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Multilevel Finite Element Riesz Bases in Sobolev Spaces

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1 Introduction

In this note we discuss some results concerning multilevel finite element schemes of hierarchical basis (HB) type in connection with discretizing and preconditioning elliptic problems in Sobolev spaces. Roughly speaking, HB-methods require the introduction of a hierarchically defined algebraic basis Ψ_j of locally supported functions for a scale of finite element discretization spaces V_j, $j \geq 0$, and aim at reducing the condition number of discretization matrices for standard elliptic problems when represented in the basis Ψ_j. Motivated by recently proposed modifications to the standard HB-method (Yserentant [Yse86]) such as the 3-point HB-method of Stevenson [Ste96, Ste97a], the coarse–grid stabilized HB-methods of Carnicer/Dahmen/Peña [CDP96], Vassilevski/Wang [VW97a, VW97b] and the L_2-semiorthogonal prewavelet methods (see [Osw94, Jun94, KO96, Ste97b]), we started in [LO96] a systematic comparison of their properties. In a first step, we considered finite element HB-methods with respect to shift-invariant, dyadically refined triangulations of $\mathbb{R}^d$, and studied the range of the smoothness parameter s for which a given HB-system $\Psi = \cup_{j\geq 0}\Psi_j$ is a Riesz basis in $H^s(\mathbb{R}^d)$. For those s, discretizations of H^s-elliptic problems in V_j with respect to Ψ_j will lead to stiffness matrices with uniformly (j-independent) condition numbers, thus resulting in an asymptotically optimal preconditioning method.

We concentrate here on the case of linear finite elements and $d \leq 3$. Section 2 contains the definitions of HB-systems and a brief survey of the connection to multilevel preconditioners. In Section 3, we report on results obtained in [LO96, LO97a] for the shift-invariant case. Future research should include extensions of the theory to realistic domains and partition sequences obtained by adaptive refinement, as well as a more quantitative investigation of work estimates (condition numbers versus arithmetical complexity per iteration). In Section 4, we provide the condition numbers for generic H_0^1– and L_2–discretizations on a square in $\mathbb{R}^2$ when using the HB-examples discussed in Section 3.

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2 HB-Systems for Linear Finite Elements

Throughout this paper, let

$$V_0 \subset V_1 \subset \ldots \subset V_j \equiv S_1^0(\mathcal{T}_j) \cap L_2(\Omega) \subset \ldots \tag{1}$$

be the sequence of linear finite element spaces with respect to uniformly and dyadically refined simplicial partitions $\mathcal{T}_j$ of element size $\approx 2^{-j}$ of a polyhedral domain Ω. Specifically, as a model case, we consider $\Omega = \mathbb{R}^d$ and the sequence of shift-invariant (2^d-1)-directional partitions $\mathcal{T}_j$. The nodal basis (NB) functions for V_j will be denoted by $\phi_{j,P}$, $P \in \mathcal{V}_j$, where $\mathcal{V}_j$ is the vertex set of $\mathcal{T}_j$. We set $\mathcal{W}_j = \mathcal{V}_j \backslash \mathcal{V}_{j-1}$ for the sets of vertices newly generated when refining $\mathcal{T}_{j-1}$, $j \geq 1$, $\mathcal{W}_0 = \mathcal{V}_0$. Points in $\mathcal{W}_j$ are the edge midpoints of $\mathcal{T}_{j-1}$. Finally, let $n_j = \#\mathcal{V}_j$, $m_j = \#\mathcal{W}_j$.

The HB-systems we look for are of the form

$$\Psi = \bigcup_{j=0}^{\infty} \{\psi_{j,P} \,:\, P \in \mathcal{W}_j\}\,, \quad \Psi_J = \bigcup_{j=0}^{J} \{\psi_{j,P} \,:\, P \in \mathcal{W}_j\}\,, \tag{2}$$

where the locally supported HB-functions

$$\psi_{j,P} = \sum_{Q \in \mathcal{V}_j} a_{j;P,Q} \phi_{j,Q}\,, \quad P \in \mathcal{W}_j\,, \tag{3}$$

are given by their masks $(a_{j,P,\cdot})$. We assume that the size of these masks (i.e., the number of nonzero coefficients in (3)) is uniformly bounded with respect to j and P. This implies that the rectangular matrices

$$\hat{I}_j = ((a_{j;P,Q}))_{Q \in \mathcal{V}_j, P \in \mathcal{W}_j} \tag{4}$$

of dimension $n_j \times m_j$ have $\mathrm{O}(m_j)$ non-zero entries. We assume that the system of level-j HB-functions $\{\psi_{j,P} \,:\, P \in \mathcal{W}_j\}$ forms an L_2-stable basis in its L_2-closed span W_j, and that V_j admits an L_2-stable direct sum decomposition $V_j = V_{j-1} \dot{+} W_j$. Here, L_2-stability means that

$$\| \sum_{P \in \mathcal{W}_j} c_{j,P} \psi_{j,P} \|_{L_2}^2 \asymp 2^{-jd} \sum_{P \in \mathcal{W}_j} c_{j,P}^2 \tag{5}$$

for all reasonable coefficient choices resp.

$$\|v_{j-1} + w_j\|_{L_2}^2 \asymp \|v_{j-1}\|_{L_2}^2 + \|w_j\|_{L_2}^2 \qquad \forall\, v_{j-1} \in V_{j-1}\,,\ \forall\, w_j \in W_j\,. \tag{6}$$

We always assume that two-sided estimates expressed by $\asymp$ hold with positive constants that are independent of parameters and functions, especially, of j. The assumptions (5), (6) are usually easy to check (since they concern only two adjacent levels), and imply that the finite sections Ψ_J of the HB-system Ψ are algebraic bases in V_J, for all $J \geq 0$.

However, there is no guarantee for uniform L_2-stability of the Ψ_J or for stability of the whole HB-system Ψ in the L_2-norm (or in other norms) under the above assumptions. This desirable property is, up to scaling, part of the definition of a Riesz basis.

Definition 1 *A system $\mathcal{F} \equiv \{f_l\} \in H$ is a Riesz basis in the (real) Hilbert space H if the mapping*

$$(c_l) \longmapsto \sum_l c_l f_l \ ,$$

which is well–defined for finite sequences (c_l), can be extended to an isomorphism between l_2 and H. In other words, $\mathcal{F}$ should be dense and minimal in H, and satisfy

$$\| \sum_l c_l f_l \|_H^2 \asymp \sum_l c_l^2 \ .$$

The best possible constants in this two-sided inequality are called Riesz bounds of $\mathcal{F}$ in H.

For properties of Riesz bases and frames (the latter generalize the stabilty concept to nonunique decompositions and generating systems) in connection with multiresolution analysis and multilevel methods, see [Dau92, Dah96, Osw97]. We quote a corollary for the finite element HB-systems introduced above when applied to variational problems in Sobolev spaces $H^s(\Omega)$. Consider the symmetric $H^s(\Omega)$-elliptic variational problem of determining $u \in H^s(\Omega)$ such that

$$a(u,v) = \langle f, v \rangle_{H^{-s} \times H^s} \qquad \forall \, v \in H^s(\Omega) \ . \tag{7}$$

We can restrict (7) to V_J: Find $u_J \in V_J$ such that

$$a(u_J, v_J) = \langle f, v_J \rangle_{H^{-s} \times H^s} \qquad \forall \, v_J \in V_J \ . \tag{8}$$

Naturally, for C^0 finite elements, $s < 3/2$ has to be assumed. For finite-dimensional V_J, (8) leads to different linear systems depending on the choice of a basis in V_J. The choice $\{\phi_{J,P} \, : \, P \in \mathcal{V}_J\}$ leads to the standard NB discretization

$$A_J x_J = f_J \ , \tag{9}$$

with $a(\phi_{J,P}, \phi_{J,Q})$ resp. $\langle f, \phi_{J,P} \rangle_{H^{-s} \times H^s}$ being the entries of the matrix resp. right-hand side of (9). Analogously, taking Ψ_J, we get

$$A_J^{\Psi} y_J = f_J^{\Psi} \ . \tag{10}$$

The solution vectors $x_J = (x_{J,P} : P \in \mathcal{V}_J)$ and $y_J = (y_{j,P} : P \in \mathcal{W}_j, j \leq J)$ represent the NB and HB coefficients of the solution u_J of (8), i.e.,

$$u_J = \sum_{P \in \mathcal{V}_J} x_{J,P} \phi_{J,P} = \sum_{j=0}^{J} \sum_{P \in \mathcal{W}_j} y_{j,P} \psi_{j,P} \ .$$

If we denote the matrix for the change of basis between HB- and NB-representations of functions from V_J by S_J^{Ψ} (e.g., $x_J = S_J^{\Psi} y_J$) then one easily sees that

$$A_J^{\Psi} = (S_J^{\Psi})^T A_J S_J^{\Psi} \ . \tag{11}$$

Note that due to (3) a multiplication by S_J^{Ψ} can be implemented in $\asymp n_J$ operations. The constants depend on the mask size bound.

As is well-known, the condition numbers of A_J exhibit exponential growth $\asymp 2^{2|s|J}$ for $s \neq 0$. A desirable feature of a HB-construction would be to get J-independent, uniformly bounded condition numbers for the HB-stiffness matrix A_J^Ψ. Using the sparse S_J^Ψ transformation, this would immediately lead to economic iterative solvers for (8). The theoretical answer is

Theorem 1 *Suppose* $\dim V_j < \infty$ *and* $s < 3/2$. *Then, the following are equivalent:*

(i) The normalized HB-system

$$\tilde{\Psi} = \bigcup_{j=0}^{\infty} \{ \|\psi_{j,P}\|_{H^s(\Omega)}^{-1} \psi_{j,P} \; : \; P \in \mathcal{W}_j \}$$

is a Riesz basis in $H^s(\Omega)$.

(ii) The HB-discretization matrices A_J^Ψ *in (10) associated with a symmetric* $H^s(\Omega)$*-elliptic variational problem (7) possess uniformly bounded condition numbers after diagonal scaling.*

The upper bound for $\kappa((D_J^\Psi)^{-1} A_J^\Psi)$, *where* D_J^Ψ *is the diagonal part of* A_J^Ψ, *depends on the Riesz bounds of* $\tilde{\Psi}$ *and the ellipticity constants of the form* $a(\cdot,\cdot)$ *(i.e., on the constants in* $a(u,u) \approx \|u\|_{H^s}^2$, $u \in H^s(\Omega)$*).*

We omit the proof which can be given by using the theory of stable subspace splittings [Osw94], compare also [Osw97]. A reformulation of Theorem 1 (ii) is that

$$\kappa(C_J^\Psi A_J) = \mathrm{O}(1)\,, \quad J \to \infty\,, \quad C_J^\Psi = S_J^\Psi (D_J^\Psi)^{-1} (S_J^\Psi)^T \,. \tag{12}$$

The product $C_J^\Psi A_J$ coincides with the matrix representation of the additive Schwarz operator associated with the splitting

$$V_J = \sum_{j=0}^{J} \sum_{P \in \mathcal{W}_j} W_{j,P} \qquad (W_{j,P} = \operatorname{span} \psi_{j,P})$$

of V_J into the direct sum of one-dimensional subspaces $W_{j,P}$ each of which is spanned by a single HB-function of some level $j \leq J$. The scalar products are induced by $a(\cdot,\cdot)$. See [Osw94, Osw97, LO96, LO97a] for more details, also on the recursive definition of the symmetric preconditioner C_J^Ψ which, besides the diagonal scaling, involves the matrices $\hat{I}_j$ (which actually describe the embedding $W_j \subset V_j$), and analogous matrices I_j describing the embedding $V_{j-1} \subset V_j$, $j = 1, \dots, J$.

3 Riesz Bases in $H^s(\mathbb{R})^d$: Examples

In general, the verification of the Riesz basis property of a given HB-system in Sobolev spaces is not trivial. It has to do with tools like Jackson-Bernstein inequalities for scales of approximating spaces (such as $\{V_j\}$) but also with the study of associated biorthogonal systems. We refer to [Dah96]. A considerable simplification is possible under the assumption of shift-invariance (i.e., we assume uniform dyadic simplicial partitions of $\Omega = \mathbb{R}^d$, $\mathcal{V}_j = 2^{-j}\mathbb{Z}^d$, and that the HB-system is actually produced by translating and dilating $2^d - 1$ ψ-functions associated with the different edge directions). This assumption is typical for wavelet analysis, and allowed us in [LO96] to obtain a number of sharp results on the s-range for which the Riesz basis property holds for particular systems. Lack of space prevents us from presenting details on the theoretical tools used to produce these s-intervals. Instead we provide examples for the linear finite element case which have been considered in more detail in [LO96], and are often modeled after HB-constructions for bounded domains taken from the literature. The results reported here can be seen as qualitative information on the interior part of the associated HB-construction for domains. The Sobolev exponents of some practical importance are $s = 1$ and $s = 0$ (second order elliptic problems (including Helmholtz terms), Fredholm integral equations of second kind), as well as $s = \pm 1/2$ (boundary integral equations of first kind, interface problems in domain decomposition methods).

Details are given for $d = 2$. In this case, the grid is three-directional (as in Figure 1 a) below). The ψ-functions associated with the different edge directions (horizontal, vertical, and diagonal) will be labeled by h, v, and d. Whenever possible, we give the corresponding HB-construction for bounded polyhedral domains, and then specialize to the shift-invariant case. Furthermore, we set $\psi_{0,P} = \phi_{0,P}$ for all $P \in \mathcal{V}_0$.

Example 1 *Standard HB* (Yserentant [Yse86]). This system is given by

$$\psi_{j,P} = \phi_{j,P}\ , P \in \mathcal{W}_j\ ,\ j \geq 1\ ,$$

and is the simplest of all HB-systems.

Theorem 2 *The normalized standard HB-system $\tilde{\Psi}$ is a Riesz basis in $H^s(\Omega)$ if and only if $d/2 < s < 3/2$ $(d \leq 2)$.*

The case $s = 1$, $d = 2$, is not included here, in coincidence with the known fact [Yse86] that the standard HB-method of Yserentant is only suboptimal: $\kappa((D_J^\Psi)^{-1} A_J^\Psi) \asymp J^2$ there.

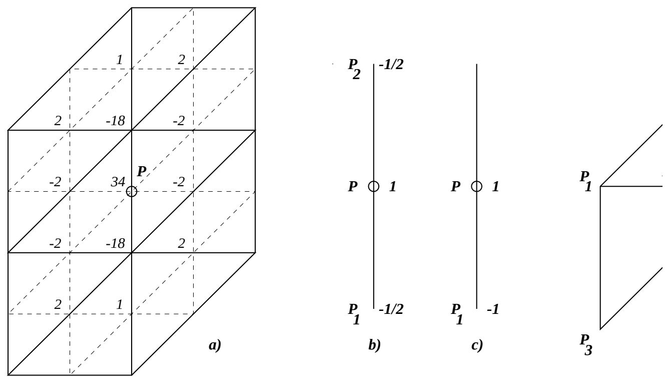
Example 2 *Extended NB system.* Though not fitting into the discussion of Riesz bases, we would like to mention the following interpretation of the optimality (see [Osw94], Section 4.2) of the BPX-preconditioner introduced by Bramble/Pasciak/Xu [BPX90]. We call $\Phi = \{\phi_{j,P}\ :\ P \in \mathcal{V}_j,\ j \geq 0\}$ an extended NB-system or BPX-system, and denote by $\tilde{\Phi}$ the corresponding $H^s(\Omega)$-normalized system. Note that Φ is not minimal. The finite sections Φ_J of this system obtained by taking only NB functions with $j \leq J$ are generating systems (not bases) for V_J. Nevertheless, we have

Theorem 3 *For arbitrary $d \geq 1$, the normalized BPX-system $\tilde{\Phi}$ is a frame in $H^s(\Omega)$ iff $0 < s < 3/2$.*

It should be mentioned that this simple enlargement of the standard HB-system not only improves the theoretical properties of the latter for $d \geq 2$. The practical performance (simplicity of implementation, operation count per preconditoning step, condition number bounds for H^1-problems) is surprisingly good.

Example 3 *L_2-semiorthogonal prewavelet systems* (Kotyczka/Oswald [Osw94, KO96], Junkherr [Jun94], Stevenson [Ste97b]). We call a HB-system an L_2-semiorthogonal prewavelet system if it is obtained by choosing the finite masks in (3) such that all $\psi_{j,P}$ are L_2-orthogonal to V_{j-1}, $j \geq 1$, and (5) still holds. It turns out (see [Osw94],

Figure 1 Masks for ψ_v: a) L_2-semiorthogonal prewavelet system, b) 3-point HB-system, c) 2-point HB-system, and d) notation for Example 5.



Section 4.4 for a similar argument) that the proof of Theorem 3 and the definition of Sobolev spaces with negative s by duality immediately imply

Theorem 4 *The normalized version* $\tilde{\Psi}$ *of an* L_2 *semiorthogonal prewavelet system is a Riesz basis for* $H^s(\Omega)$ *if* $-3/2 < s < 3/2$.

Thus, such systems cover most of the potential applications in an asymptotically optimal sense. For the shift-invariant case, an example with smallest possible support of the $\psi_{j,P}$ has been constructed in [KO96] for $d = 2$. Figure 1 a) shows the mask for ψ_v. The other two masks are obtained by suitable rotation. For general $\Omega \subset \mathbb{R}^2$, a mask construction has been proposed in [LO96], however, there is no rigorous proof of (5) for this case. Generalizations to $d \geq 3$ along the lines of [KO96, Jun94] do not seem to be of practical interest. One reason is the relatively large masks (e.g., an average of 29 non-zero coefficients is needed to satisfy the orthogonality constraint for $d = 3$ (uniform refinement case)). Very recently, Stevenson [Ste97b] came up with an alternative construction of L_2-semiorthogonal prewavelets which is suitable for all $d \leq 3$. However, the s-values of interest are also covered by much simpler HB-constructions (see below).

Finally let us mention that, from a theoretical (and practical) point of view, the construction of HB-systems consisting of functions $\psi_{j,P}$ which are orthogonal resp. semiorthogonal with respect to the variational scalar product $a(\cdot,\cdot)$ would be desirable. However, for $d \geq 2$ and the $H^s(\mathbb{R}^d)$ scalar product ($s > 0$) such systems cannot have uniformly bounded mask size, see [LO97b].

Example 4 *3-point HB-system* (Stevenson [Ste96, Ste97a]). If L_2-semiorthogonality is weakened, simplifications are possible. Let P_1, P_2 denote the endpoints of the edge in $\mathcal{T}_{j-1}$ which contains $P \in \mathcal{W}_j$ as midpoint. Set

$$\psi_{j,P} = \psi_{j,P} + a_1\psi_{j,P_1} + a_2\psi_{j,P_2}$$

and choose a_1, a_2 such that

$$\sum_{Q\in\mathcal{V}_j} \mu_{j,Q}\psi_{j,P}(Q)u_{j-1}(Q) = 0 \qquad \forall u_{j-1} \in V_{j-1} \ .$$

Here, the choice $\mu_{j,Q} = |\mathrm{supp}\,\phi_{j,Q}|/3$ guarantees that the corresponding quadrature formula $\sum_Q \mu_{j,Q}u(Q)$ is exact w. r. t. V_j. Thus, the construction can be interpreted as replacing L_2-orthogonality by discrete L_2-orthogonality.

This HB-system has been studied, both theoretically and numerically, by Stevenson [Ste96, Ste97a] for $d = 2, 3$. For the shift-invariant case (the masks for this case are edge- and d-independent, see Figure 1 b)), we showed in [LO96]

Theorem 5 *The normalized 3-point HB-system of Stevenson is a Riesz basis in* $H^s(\mathbb{R}^d)$, $d \le 3$, *iff* $-0.992036 < s < 3/2$.

For partial results and numerical evidence in the case of general Ω, see [Ste96, Ste97a]. It turns out that in the shift-invariant case, one can construct a whole family of analogous, edge-oriented, HB-systems (see [LO96]Section 4.1, [LO97a]Section 3.2). The simplest one leads to a 2-point HB-system, see Figure 1 c) for the mask of ψ_v, with the corresponding s-interval still covering the L_2-case: $-0.044117 < s < 3/2$. It is not quite clear at the moment what the correct 2-point HB-definition is for general Ω.

Example 5 *Coarse-grid stabilized HB-systems* (Carnicer/Dahmen/P̃ena [CDP96], Vassilevski/Wang [VW97a, VW97b]). The common idea is to define

$$\psi_{j,P} = (Id_j - Q_{j-1})\phi_{j,P} \ , \quad P \in \mathcal{W}_j \ , \ \ j \ge 1 \ ,$$

where $Q_{j-1} : V_j \to V_{j-1}$, and Id_j denotes the identity on V_j. For Q_{j-1}, quasi-interpolant operators are suggested in [CDP96], Section 4.2, while [VW97b] prefers the use of approximations to the exact L_2-orthogonal projection obtained by approximately inverting the L_2-Gram matrix of the nodal basis in V_{j-1}. The most economical proposals from these papers lead to

$$\psi_{j,P} = \phi_{j,P} - \sum_{i=1}^{4} a_i\phi_{j-1,P_i} \ , \quad P \in \mathcal{W}_j \ , \tag{13}$$

where P_i denote the vertices of the two triangles in $\mathcal{T}_{j-1}$ sharing the edge e_P (with obvious modifications for P near the boundary). Compare Figure 1 d). As a rule all a_i are non-zero, thus, these proposals are essentially 5-point HB-systems.

For $d = 2$, the shift-invariant case was analyzed in [LO96], where we concentrated on the specific, one-parameter family of masks given by

$$a_1 = a_2 = a \ , \quad a_3 = a_4 = 1/8 - a \qquad (a \in \mathbb{R}) \ . \tag{14}$$

This class is remarkable in that the ψ-functions satisfy moment conditions of order 2, a property, which is desirable if stiffness matrix compression is an issue (e.g., for integral equation applications). The following table shows the s-range for which the scaled coarse-grid stabilized HB-system $\tilde{\Psi}$ specified by (13), (14), is a Riesz basis in $H^s(\mathbb{R}^d)$ for some a (the reader may view this as the last theorem of this note). In

Table 1 Coarse-grid stabilized HB-systems: Results

a-value	s-range	comments
5/48	$0.248994 < s < 3/2$	$m = \beta = 1$ in [VW97b]
1/8	$0.022818 < s < 3/2$	complexity as 3-point HB
1/6	$-0.357680 < s < 3/2$	[CDP96]
3/16	$-0.440765 < s < 3/2$	maximal s-range,[CS93]
1/4	$0.396793 < s < 3/2$	

[LO97a]Section 3.2, a more detailed table is given. E.g., we have found that for these systems the Riesz basis property holds in $L_2(\mathbb{R}^d)$ if $0.1271146 < a < 0.220647$ resp. in $H^1(\mathbb{R}^d)$ if $0.028759 < a < 0.3014364$. This shows a certain robustness of such constructions (provided that the moment conditions are preserved).

The intervals in Table 1 suggest that the counterparts for general Ω might well work, at least, for second order elliptic problems ($s = 1$). However, there are no definite results in this direction so far. Compare [VW97a] for theoretical results on the existence (with possibly quite large masks) of coarse-grid stabilized HB-Riesz bases for $H^s(\Omega)$, $s > 0$. A crude message from the above examples is that fine-grid corrected HB of simple structure seem to have better properties than coarse-grid stabilized HB-systems. However, the condition number computations presented below will slightly correct this impression.

4 Condition Numbers

The impression that a "larger s-interval" for the Riesz basis property to hold means "better practical performance" is misleading (though, by some kind of interpolation argument, one might expect good preconditioning effects if the s corresponding to a given variational problem is in the central part of the computed interval). On the other hand, when elliptic operators including parts of different order are the main concern, a large s-interval might be of benefit. In any case, numerical estimations of Riesz bounds resp. condition numbers $\kappa((D_J^\Psi)^{-1}A_J^\Psi)$ and other performance testing are recommended.

The following tables serve as an orientation for the more practically interested reader. We only present calculations of condition numbers $\kappa((D_J^\Psi)^{-1}A_J^\Psi)$ for $d = 2$ and $s = 0, 1$ (boundary integral equations, where $s = \pm 1/2$, are not addressed). The domain is the unit square, computations are done on standard uniform dyadic triangulations, and zero boundary conditions were imposed on the spaces V_j. The bilinear forms is the L_2-scalar product ($s = 0$) resp. is induced by the Laplace operator ($s = 1$). The index $j = 0$ corresponds to stepsize $h_0 = 1/2$, thus resulting in a one-dimensional V_0. The largest problems ($J = 7$) have dimension 65025. Everything else is implemented exactly as described above. For the 2-point HB-system, the choice of P_1 is to the right of and/or above P (compare Figure 1 c) for the notation). In the remaining boundary strip, the HB functions of Example 1 have been taken.

Table 2 H_0^1-case: Condition numbers $\kappa((D_J^\Psi)^{-1}A_J^\Psi)$

J	stHB	BPX	3ptHB	2ptHB	$a=5/48$	$a=1/8$	$a=1/6$	$a=3/16$
1	4.56	2.87	4.96	6.60	3.45	3.37	3.39	3.45
2	10.59	5.31	8.81	19.83	6.51	5.51	5.28	5.31
3	19.53	7.06	11.60	38.45	10.48	8.26	6.66	6.57
4	31.85	8.27	13.56	53.03	14.20	10.68	8.10	7.81
5	47.14	9.22	15.22	63.36	17.41	12.76	9.17	8.72
6	65.38	9.99	16.44	71.39	20.33	14.52	10.05	9.51
7	86.15	10.64	17.25	77.54	22.76	15.87	10.81	10.17

Table 3 L_2-case: Condition numbers $\kappa((D_J^\Psi)^{-1}A_J^\Psi)$

J	stHB	BPX	3ptHB	2ptHB	$a=5/48$	$a=1/8$	$a=1/6$	$a=3/16$
1	22	3.94	2.86	7.2	6.3	5.63	4.73	4.50
2	111	8.24	4.02	22.1	16.2	11.74	7.16	7.34
3	543	12.60	4.70	43.7	33.9	20.56	10.08	9.44
4	2565	16.75	5.10	72.5	60.3	30.73	12.26	11.37
5	11852	20.75	5.41	138.2	100.5	42.86	14.30	13.00
6	–	24.68	5.66	188.2	160.1	57.07	16.13	14.39

We finish with two observations. First, for all examples included in Tables 2 and 3 (standard HB, BPX, 3-point HB, 2-point HB, and coarse-grid stabilized HB for $a = 5/48$ (from [VW97b]), $a = 1/8$, $a = 1/6$ (from [CDP96]), and $a = 3/16$ (see [CS93])), the arithmetical costs per pcg-iteration would be almost the same if one neglects the overhead for computing masks. The pcg-step for the 5-point examples (Example 5) is asymptotically more expensive than the pcg-step only by a factor 1.2 for the cheapest method (standard HB). Similar considerations can be found in [Ste97b].

Secondly, when experimenting we found (for the first time) some HB-proposals which give for the H_0^1-case the same condition number behavior as in the BPX-method. This was interesting to us because so far all numerical evidence (also with other, wavelet based additive preconditioners) showed the superiority of the simple, frame-based BPX-algorithm by a factor of about 2, at least. Compare also the experiments in [VW97b] with different variants of coarse-grid stabilized HB-methods for $d = 2$ and $d = 3$. On the other hand, some cheap HB-proposals such as the 2-point HB-system which was mentioned in connection with Example 4, did not fulfill our expectations.

Further theoretical work and numerical testing is planned, in particular, for more general partitions, $d = 3$, and including the multiplicative (i.e., multigrid V-cycle) versions of the considered HB-methods. Another aspect which we wish to take up in the future is the design of HB-systems with small mask size and sufficiently many moment conditions for applications to integral equations where one wishes to cover the Sobolev spaces with $s = 0$ and $s = \pm 1/2$. In the shift-invariant case, some simple proposals based on P0-elements have been discussed in [LO97a]. The methods of [LO96, LO97a] also allow similar investigations for C^1-elements.

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