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A Stable Spectral Multi-Domain Method for the Unsteady, Compressible Navier-Stokes Equations

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1 Introduction

In this paper we develop a multi-domain scheme, based on quadrilaterals as the building-block, for stable approximation of the two-dimensional compressible Navier-Stokes equations on conservation form. Although the presentation here is self-contained, the results rely heavily on a series of recent papers [HG96, Hes97a, Hes97b], to which we also refer for proofs and theoretical details. For ease of exposure, we have chosen to restrict the attention to schemes for two-dimensional subsonic flows. Details for supersonic flows and three-dimensional schemes can be found in [Hes97b].

Previous work on spectral multi-domain methods for the compressible Navier-Stokes equations is rather sparse. Only recently have several methods appeared [Kop93, Han93, KK96] with the emphasis being on methods for steady state problems. All previous methods for viscous flows are based on a treatment of the inviscid part of the equation, in most cases by applying methods known from the Euler equations, and a separate treatment of the viscous part of the equation. This second contribution is then applied as a correction to the result obtained from the inviscid patching.

The main difference between previously proposed methods and the one introduced here is that we develop a patching scheme which accounts for the inviscid and viscous part of the equation simultaneously. This approach is made possible by implementing the interface conditions using a penalty term [FG88], hence allowing for boundary conditions of a general type.

In Section 2 we introduce some background and notation. Section 3 introduces the complete scheme and theorems for well-posedness in a general plane domain and asymptotic stability of the scheme in a curvilinear quadrilateral. An example of the performance of the scheme for a non-trivial test case is presented in Section 4, which also contains a few concluding remarks.

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2 General Background and Notation

We wish to devise a scheme for approximating wave dominated problems in the domain, $\Omega \subset \mathsf{R}^2$, enclosed by the boundary $\delta\Omega$. To obtain such solutions we employ polynomial expansions to approximate the unknown functions and their spatial derivatives. As is well known, the most natural and computationally efficient way of applying polynomial expansions in several dimensions is through the use of tensor products. This procedure, however, requires that the computational domain is diffeomorphic to the unit square. To surmount this limitation, we construct Ω using K non-overlapping general quadrilaterals, $\mathsf{D}^k \subset \mathsf{R}^2$, such that $\Omega = \bigcup_{k=1}^{K} \mathsf{D}^k$. In what remains the emphasis will be on schemes for addressing problems in D^k and for simplicity we will by D with boundary $\delta\mathsf{D}$, refer to any quadrilateral domain unless clarification is deemed necessary.

To apply the tensor product formulation we require that there exists a diffeomorphism, $\Psi : \mathsf{D} \to \mathsf{I}$, where $\mathsf{I} \subset \mathsf{R}^2$ is the unit square, i.e., $\mathsf{I} \in [-1, 1]^2$. We will return to the specification of the map, Ψ, shortly. For convenience, we term the coordinates, $\boldsymbol{x} \in \mathsf{D}$, as (x, y) and (x_1, x_2) interchangeably. Likewise, we introduce the coordinates, $\boldsymbol{\xi} \in \mathsf{I}$, named (ξ, η).

As mentioned briefly, the map, $\Psi : \mathsf{D} \to \mathsf{I}$, plays an important role in the application of polynomial methods to problems in general geometries. To establish a one to one correspondence between the unit square and the general quadrilateral we construct the global map using transfinite blending functions as originally suggested in [GH73]. We refer to [Hes97b] for a thorough account of this procedure within the present context.

Once the global map, Ψ, has been constructed, we compute the metric of the mapping, the corresponding transformation Jacobian and outward pointing normal vectors at all points of the enclosing edges of the quadrilateral. Spatial derivatives are obtained through the chain rule and the relevant operators are all expressed in general curvilinear coordinates.

Approximation in I is done by a standard pseudospectral method using tensor products of interpolating Lagrange polynomials based on the Gauss-Lobatto nodal sets of Jacobi polynomials. We refer to [Fun92] for a general discussion of these techniques and to [HG96, Hes97a, Hes97b] for a thorough discussion within the present context.

3 A Stable Scheme for Navier-Stokes Equations

Consider the non-dimensional, compressible Navier-Stokes equations on conservation form

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla \cdot \boldsymbol{\Pi} = \frac{1}{\mathrm{Re}_{\mathrm{ref}}} \nabla \cdot \boldsymbol{\Pi}_\nu \ , \tag{1}$$

where we introduce the state vector, $\boldsymbol{q} = [\rho, \rho u, \rho v, E]^T$, and the inviscid flux tensor, $\boldsymbol{\Pi} = (\boldsymbol{F}_1, \boldsymbol{F}_2)$, with the elements $\boldsymbol{F}_1 = [\rho u, \rho u^2 + p, \rho uv, (E + p)u]^T$ and likewise $\boldsymbol{F}_2 = [\rho v, \rho uv, \rho v^2 + p, (E + p)v]^T$. Here ρ is the density, $\boldsymbol{u} = (u, v)$ is the Cartesian velocity, E is the total energy and p is the pressure.

The total energy, $E = \rho T + \frac{1}{2}\rho \boldsymbol{u} \cdot \boldsymbol{u}$, and the pressure are assumed to be related through the ideal gas law, $p = (\gamma - 1)\rho T$, where T is the temperature field and

$\gamma = c_p/c_v$ is the ratio between the heat capacities at constant pressure (c_p) and volume (c_v), respectively, and is assumed constant.

The elements of the viscous flux tensor, $\mathbf{\Pi}_\nu = (\boldsymbol{F}_1^\nu, \boldsymbol{F}_2^\nu)$, are given as $\boldsymbol{F}_1^\nu = [0, \tau_{xx}, \tau_{yx}, \tau_{xx}u + \tau_{yx}v + \frac{\gamma k}{\mathrm{Pr}}\frac{\partial T}{\partial x}]^T$ and also $\boldsymbol{F}_2^\nu = [0, \tau_{xy}, \tau_{yy}, \tau_{xy}u + \tau_{yy}v + \frac{\gamma k}{\mathrm{Pr}}\frac{\partial T}{\partial y}]^T$. Considering only Newtonian fluids, the stress tensor elements are

$$\tau_{x_i x_j} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda\sum_{k=1}^{2}\frac{\partial u_k}{\partial x_k} \quad ,$$

where δ_{ij} is the Kronecker delta-function and $(u_1, u_2) = \boldsymbol{u}$. Here μ is the dynamic viscosity, λ is the bulk viscosity and k is the coefficient of thermal conductivity.

The equations are normalized using the reference values, $u_{\mathrm{ref}} = u_0,\ \rho_{\mathrm{ref}} = \rho_0,\ p_{\mathrm{ref}} = \rho_0 u_0^2,\ T_{\mathrm{ref}} = u_0^2/c_v$ and a reference length L, where (ρ_0, u_0) is a given characteristic state. This yields a Reynolds number as $\mathrm{Re} = \rho_0 u_0 L/\mu_0$ and a Prandtl number as $\mathrm{Pr} = c_p\mu_0/k_0$. Note, that the Reynolds number in Eq.(1), $\mathrm{Re}_{\mathrm{ref}}$, based on the reference values, in general is different from Re. In the remaining part of the paper we shall refer to the latter as the Reynolds number. With this normalization we need to specify the Mach number, M, the Reynolds number, Re, the length scale, L, and a dimensional temperature, T_0.

We consider only atmospheric air and take $\gamma = 1.4$ and $\mathrm{Pr} = 0.72$. To model the temperature dependence of the dynamic viscosity we use Sutherland's viscosity law [Sch79]. Assuming that the Prandtl number is constant allows for modeling the temperature dependency of the coefficient of thermal conductivity similarly and we adopt Stokes hypothesis (see e.g. [Sch79]) to obtain $\lambda = -\frac{2}{3}\mu$ in all simulations.

Well-posed Patching Conditions

To derive a set of well-posed boundary conditions for the compressible Navier-Stokes equations on a general plane surface, we introduce the transformation derivatives

$$\mathcal{A}_i = \frac{\partial \boldsymbol{F}_i}{\partial \boldsymbol{q}} \ \text{ and } \ \mathcal{B}_{ij} = \frac{1}{2}\left(\frac{\partial \boldsymbol{F}_i^\nu}{\partial \boldsymbol{q}_{x_j}} + \frac{\partial \boldsymbol{F}_j^\nu}{\partial \boldsymbol{q}_{x_i}}\right) \quad ,$$

where $\boldsymbol{q}_{x_i} = \partial \boldsymbol{q}/\partial x_i$. To arrive at the proper boundary operator, we find it convenient also to introduce the operators

$$\mathcal{A} = \sum_{i=1}^{2}\mathcal{A}_i n_i \ \text{ and } \ \mathcal{B}_{x_i} = \sum_{j=1}^{2}\mathcal{B}_{ij}n_j \quad ,$$

where $\boldsymbol{n} = (n_1, n_2) = |\boldsymbol{n}|\hat{\boldsymbol{n}}$ is an outward pointing normal vector at $\delta\mathsf{D}$ of length $|\boldsymbol{n}|$.

Provided the solution, $\boldsymbol{q}$, is smooth it is sufficient to consider well-posedness and stability of the linearized and localized set of equations as discussed in [KL89] and applied extensively in [HG96, Hes97a, Hes97b].

A diagonalizing similarity transformation for arbitrary $\boldsymbol{n}$ for the constant coefficient operator, $\mathcal{A}(\boldsymbol{q}_0)$, was given by Warming et al. [WBH75]. Applying this transformation yields the diagonal matrix, $\mathcal{A}^{\mathbf{n}} = (\mathcal{S}^{\mathbf{n}})^{-1}\mathcal{A}\mathcal{S}^{\mathbf{n}}$, with the diagonal elements

$$\lambda_1^{\mathbf{n}} = \boldsymbol{u}_0\cdot\boldsymbol{n} + c_0|\boldsymbol{n}| \ , \ \lambda_2^{\mathbf{n}} = \lambda_3^{\mathbf{n}} = \boldsymbol{u}_0\cdot\boldsymbol{n} \ , \ \lambda_4^{\mathbf{n}} = \boldsymbol{u}_0\cdot\boldsymbol{n} - c_0|\boldsymbol{n}| \ ,$$

representing the advective velocities of the characteristic functions, $\boldsymbol{R}^{\mathbf{n}} = (\mathcal{S}^{\mathbf{n}})^{-1}\boldsymbol{q}$, along the direction given by $\boldsymbol{n}$, given as

$$\boldsymbol{R}^{\mathbf{n}} = \begin{bmatrix} \boldsymbol{m}\cdot\hat{\boldsymbol{n}} + \frac{\gamma-1}{c_0}\left(E + \frac{1}{2}\rho q_0^2 - \rho\boldsymbol{u}_0\cdot\boldsymbol{u}\right) \\ \rho - \frac{\gamma-1}{c_0^2}\left(E + \frac{1}{2}\rho q_0^2 - \rho\boldsymbol{u}_0\cdot\boldsymbol{u}\right) \\ \boldsymbol{m}\cdot\hat{\boldsymbol{k}} \\ -\boldsymbol{m}\cdot\hat{\boldsymbol{n}} + \frac{\gamma-1}{c_0}\left(E + \frac{1}{2}\rho q_0^2 - \rho\boldsymbol{u}_0\cdot\boldsymbol{u}\right) \end{bmatrix} ,$$

where we introduce the linearized momentum, $\boldsymbol{m} = \rho(\boldsymbol{u} - \boldsymbol{u}_0)$ and the tangential vector, $\boldsymbol{k} = |\boldsymbol{n}|\hat{\boldsymbol{k}} = |\boldsymbol{n}|(-\hat{n}_2, \hat{n}_1)$. Here $c_0 = \sqrt{\gamma p_0/\rho_0}$ represents the sound speed at the linearizing state.

Likewise, we also define the transformed viscous matrices, $\mathcal{B}^{\mathbf{n}}_{x_i} = (\mathcal{S}^{\mathbf{n}})^{-1}\mathcal{B}_{x_i}\mathcal{S}^{\mathbf{n}}$, to finally obtain the viscous correction vector

$$\boldsymbol{G}^{\mathbf{n}} = \mathcal{B}^{\mathbf{n}}_x\frac{\partial\boldsymbol{R}^{\mathbf{n}}}{\partial x} + \mathcal{B}^{\mathbf{n}}_y\frac{\partial\boldsymbol{R}^{\mathbf{n}}}{\partial y} = |\boldsymbol{n}|\begin{bmatrix} \frac{k_0(\gamma-1)}{2\mathrm{Pr}\rho_0}\boldsymbol{V}_1\cdot\hat{\boldsymbol{n}} + \frac{\lambda_0+2\mu_0}{2\rho_0}\boldsymbol{V}_2\cdot\hat{\boldsymbol{n}} - \frac{\lambda_0+\mu_0}{2\rho_0}\boldsymbol{V}_3\cdot\hat{\boldsymbol{k}} \\ -\frac{k_0(\gamma-1)}{2\mathrm{Pr}c_0}\boldsymbol{V}_1\cdot\hat{\boldsymbol{n}} \\ -\frac{\mu_0}{\rho_0}\boldsymbol{V}_3\cdot\hat{\boldsymbol{n}} + \frac{\lambda_0+\mu_0}{4\rho_0}\boldsymbol{V}_2\cdot\hat{\boldsymbol{k}} \\ \frac{k_0(\gamma-1)}{2\mathrm{Pr}\rho_0}\boldsymbol{V}_1\cdot\hat{\boldsymbol{n}} - \frac{\lambda_0+2\mu_0}{2\rho_0}\boldsymbol{V}_2\cdot\hat{\boldsymbol{n}} + \frac{\lambda_0+\mu_0}{2\rho_0}\boldsymbol{V}_3\cdot\hat{\boldsymbol{k}} \end{bmatrix} .$$

Here we have, for simplicity, introduced the vectors

$$\boldsymbol{V}_1 = \nabla R^{\mathbf{n}}_1 + \nabla R^{\mathbf{n}}_4 - \frac{2c_0}{(\gamma-1)}\nabla R^{\mathbf{n}}_2 \ , \ \boldsymbol{V}_2 = \nabla R^{\mathbf{n}}_1 - \nabla R^{\mathbf{n}}_4 \ , \ \boldsymbol{V}_3 = \nabla R^{\mathbf{n}}_3 \ ,$$

where $\boldsymbol{V}_1$ accounts for the normal heat flux, $\boldsymbol{V}_2$ for the normal stress and $\boldsymbol{V}_3$ for the effects of the tangential stress.

We are now in a position to state the following

Theorem 14.1 *Assume there exists a solution, $\boldsymbol{q}$, to the compressible Navier-Stokes equations on a general plane surface, D, enclosed by an almost smooth boundary, $\delta\mathsf{D}$, with the outward pointing normal vector, $\boldsymbol{n}$, uniquely defined at all points with the exception of a finite number of sets having measure zero in R.*

Assume also that the fluid properties are constrained as

$$\mu \geq 0 \ , \ \lambda \leq 0 \ , \ \mu + \lambda \geq 0 \ , \ \frac{\gamma k}{\mathrm{Pr}} \geq 0 \ , \ \gamma \geq 1 \ .$$

Provided the boundary operator is constructed such that

$$\forall\boldsymbol{x}\in\delta\mathsf{D}, \forall i\in[1,4]: \ R^{\mathbf{n}}_i\left[-\frac{1}{2}\lambda^{\mathbf{n}}_i R^{\mathbf{n}}_i + \frac{1}{\mathrm{Re}_{\mathrm{ref}}}G^{\mathbf{n}}_i\right] \leq 0 \ ,$$

where $R^{\mathbf{n}}_i$ and $G^{\mathbf{n}}_i$ represents the components of the vectors, $\boldsymbol{R}^{\mathbf{n}}$ and $\boldsymbol{G}^{\mathbf{n}}$, respectively, the constant coefficient problem is well-posed.

From this result it is straightforward to obtain a set of maximal dissipative boundary conditions of the form

$$\mathcal{R}^{\mathbf{n}}_{\pm}\boldsymbol{R}^{\mathbf{n}} + \frac{1}{\mathrm{Re}_{\mathrm{ref}}}\mathcal{G}_{\pm}\boldsymbol{G}^{\mathbf{n}} = 0 \ ,$$

where the subscript $\pm$ refers to the situations for which the boundary is an inflow, $\boldsymbol{u}_0 \cdot \boldsymbol{n} < 0$, or an outflow, $\boldsymbol{u}_0 \cdot \boldsymbol{n} > 0$, boundary and we introduce the four matrices, $\mathcal{R}^{\mathbf{n}}_{\pm}$ and $\mathcal{G}_{\pm}$, to construct the appropriate boundary operator.

For the subsonic inflow case, well-posedness appears for $\mathcal{R}^{\mathbf{n}}_{-} = \mathrm{diag}\,(0, |\lambda^{\mathbf{n}}_2|, |\lambda^{\mathbf{n}}_3|, |\lambda^{\mathbf{n}}_4|)$ and $\mathcal{G}_{-} = \mathrm{diag}(1,1,1,1)$. Likewise, for the subsonic outflow case we obtain the operator as $\mathcal{R}^{\mathbf{n}}_{+} = \mathrm{diag}\,(0,0,0,|\lambda^{\mathbf{n}}_4|)$ and $\mathcal{G}_{+} = \mathrm{diag}(0,1,1,1)$. The matrices corresponding to supersonic inflow and outflow are given in [Hes97b].

The singular nature of $\mathcal{G}_{+}$ is a consequence of the fact that for $G^{\mathbf{n}}_2 = 0$ we obtain that $G^{\mathbf{n}}_1 = -G^{\mathbf{n}}_4$. Consequently, only three conditions are required at outflow.

Similar to what was discussed in [HG96], we observe that the number of necessary boundary conditions at inflow (4) and outflow (3) conforms with results reported in [Str77]. We also recall that the boundary operator remains well-posed even in the case where the Reynolds number approaches infinity and we obtain the characteristic boundary conditions for the inviscid, compressible Euler equations.

The Stable Semi-Discrete Scheme

Establishing the boundary operator leading to a well-posed problem when considering the solution of the compressible Navier-Stokes equations in a general domain, allows us to develop an asymptotically stable scheme for approximating the equations in a general curvilinear domain. Although a similar approach may be applied for constructing schemes in general domains, we restrict the attention to the quadrilateral domain.

We propose to solve the compressible Navier-Stokes equations in a quadrilateral using a collocation method as

$$\begin{aligned}\frac{\partial \boldsymbol{q}}{\partial t} + \nabla\cdot\boldsymbol{\Pi} = \quad & \frac{1}{\mathrm{Re}_{\mathrm{ref}}}\nabla\cdot\boldsymbol{\Pi}_{\nu} \\ & -\tau Q(\boldsymbol{x})\mathcal{S}^{\mathbf{n}}\left[\mathcal{R}^{\mathbf{n}}_{\pm}\left(\boldsymbol{R}^{\mathbf{n}} - \boldsymbol{R}^{\mathbf{n}}_{BC}\right) + \frac{1}{\mathrm{Re}_{\mathrm{ref}}}\mathcal{G}_{\pm}\left(\boldsymbol{G}^{\mathbf{n}} - \boldsymbol{G}^{\mathbf{n}}_{BC}\right)\right] \quad ,\end{aligned} \tag{2}$$

where we introduce $\boldsymbol{R}^{\mathbf{n}}_{BC}$ and $\boldsymbol{G}^{\mathbf{n}}_{BC}$ to account for the boundary conditions in characteristic form at the various boundaries, be they sub-domain boundaries or open boundaries. The matrix, $\mathcal{S}^{\mathbf{n}}$, coming from the similarity transform of $\mathcal{A}$ along $\boldsymbol{n}$, is given as

$$\mathcal{S}^{\mathbf{n}} = \begin{bmatrix} \alpha & 1 & 0 & \alpha \\ \alpha(u + c\hat{n}_1) & u & -\hat{n}_2 & \alpha(u - c\hat{n}_1) \\ \alpha(v + c\hat{n}_2) & v & \hat{n}_1 & \alpha(v - c\hat{n}_2) \\ \alpha(H + c\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) & \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} & \boldsymbol{u}\cdot\hat{\boldsymbol{k}} & \alpha(H - c\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) \end{bmatrix} \quad ,$$

where we have the constant, $\alpha = 1/(2c)$, and the specific stagnation enthalpy, $H = (E+p)/\rho$. In $\mathcal{S}^{\mathbf{n}}$, all physical variables refer to the state, $\boldsymbol{q}_0$, around which we have linearized. The function $Q(\boldsymbol{x})$ is defined as

$$Q(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \delta\mathsf{D} \\ 0 & \text{otherwise} \end{cases} \quad ,$$

ensuring that Eq.(2) is modified at the boundaries only.

The conditions on τ ensuring asymptotic stability are given in the following Theorem.

Theorem 14.2 *Assume there exists a solution, $\boldsymbol{q}$, to the compressible Navier-Stokes equations in a general curvilinear quadrilateral domain, D, enclosed by the almost smooth boundary, $\delta\mathsf{D}$ and that the flow is purely subsonic. Assume also that there exists an diffeomorphism, $\Psi : \mathsf{D} \to \mathsf{I}$, which maps D onto the unit square, I.*

The fluid properties must by constrained as

$$\mu \geq 0 \ , \quad \lambda \leq 0 \ , \quad \mu + \lambda \geq 0 \ , \quad \frac{\gamma k}{\mathrm{Pr}} \geq 0 \ , \quad \gamma \geq 1 \ .$$

Let

$$\kappa = \frac{1}{\mathrm{Re}_{\mathrm{ref}}} \frac{|\boldsymbol{n}|^2}{2\tilde{\omega}} \frac{1}{\rho_0 |\boldsymbol{u}_0 \cdot \boldsymbol{n}|} \max\left(\mu_0, 2\mu_0 + \lambda_0, \frac{\gamma k_0}{\mathrm{Pr}}\right) \ .$$

Approximating the solution of the linearized constant coefficient version of Eq.(2) using a collocation method yields an asymptotically stable scheme provided the penalty parameters are bounded as

$$\frac{1}{\tilde{\omega}\kappa}\left(1 + \kappa - \sqrt{1+\kappa}\right) \leq \tau \leq \frac{1}{\tilde{\omega}\kappa}\left(1 + \kappa + \sqrt{1+\kappa}\right) \ .$$

The correct choice of $\tilde{\omega}$ and the outward pointing normal vector, $\boldsymbol{n}$, depending on whether an edge or a vertex, is considered is given in the Appendix.

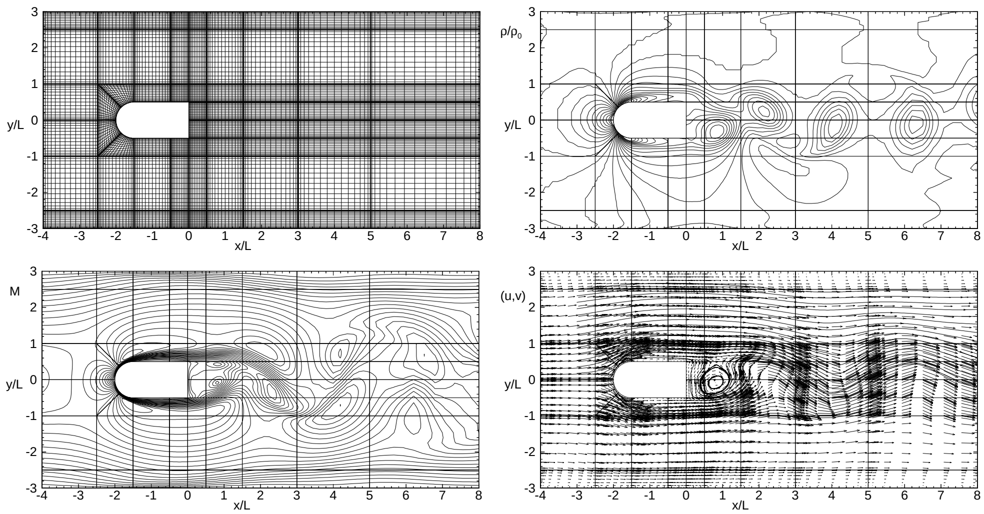
We note that the above result is strictly valid only in the case in which the Jacobian is a constant. However, as we will show shortly, the scheme remains stable also for non-constant Jacobians, thus establishing a stable method for approximating the compressible Navier-Stokes equations in a general quadrilateral domain. A similar result has been established for supersonic inflow and outflow conditions and can be found in [Hes97b].

The result stated in Theorem 14.2 is valid also for the vertices of the quadrilateral. However, at a point where several vertices meet, one has to determine which element is upstream and which is downstream in order to pass the appropriate information between the vertices. For this purpose we define the two vectors, $\boldsymbol{n}_\xi = \xi\nabla\xi$ and $\boldsymbol{n}_\eta = \eta\nabla\eta$. A vertex, say $(\xi, \eta) = (-1, -1)$, can then be identified as the upstream vertex provided $\boldsymbol{u} \cdot \boldsymbol{n}_\xi > 0$ and $\boldsymbol{u} \cdot \boldsymbol{n}_\eta > 0$. In a similar fashion, we may identify the downstream element by reversing the inequalities. The conditions for this test are summarized in the Appendix. Contrary to some previously proposed schemes (see e.g. [Kop91]), this approach handles any number of domains coming together, as the upstream and downstream domains are uniquely identified through the signs of the above scalar-products. For the boundary conditions of the viscous part, we use the average of the Cartesian derivatives across the vertex.

For temporal integration, we use a 3rd-order Runge-Kutta with the boundary conditions being imposed at the intermediate time-steps. Following completion of each time-step, we enforce global continuity and we use the solution at the previous time-step as the solution around which we linearize at the sub-domain boundaries, while the exact solution is used at the open boundaries. The time-step is computed adaptively as

$$\Delta t \leq \mathrm{CFL} \times \min_{\mathbf{x}\in\Omega} \left[|\chi \cdot \boldsymbol{u}| + c\sqrt{\chi \cdot \chi} + \frac{2\gamma}{\mathrm{Pr}\mathrm{Re}_{\mathrm{ref}}} \frac{\mu}{\rho} \chi \cdot \chi \right]^{-1} \ ,$$

Figure 1 Fragment of the grid used for the flame holder computation. Also shown is the instantaneous density, ρ/ρ_0, the Mach number, M, and the velocity field, $\mathbf{u}$.



where χ is the grid-distortion vector, $\boldsymbol{\chi} = [|\xi_x|/\Delta\xi_i + |\eta_x|/\Delta\eta_j, |\xi_y|/\Delta\xi_i + |\eta_y|/\Delta\eta_j]$ with $\Delta\xi_i$ signifying the local grid size and similarly for $\Delta\eta_j$.

4 Numerical Examples and Remarks

We have implemented the proposed scheme in order to confirm the theoretical results obtained for the linearized, constant coefficient Navier-Stokes equations. In [Hes97a, Hes97b] we presented several solutions of steady state flows, confirming the spectral accuracy of the proposed scheme. However, to emphasize the ability to handle truly unsteady flow, we consider here a problem of some practical importance.

We consider the flow around a flame holder embedded in a narrow channel. This geometry can be viewed as a prototype combustion chamber in a high-speed ram-jet. However, although the engine is designed to perform at supersonic speeds, the flow in the combustion chamber remains purely subsonic. We consider the geometry pictured in Fig. 1, with the base hight of the flame holder being, $L = 12.7$ mm. The flame holder is embedded in a narrow channel with a total height of only $6L$. The full length of the computational domain is $25L$, i.e., Fig. 1 shows only a part of the computational domain.

All walls are assumed to be isothermal, no-slip wall, being held at a stagnation temperature of $T_0 = 300°K$. The free-stream Reynolds number is 250 and the Mach number is 0.4, ensuring that the flow remains subsonic.

The total computation uses 104 elements, each employing a polynomial expansion of order 14. The open boundaries are held at the free-stream values with a laminar, parabolic inflow and outflow velocity profile and the pressure drop computed self-consistently.

Figure 1 clearly illustrates the well known von Karman vortex street rear of the bluff body and the boundary layers at the wall. We also note that all fields are smooth across sub-domain boundaries, including the vertices.

Although these results are of a qualitative nature they confirm the stability of the complete scheme for general curvilinear elements, the validity of the treatment of the vertices and the efficacy of the scheme for the study of unsteady compressible flows in complex geometries.

We have not addressed the question of efficient implementation. However, we recall that the patching of sub-domains and treatment of physical boundaries is purely local in time and space, i.e., the algorithm lends it self to efficient implementation on parallel computers with distributed memory. This will be of significant importance when future attention is directed towards the solution of unsteady three-dimensional problems in complex geometries.

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Appendix

To ensure stability of the semi-discrete scheme we must choose the parameters, $\boldsymbol{n}$ and $\tilde{\omega}$, appropriately. Moreover, we need to establish the proper conditions for identifying a vertex as an upstream or downstream vertex.

Let us first define the vectors, $\boldsymbol{n}_\xi = \xi \nabla \xi$ and $\boldsymbol{n}_\eta = \eta \nabla \eta$. We will also introduce the two variables, ω_ξ and ω_η. The actual value of these parameters are resolution as well as method dependent.

For *Legendre* methods, we have $\omega_\xi = 2/(N_\xi(N_\xi + 1))$ and $\omega_\eta = 2/(N_\eta(N_\eta + 1))$, where N_ξ and N_η represents the resolution along ξ and η, respectively.

For *Chebyshev* methods, on the other hand, we have $\omega_\xi = N_\xi^{-2}$ and $\omega_\eta = N_\eta^{-2}$. The appropriate values of the parameter, $\tilde{\omega}$, and the outward pointing normal vector, $\boldsymbol{n}$, required to construct stable schemes along edges and vertices of the quadrilateral is given below.

We also give the condition for determining whether a vertex is indeed upstream. For this purpose, we introduce the convective velocity, $\boldsymbol{u}$. The conditions for naming a purely downstream vertex is obtained by reversing the inequalities.

		τ – Parameters		Outflow Conditions	
ξ	η	$\tilde{\omega}$	$\boldsymbol{n}$	$\boldsymbol{u} \cdot \boldsymbol{n}_\xi$	$\boldsymbol{u} \cdot \boldsymbol{n}_\eta$
± 1	$\cdot$	ω_ξ	$\boldsymbol{n}_\xi$	> 0	–
$\cdot$	± 1	ω_η	$\boldsymbol{n}_\eta$	–	> 0
± 1	± 1	$\omega_\xi \omega_\eta$	$\omega_\eta \boldsymbol{n}_\xi + \omega_\xi \boldsymbol{n}_\eta$	> 0	> 0

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