i.

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Dipartimento di Metodi e Modelli Matematici, Università di Roma, La Sapienza, Via Antonio Scarpa 16, 00161 Roma

E-mail address: savo@dmmm.uniroma1.it