Second Meeting on
Quaternionic Structures
in Mathematics and Physics
Roma, 6-10 September 1999

THEOREMS OF EXISTENCE OF LOCAL AND GLOBAL SOLUTIONS OF PDEs IN THE CATEGORY OF NONCOMMUTATIVE QUATERNIONIC MANIFOLDS

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ABSTRACT - In this paper we apply our recent geometric theory of noncommutative (quantum) manifolds and noncommutative (quantum) PDEs [7,8,12] to the category of quantum quaternionic manifolds. These are manifolds modelled on spaces built starting from quaternionic algebras. For PDEs considered in such category we determine theorems of existence of local and global quantum quaternionic solutions. We shaw also that such a category of quantum quaternionic manifolds properly contains that of manifolds with (almost) quaternionic structure. So our theorems of existence of quantum quaternionic manifolds for PDEs produce a cascade of new solutions with nontrivial topology.

1 - QUANTUM MANIFOLDS AND QUANTUM PDEs

In order to give a geometrical model for quantum physics we introduced in some recents works [7,8,12] a new category of noncommutative manifolds, (**quantum manifolds**), where the "first brick" used to build them is a suitable structured noncommutative Frèchet algebra, (**quantum algebra**). The aim of this paper is to show that the category of quantum manifolds contains subcategories of great interes whose noncommutative manifolds have the quaternionic algebra as a fundamental structure element. (We call such manifolds **noncommutative quaternionic manifolds**). Then for PDEs built in such subcategories we shall apply our geometric theory of QPDEs and obtain theorems of existence of local and global solutions for noncommutative quaternionic PDEs.

Set $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$. Let us recall some fundamental definitions and results on quantum manifolds as given by us in refs.[7,8,12]. A **quantum algebra** is a triplet (A, ϵ, c), where: (i) A is a metrizable, complete, Hausdorff, locally convex topological $\mathbf{K}$-vector space, that is also a ring with unit; (ii) $\epsilon : \mathbf{K} \to A_0 \subset A$ is a ring homomorphism, where A_0 is the centre of A; (iii) $c : A \to \mathbf{K}$ is a $\mathbf{K}$-linear morphism, with $c(e) = 1$, e =unit of A. For any $a \in A$ we call $a_C \equiv c(a) \in \mathbf{K}$ the **classic limit** of a; (iv) A is an associative $\mathbf{K}$-algebra. A **quantum vector space** of dimension $(m_1, \dots, m_s) \in \mathbf{N}^s$, built on the quantum algebra $A \equiv A_1 \times \dots \times A_s$, is a locally convex topological $\mathbf{K}$-vector space E isomorphic to $A_1^{m_1} \times \dots A_s^{m_s}$. A **quantum manifold** of dimension $(m_1, \dots, m_s)$ over a quantum algebra $A \equiv A_1 \times \dots \times A_s$ of class Q^k_w, $0 \le k \le \infty, \omega$, is a locally convex manifold M modelled on E and with a Q^k_w-atlas of local coordinate mappings, i.e., the transiction functions $f : U \subset E \to U' \subset E$ define a pseudogroup of local Q^k_w-homeomorphisms on E, where Q^k_w means C^k_w, i.e., weak differentiability [5], and derivatives A_0-linaires. So for each open coordinate set $U \subset M$ we have a set of $m_1 + \dots + m_s$ coordinate functions $x^A : U \to A$, (**quantum coordinates**). The **tangent space** T_pM at $p \in M$, is the vector space of the equivalence classes $v \equiv [f]$ of C^1_w (or equivalently C^1) curves $f : I \to M$, $I \equiv$ open neighborhood of $0 \in \mathbf{R}$, $f(0) = p$; two curves f, f'

are equivalent if for each (equivalently, for some) coordinate system μ around p the functions $\mu \circ f$, $\mu \circ f' : I \to A_1^{m_1} \times \ldots \times A_s^{m_s}$ have the same derivative at $0 \in \mathbf{R}$. Then, derived tangent spaces associated to a quantum manifold M can be naturaly defined. (For details see refs.[7,8,12].) We say that a quantum manifold of dimension $(m_1, \ldots, m_s)$ is **classic regular** if it admits a projection $c : M \to M_C$ on a n-dimensional manifold M_C. We will call M_C the **classic limit** of M and in order to emphasize this structure we say that the dimension of M is $(n \downarrow m_1, \ldots, m_s)$. A **quantum PDE** (QPDE) of order k on the fibre bundle $\pi : W \to M$, defined in the category of quantum manifolds, is a subfibrebundle $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ of the jet-quantum derivative space $J\hat{\mathcal{D}}^k(W)$ over M. ($J\hat{\mathcal{D}}^k(W)$ is, in the category of quantum manifolds, the corresponding of the jet-derivative space for usual manifolds. For more details see refs.[7,8,12].) In refs.[8,12] we have formulated also a geometric theory for quantum PDEs that generalizes the theory of PDEs for usual manifolds. In particular in the following we shall emphasize some important definitions and results about. A QPDE $\hat{E}_k$ is **quantum regular** if the r-quantum prolongations $\hat{E}_{k+r} \equiv J\hat{\mathcal{D}}^r(\hat{E}_k) \cap J\hat{\mathcal{D}}^{k+r}(W)$ are subbundles of $\pi_{k+r,k+r-1} : J\hat{\mathcal{D}}^{k+r}(W) \to J\hat{\mathcal{D}}^{k+r-1}(W)$, $\forall r \geq 0$. Furthermore, we say that $\hat{E}_k$ is **formally quantumintegrable** if $\hat{E}_k$ is quantum regular and if the mappings $\hat{E}_{k+r+1} \to \hat{E}_{k+r}$, $\forall r \geq 0$, and $\pi_{k,0} : \hat{E}_k \to W$ are surjective. In the following we shall consider QPDEs on a fiber bundle $\pi : W \to M$, where M is a quantum manifold of dimension m on the quantum algebra A and W is a quantum manifold of dimension (m, s) on the quantum algebra $B \equiv A \times E$, where E is also an A_0-algebra. The **quantum symbol** $\dot{g}_{k+r}$ of $\hat{E}_{k+r}$ is a family of A_0-modules over $\hat{E}_k$ characterized by means of the following short exact sequence of A_0-modules: $0 \to \pi^*_{k+r}\dot{g}_{k+r} \to vT\hat{E}_{k+r} \to \pi^*_{k+r,k+r-1}vT\hat{E}_{k+r-1}$. Then one has the following complex of A_0-modules over $\hat{E}_k$ (**δ-quantum complex**):

$$0 \to \dot{g}_m \xrightarrow{\delta} TMO\dot{g}_{m-1} \xrightarrow{\delta} \dot{\Lambda}_0^2 MO\dot{g}_{m-2} \xrightarrow{\delta} \ldots \xrightarrow{\delta} \dot{\Lambda}_0^{m-k} MO\dot{g}_k \xrightarrow{\delta} \delta(\dot{\Lambda}_0^{m-k} MO\dot{g}_k) \to 0$$

where $\dot{\Lambda}_0^s M$ is the skewsymmetric subbundle of $\dot{T}_0^r M \equiv TM \otimes_{A_0} \cdots r \cdots \otimes_{A_0} TM$. We call **Spencer quantumcohomology** of $\hat{E}_k$ the homology of such complex. We denote by $\{H_q^{m-j,j}\}_{q \in \hat{E}_k}$ the homology at $(\dot{\Lambda}_0^j MO\dot{g}_{m-j})_q$. We say that $\hat{E}_k$ is **r-quantumacyclic** if $H_q^{m,j} = 0$, $m \geq k$, $0 \leq j \leq r$, $\forall q \in \hat{E}_k$. We say that $\hat{E}_k$ is **quantuminvolutive** if $H_q^{m,j} = 0$, $m \geq k$, $j \geq 0$. We say that $\hat{E}_k$ is **δ-regular** if there exists an integer $\kappa_0 \geq \kappa$, such that $\dot{g}_{\kappa_0}$ is quantum involutive or 2-quantumacyclic.

THEOREM 1.1 - (δ-POINCARÉ LEMMA FOR QUANTUM PDEs)[12]. *Let $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ be a quantum regular QPDE. If A_0 is a Noetherian* **K***-algebra, then $\hat{E}_k$ is a δ-regular QPDE.*

THEOREM 1.2 - (CRITERION OF FORMAL QUANTUM INTEGRABILITY)[12]. *Let $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ be a quantum regular, δ-regular QPDE. Then if $\dot{g}_{k+r+1}$ is a bundle of A_0-modules over $\hat{E}_k$, and $\hat{E}_{k+r+1} \to \hat{E}_{k+r}$ is surjective for $0 \leq r \leq m$, then $\hat{E}_k$ is formally quantumintegrable.*

An **initial condition** for QPDE $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ is a point $q \in \hat{E}_k$. A **solution** of $\hat{E}_k$ passing for the initial condition q is a m-dimensional quantum manifold $N \subset \hat{E}_k$ such that $q \in N$ and such that N can be represented in a neighboroud of any of its points $q' \in N$, except for a nonwhere dense subset $\Sigma(N) \subset N$ of dimension $\leq m-1$, as image of the k-derivative $D^k s$ of some Q^k_w-section s of $\pi : W \to M$. We call $\Sigma(N)$ the set of **singular points** (of Thom-Bordman type) of N. If $\Sigma(N) \neq \emptyset$ we say that N is a **regular solution** of $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$. Furthermore, let us denote by $\hat{J}^k_m(W)$ the k-jet of m-dimensional quantum manifolds (over A) contained into W. One has the natural embeddings $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W) \subset \hat{J}^k_m(W)$. Then, with respect to the embedding $\hat{E}_k \subset \hat{J}^k_m(W)$ we can consider solutions of $\hat{E}_k$ as m-dimensional (over A) quantum manifolds $V \subset \hat{E}_k$ such that V can be representable in the neighborhood of any of its points $q' \in V$, except for a nonwhere dense subset $\Sigma(V) \subset V$, of dimension $\leq m-1$, as $N^{(k)}$, where $N^{(k)}$ is the k-quantum prolongation of a m-dimensional (over A) quantum manifold $N \subset W$. In the case that $\Sigma(V) = \emptyset$, we say that V is a **regular solution** of $\hat{E}_k \subset \hat{J}^k_m(W)$. Of course, solutions V of $\hat{E}_k \subset \hat{J}^k_m(W)$, even if regular ones, are not, in general diffeomorphic to their projections $\pi_k(V) \subset M$, hence are not representable by means of sections of $\pi : W \to M$.

Therefore, above theorem allows us to obtain existence theorems of local solutions. Now, in order to study the structure of global solutions it is necessary to consider the integral bordism groups of QPDEs. In refs.[7,8,12] we extended to QPDEs our previous results on the determination of integral bordism groups of PDEs [6-11,13]. Let us denote by $\Omega_p^{\hat{E}_k}$, $0 \le p \le m-1$, the integral bordism groups of a QPDE $\hat{E}_k \subset \hat{J}^k_m(W)$ for closed integral quantum submanifolds of dimension p, over a quantum algebra A, of $\hat{E}_k$. The structure of smooth global solutions of $\hat{E}_k$ are described by the integral bordism group $\Omega_{m-1}^{\hat{E}_\infty}$ corresponding to the ∞-quantumprolongation $\hat{E}_\infty$ of $\hat{E}_k$. Beside the groups $\Omega_p^{\hat{E}_k}$, $0 \le p \le m-1$, we can also introduce the **integral singular p-bordism groups** ${}^B\Omega_{p,s}^{\hat{E}_k}$, $0 \le p \le m-1$, where B is a quantum algebra. Then one can prove [12] that ${}^B\Omega_{p,s}^{\hat{E}_k} \cong \Omega_{p,s}^{\hat{E}_k} \otimes_{\mathbf{K}} B$, where $\Omega_{p,s}^{\hat{E}_k}$ are the integral singular bordism groups for $B = \mathbf{K}$. Furthermore, the equivalence classes in the groups ${}^B\Omega_{p,s}^{\hat{E}_k}$ are characterized by means of suitable characteristic numbers (belonging to B), similarly to what happens for PDEs [7-11]. In ref.[12] we given also a general method to explicitly calculate such bordism groups for quantum PDEs.

2 - THE CATEGORY OF QUANTUM QUATERNIONIC MANIFOLDS

Let us first recall some fundamental definitions and results on quaternionic algebra [2]. Let $\mathbf{K} = \mathbf{R}, \mathbf{C}$. Let R be a commutative ring. Let $\alpha, \beta \in R$, (e_1, e_2) the canonical basis of the R-module R^2. We say **quadratic algebra of type** (α, β) over R the R-module R^2 endowed with the structure of algebra defined by means of the following multiplication: $(\clubsuit)$ $e_1^2 = e_1, e_1e_2 = e_2e_1 = e_2, e_2^2 = \alpha e_1 + \beta e_2$. Any R-algebra E, isomorphic to a quadratic algebra is called a quadratic algebra too. (Any R-algebra E that admits a basis of two elements (one being the identity) is a quadratic algebra.) Then the basis is called a **basis of type** (α, β). A quadratic algebra E is associative and commutative. Let E be a quadratic R-algebra, e its unit. Let $u \in E$ and $T(u)$ the trace of the endomorphism $x \mapsto ux$ of the free R-module E. Then the application $s : E \to E$, $s(u) = T(u).e - u$, is an endomorphism of the R-algebra E and one has $s^2(u) = u$, $\forall u \in E$. A **Cayley algebra** on R is a couple (E, s), where E is a R-algebra, with unit $e \in E$, and s is a skewendomorphism of E such that: (a) $u + \bar{u} \in Re$, (b) $u.\bar{u} \in Re$, with $\bar{u} \equiv s(u)$, $\forall u \in E$. s is called **conjugation** of the Cayley algebra E and $s(u) \equiv \bar{u}$ is the **conjugated** of u. From the condition (a) it follows that $u\bar{u} = \bar{u}u$. One defines **Cayley trace** and **Cayley norm** respectively the following maps: $T : E \to Re$, $u \mapsto T(u) = u + \bar{u}$; $N : E \to Re$, $u \mapsto N(u) = u.\bar{u}$. One has the following properties: (1) $\bar{e} = e$; (2) $s(u + s(u)) = u + s(u) \Rightarrow s(u) + s^2(u) = u + s(u) \Rightarrow s^2(u) = u \Rightarrow s^2 = id_E$; (3) $T(\bar{u}) = T(u)$; (4) $N(\bar{u}) = N(u)$; (5) $(u - u)(u - \bar{u}) = 0 \Rightarrow u^2 - T(u).u + N(u) = 0$; (6) Let E be a R-algebra and let s, s' be skewendomorphisms of E such that (E, s) and (E, s') are Cayley algebras. If E admits a basis containing E, one has $s = s'$; (7) $\overline{u + v} = \bar{u} + \bar{v}$; $\overline{\alpha u} = \alpha\bar{u}$; $\overline{u.v} = \bar{v}.\bar{u}$, $\forall \alpha \in R$, $\forall u, v \in E$; (8) $T(e) = 2e$; $N(e) = e$; (9) $T(uv) = T(vu)$; (10) $T(v\bar{u}) = T(u\bar{v}) = N(u + v) - N(u) - N(v) = T(u)T(v) - T(uv)$; (11) $N(\alpha u) = \alpha^2 N(u)$; (12) $(T(u))^2 - T(u^2) = 2N(u)$; (13) T is a linear form on E and N is a quadratic form on E.

EXAMPLE 2.1 - (CAYLEY EXTENSION OF A CAYLEY ALGEBRA (E,s) DEFINED BY AN ELEMENT $\gamma \in R$). 1) Let (E,s) be a Cayley algebra and let $\gamma \in R$. Let F be the R-algebra with underlying module $E \times E$ and with multiplication $(x,y)(x',y') = (xx' + \gamma\bar{y}'y, y\bar{x}' + y'x)$. Then $(e,0)$ is the unit of F and $E \times \{0\}$ is a subalgebra of F isomorphic to E that can be identified with E. Let t be the permutation of F defined by $t(x,y) = (\bar{x}, -y)$, $\forall x,y \in E$. Then the couple (F,t) is a Cayley algebra over R. Set $j = (0,e)$. So we can write $(x,y) = (x,0)(e,0) + (0,y)(0,e) = xe + yj$. One has $yj = j\bar{y}$, $x(yj) = (yx)j - (xj)y = (x\bar{y})j$, $(xj)(yj) = \bar{y}xe$, $j^2 = e$. Furthermore, one has $T_F(xe + yj) = T(x)$, $N_F(xe + yj) = N(x) - \gamma N(y)$. F is associative iff E is associative and

commutative.
2) As a particular case one has: If $E=R$ (hence $s=id_R$), the Cayley extension of $(R.id_R)$ by an element $\gamma\in R$ is a **quadratic R-algebra** with basis (e,j) with $j^2=\gamma e$.
3) Another particular case is the following. Let E be a quadratic algebra of type (α,β) such that the underlying module is R^2 with multiplication rule given by means of (♣) for the canonical basis. Let the conjugation s be the conjugation in E. Then for any $\gamma\in R$, the Cayley extension F of (E,s) by means of γ is called **quaternionic algebra of type** (α,β,γ). (This is an associative algebra.) The underlying module is R^4. Let us denote by $(0,i,j,k)$ the canonical basis of R^4. Then the corresponding multiplication rule is given by the following table. (In the same table it are also reported the trace and norm formulas.)

TAB.2.2 - Multiplication table and trace and norm formulas.

	i	j	k	trace and norm formulas
i	$\alpha e+\beta i$	k	$\alpha j+\beta\kappa$	$T_F(u)=2\rho+\beta\xi$
j	$\beta j-\kappa$	γe	$\beta\gamma-\gamma i$	$N_F(u)=\rho^2+\beta\rho\xi-\alpha\xi^2-\gamma(\eta^2+\beta\eta\zeta-\alpha\zeta^2)$
k	$-\alpha j$	γi	$-\alpha\gamma e$	$N_F(uv)=N_F(u)N_F(v)$

$u=\rho e+\xi i+\eta j+\zeta\kappa,\quad \rho,\xi,\eta,\zeta\in\mathbf{K};\quad \bar{u}=(\rho+\beta\xi)e-\xi i-\eta j-\zeta\kappa.$

4) An A-algebra isomorphic to a quaternionic algebra is called a quaternionic algebra; if a basis of such an algebra has the multiplication to be (0) then it is called of type (α,β,γ).
5) If $\beta=0$ we say that the quaternionic algebra is of type (α,γ). One has:

TAB.2.3 - Multiplication table and trace and norm formulas.

	i	j	k	trace and norm formulas
i	αe	k	αj	$T_F(u)=2\rho$
j	$-\kappa$	γe	$-\gamma i$	$N_F(u)=\rho^2-\alpha\xi^2-\gamma\eta^2+\alpha\gamma\zeta^2$
k	$-\alpha j$	γi	$-\alpha\gamma e$	

$u=\rho e+\xi i+\eta j+\zeta\kappa,\quad \rho,\xi,\eta,\zeta\in\mathbf{K};\quad \bar{u}=\rho e-\xi i-\eta j-\zeta\kappa.$

(Of course as $-1\neq 1$ this algebra is not commutative.)
☐ In particular if $A=\mathbf{K}=\mathbf{R}$, $\alpha=\gamma=-1$, $\beta=0$, F is called the **Hamiltonian quaternionic algebra** and is denoted by $\mathbf{H}$. In this case $N(u)\neq 0$, hence u admits an inverse $u^{-1}=N(u)^{-1}\bar{u}$ in $\mathbf{H}$, therefore $\mathbf{H}$ is a noncommutative corp. Any finte $\mathbf{R}$-algebra that is also a corp (noncommutative) is isomorphic to $\mathbf{H}$. Any quaternion $q\in\mathbf{H}$ can be represented by $q=\rho e+\xi i+\eta j+\zeta k$, where i,j,k are linearly independent symbols that satisfy the following multiplication rules: $ij=k=-ji$, $jk=i=-kj$, $ki=j=-ik$, $i^2=j^2=k^2=-1$. One has the following $\mathbf{R}$-algebras homomorphism: $A:\mathbf{H}\to M(2;\mathbf{C})$, $q\mapsto\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi\end{pmatrix}$, where i is the imaginary unity of $\mathbf{C}$. The matrices $\sigma_x\equiv -iA(k)$, $\sigma_y\equiv -iA(j)$, $\sigma_z\equiv -iA(i)$, where $A(i)=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$, $A(j)=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$, $A(k)=\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}$, are called **Pauli matrices** and satisfy $\sigma_x^2=\sigma_y^2=\sigma_z^2=1$, $\sigma_x\sigma_y=-\sigma_y=\sigma_x=i\sigma_z$. The set $\mathbf{H}_1\equiv N_{\mathbf{H}}^{-1}(1)$ of quaternions of norm 1 is isomorphic to the group $SU(2)$: $\mathbf{H}_1\cong SU(2)$. The **n-dimensional quaternionic space** H^n has a canonical basis $\{e_k\}_{1\leq k\leq n}$, $e_k\in\mathbf{H}$, and any $v\in\mathbf{H}^n$ can be represented in the form $v=\sum_{1\leq k\leq n}q^k e_k$, $q^k\in\mathbf{H}$, ($q^k\equiv$**quaternionic components**). As any quaternionic number q admits the following representation $q=x+yj=x+j\bar{y}$, with $x=\rho e+\xi i$, $y=\eta+\zeta i$, where x and y can be considered complex

numbers, then one has the following isomoprphism $\mathbf{H}^n\cong\mathbf{C}^{2n}$, $(q^k)\mapsto(x^k,y^k)$, where $\mathbf{C}^{2n}$ has the following basis $(e_1,\dots,e_n,je_1,\dots,je_n)$. We write $\dim_{\mathbf{H}}\mathbf{H}^n=n$, $\dim_{\mathbf{C}}\mathbf{H}^n=2n$. By using different quaternionic bases in $\mathbf{H}^n$ one has that the quaternionic components of any vector $v\in\mathbf{H}^n$ transform by means of the following rule $q'^k=\sum_{1\le i\le n}q^i\lambda_i^k$, $(\lambda_i^k)\in GL(n,\mathbf{H})$. Furthermore, the corresponding complex components transform in the following way:

$$\{x'^h=x^l a_l^h-y^l\bar{b}_l^h,\quad y'^h=x^l b_l^h+y^l\bar{a}_l^h\},\quad \lambda_l^h=a_l^h+b_l^h j.$$

Then one has a group-homomorphism $GL(n,\mathbf{H})\xrightarrow{c}GL(2n,\mathbf{C})$, such that if $\Lambda=A+Bj\in GL(n,\mathbf{H})$, then $c(\Lambda)=\begin{pmatrix}A & B\\ -\bar{B} & \bar{A}\end{pmatrix}$. On $\mathbf{H}^n$ there is a canonical quadratic form $|v|^2=\sum_{1\le k\le n}|q^k|^2=\sum_{1\le k\le n}q^k\bar{q}^k=\sum_{1\le k\le n}(|x^k|^2+|y^k|^2)\in\mathbf{R}$, where $v=q^k e_k$, $q^k\in\mathbf{H}$, $q^k=x^k+y^k j$, $x^k,y^k\in\mathbf{C}$. So such a quadratic form coincides with the ordinary norm of the vector space $\mathbf{C}^{2n}$. Furthermore one has on $\mathbf{H}^n$ the following form $<v_1,v_2>_{\mathbf{H}}=\sum_{1\le k\le n}q_1^k\bar{q}_2^k\in\mathbf{H}$, $v_i=\sum_{1\le k\le n}q_i^k e_k$, $i=1,2$. The quaternionic transformations of $\mathbf{H}^n$, that conserve above form form a group $Sp(n)\subset GL(n,\mathbf{H})$. As we can write $<v_1,v_2>_{\mathbf{H}}$ in the following way:

$$<v_1,v_2>_{\mathbf{H}}=\begin{cases}\sum_{1\le k\le n}(x_1^k\bar{x}_2^k+y_1^k\bar{y}_2^k)=<v_1,v_2>_{\mathbf{C}}= & \text{hermitiam form in } \mathbf{C}^{2n}\\ \sum_{1\le k\le n}(y_1^k x_2^k-x_1^k y_2^k)j=\sigma(v_1,v_2)_{\mathbf{C}}= & \text{skewsymmatric form in } \mathbf{C}^{2n}\end{cases}$$

we see that $\Lambda\in Sp(n)$ preserves the hermitian form and the skewsymmetric form. Therefore $c(Sp(n))\subset U(2n)$ and it is formed by the unitary transformations of $\mathbf{C}^{2n}$ that preserves the antisymmatric form $\sigma(v_1,v_2)_{\mathbf{C}}$.

EXAMPLE 2.2 - $Sp(1)\cong SU(2)\subset U(2)$. So all the transformations contained in $c(Sp(1))$ are unimodular. ■

DEFINITION 2.1 - *Let B be a quantum algebra. We define* **Cayley B-quantum algebra** *any quantum algebra C that is obtained from a Cayley* **K***-algebra A, by "extending the scalars" from* **K** *to B, i.e., $C\cong B\otimes_{\mathbf{K}}A$.*

EXAMPLE 2.3 - The noncommutative B-quaternionic algebra $B\otimes_{\mathbf{K}}\mathbf{H}$ is a Cayley B-quantum algebra over $\mathbf{K}=\mathbf{R}$ endowed with the natural topology of Banach space and considering the **K**-linear morphism $c=c_B\otimes\frac{1}{2}T:B\otimes_{\mathbf{K}}\mathbf{H}\to\mathbf{K}$, where T is the trace of **H**. (Another possibility is to take $c=c_B\otimes N$, where N is the norm of **H**. In this last case, whether B is an augmented quantum algebra, then $B\otimes_{\mathbf{K}}\mathbf{H}$ becomes an augmented quantum algebra too.) ■

DEFINITION 2.2 - *A* **quantum B-quaternionic manifold** *of dimension n and class Q^k_w, $0\le k\le\infty,\omega$, is a quantum manifold M of dimension n and class Q^k_w over the B-quantum algebra $C\equiv B\otimes_{\mathbf{K}}\mathbf{H}$. Then the quantum coordinates in an open coordinate subset $U\subset M$ are called* **B-quaternionic coordinates**, *$\{q^k\}_{1\le k\le n}$, $q^k:U\to C$.*[1]

DEFINITION 2.3 - *The* **category $\mathcal{C}^B_{\mathbf{H}}$, of quantum B-quaternionic manifolds** *of class Q^k_w, is defined by considering as morphisms maps of class Q^k_w, between quantum B-quaternionic manifolds.*

EXAMPLE 2.4 - **Quantum quaternionic Möbius strip**. Let us denote $I\equiv[-\pi,\pi]\subset\mathbf{R}$, $N\equiv I\times\mathbf{H}$. Let us introduce the following equivalence relation in N: $(x,y)\sim(x,y)$ if $x\ne-\pi,\pi$, $(-\pi,-y)\sim(\pi,y)$. Then $N/\sim\equiv M$ is called noncommutative quaternionic Möbius strip. One has a natural projection $p:M\to S^1$, given by $p([x,y])=x\in S^1$ if $x\ne-\pi,\pi$, and $p([\pi,y])=*\in S^1$, where $*$ is the point of $S^1\equiv I/\{-\pi,\pi\}$, corresponding to $\{-\pi,\pi\}$. One can recover M with two open sets:

$$\{\Omega_1\equiv p^{-1}(U_1),\quad U_1\equiv]-\pi,\pi[;\quad \Omega_2\equiv p^{-1}(U_2),\quad U_2\equiv S^1\setminus\{0\}\}.$$

We put quaternionic coordinates on Ω_i, $i=1,2$, in the following way. On Ω_1, $\{x^1[x,y]=p[x,y]=x\in\mathbf{R},x^2[x,y]=y\in\mathbf{H}\}$. On Ω_2, if $x\ne-\pi,\pi$, $\bar{x}^1[x,y]=-\pi+x\in\mathbf{R}$, $\bar{x}^1[-x,y]=\pi-x\in\mathbf{R}$,

[1] As a particular case we can take $B=\mathbf{K}$. In this case $C=\mathbf{H}$ and we call such quantum $\mathbf{K}$-quaternionic manifolds simply **quantum quaternionic manifolds**.

$\bar{x}^2[\pm x,y]=y\in\mathbf{H}$; $\bar{x}^1[-\pi,-y]=\bar{x}^1[\pi,y]=0$, $\bar{x}^2[-\pi,-y]=\bar{x}^2[\pi,y]=|y|\in c^{-1}(\mathbf{R}^+)\subset\mathbf{H}$, with $c=\frac{1}{2}T$. The change of coordinates is given by:

$$\forall q\in\Omega_1\cap\Omega_2,\quad (p(q)\in S^1\setminus\{*,0\}\equiv U_+\dot{\bigcup}U_-,\quad \left\{\begin{array}{l}\bar{x}^1|_{U_-}=\pi-x^1\in\mathbf{R}\\ \bar{x}^1|_{U_+}=-\pi+x^1\in\mathbf{R}\\ \bar{x}^2=x^2\in\mathbf{H}\end{array}\right\}.$$

The structure group, i.e. the group of the jacobian matrix, is isomorphic to $\mathbf{Z}_2$. In fact one has:

$$\left\{(\partial x_j.\bar{x}^k)=\begin{pmatrix}(\partial x_1.\bar{x}^1) & (\partial x_1.\bar{x}^2)\\ (\partial x_2.\bar{x}^1) & (\partial x_2.\bar{x}^2)\end{pmatrix}=\begin{pmatrix}\pm 1 & 0\\ 0 & 1\end{pmatrix}\subset GL(2;\mathbf{R})\right\}.$$

Therefore, M is a quantum quaternionic manifold modelled on $\mathbf{R}\times\mathbf{H}\subset\mathbf{H}^2$, hence $\dim_{\mathbf{H}}M=2$. Furthermore one has the canonical projection $M\to S^1$, therefore M is a regular quantum manifold of dimension $(1\downarrow 2)$. Finally remark that as M is not covered by a global chart, it is a non trivial example of quantum quaternionic manifold. Of course this can be also seen by means of homological arguments. In fact one has $H_1(M;\mathbf{R})\cong H_1(M_C;\mathbf{R})\cong H_1(S^1;\mathbf{R})\cong\mathbf{R}$. Therefore, M is not homotopy equivalent to $\mathbf{R}^5$, as $H_1(\mathbf{R}^5;\mathbf{R})=0$. ■

EXAMPLE 2.5 - **Quaternionic manifolds** [3,15,16]. The category $\mathcal{C}_{\mathbf{H}}$ of quaternionic manifolds is a subcategory of $\mathcal{C}_{\mathbf{H}}^{B\equiv\mathbf{R}}$, where the morphisms are quaternionic affine maps [15]. Therefore any of such morphisms $f\in Hom_{\mathcal{C}_{\mathbf{H}}}(M,N)$, where $\dim M=4m$, $\dim N=4n$, are locally represented by formulas like the following: $f^k=A^k_jq^j+r^k$, $A^k_j,q^j,r^k\in\mathbf{H}$, $1\le k\le n$, $1\le j\le m$. A^k_j identify $m\times n$ matrices with entries in $\mathbf{H}$, or equivalently, real matrices of the form

$$(A^k_j)=\begin{pmatrix}\hat{A}^1_1 & \dots & \hat{A}^m_1\\ \dots & & \dots\\ \hat{A}^1_n & \dots & \hat{A}^m_n\end{pmatrix},\quad \hat{A}^j_i=\begin{pmatrix}a & b & c & d\\ -b & a & d & -c\\ -c & -d & a & b\\ -d & c & b & a\end{pmatrix},\quad a,b,c,d\in\mathbf{R}.$$

The set of such matrices is denoted by $M(n,m;\mathbf{H})$. The structure group of a $4n$-dimensional quaternionic manifold is $GL(n;\mathbf{H})$, (that is the subset of $M(n,m;\mathbf{H})$ of invertible matrices). Therefore, quaternionic manifolds are quantum quaternionic manifolds where the local maps $f:U\subset\mathbf{H}^n\to\bar{U}\subset\mathbf{H}^n$, change of coordinates, are $\mathbf{H}$-linear. Hence $Df(p)\in\mathbf{H}^{n^2}$, $\forall p\in U$. In fact, one has the following commutative diagram:

$$\begin{array}{ccccc}
Hom_{\mathbf{H}}(\mathbf{H}^n;\mathbf{H}^n) & \hookrightarrow & Hom_{\mathbf{R}}(\mathbf{H}^n;\mathbf{H}^n) & \cong & Hom_{\mathbf{R}}(\mathbf{H};\mathbf{H})^{n^2}\\
\wr\| & & & & \wr\|\\
Hom_{\mathbf{H}}(\mathbf{H};\mathbf{H})^{n^2} & & & & Hom_{\mathbf{R}}(\mathbf{R}^4;\mathbf{R}^4)^{n^2}\\
\wr\| & & & & \wr\|\\
\mathbf{H}^{n^2} & \cong & \mathbf{R}^{4n^2} & \hookrightarrow & \mathbf{R}^{4^2n^2}
\end{array}$$

On the other hand the tangent space T_pM has a natural structure of $\mathbf{H}$-module iff M is an affine manifold. (As in this case the action of $\mathbf{H}$ on $T_pM\cong\mathbf{H}^n$ does not depend on the coordinates used to obtain the identification of T_pM with $\mathbf{H}^n$.) Hence the category $\mathcal{C}_{\mathbf{H}}$ is the subcategory of $\mathcal{C}_{\mathbf{H}}^{\mathbf{R}}$ of affine quantum quaternionic manifolds. A trivial example of quaternionic manifold is $\mathbf{R}^{4n}\cong\mathbf{H}^n$. If $\{x^i,y^i,u^i,v^i\}_{1\le i\le n}$ are real coordinates on $\mathbf{R}^{4n}$, then the almost quaternionic structure given by

$$\left\{\begin{array}{llll}J(\partial x_i)=\partial y_i, & J(\partial y_i)=-\partial x_i, & J(\partial u_i)=-\partial v_i, & J(\partial v_i)=\partial u_i\\ K(\partial x_i)=\partial u_i, & K(\partial y_i)=\partial v_i, & K(\partial u_i)=-\partial x_i, & J(\partial v_i)=-\partial y_i\end{array}\right\}$$

is called the **standard right quaternionic structure** on $\mathbf{R}^{4n}$.[2] A non trivial example of quaternionic manifold is $\mathbf{R}^{4n}$ with the standard quaternionic structure quotiented by a discrete translation group that gives a torus. ∎

EXAMPLE 2.6 - **Almost quaternionic manifolds**. The category $\widetilde{\mathcal{C}_{\mathbf{H}}}$ of almost quaternionic manifolds is a subcategory of $\mathcal{C}_{\mathbf{H}}^{B\equiv\mathbf{R}}$, where the structure group of a $4n$-dimensional quantum quaternionic manifold is $GL(n;\mathbf{H})Sp(1)\subset GL(4n;\mathbf{R})$. The category $\widetilde{\mathcal{C}_{\mathbf{H}}}$ properly contains $\mathcal{C}_{\mathbf{H}}$. An example of almost quaternionic manifold, that is not contained into $\mathcal{C}_{\mathbf{H}}$, is the quaternionic projective space $\mathbf{H}P^1$. This cannot be a quaternionic manifold, since it does not admit a structure of complex manifold.∎

REMARK 2.1 - As the centre C_0 of $C\equiv B\otimes_{\mathbf{K}}\mathbf{H}$ is isomorphic to B_0, that is the centre of B, we get that, whether B_0 is Noetherian, one can apply above Theorem 1.1 and Theorem 1.2 for QPDEs, in order to state the formal quantum-integrability for quantum B-quaternionic PDEs. Note that in such a way we obtain as solutions submanifolds that have natural structures of quantum B-quaternionic manifolds. Then applying our theorems on the integral bordism groups for quantum PDEs [12], we can also calculate theorem of existence of global solutions for quantum B-quaternionic PDEs.

EXAMPLE 2.7 - **Quantum B-quaternionic heat equation**. Let us consider the fiber bundle $\pi: W\equiv C^3\to C^2\equiv M$ with coordinates $(t,x,u)\mapsto(t,x)$. The quantum B-quaternionic heat equation is the following QPDE: $\widehat{(HE)}_C\subset\hat{J}\mathcal{D}^2(W)\subset\hat{J}^2_2(W)$: $u_{xx}-u_t=0$. This is a formally (quantum)integrable QPDE. Hence, for $\widehat{(HE)}_C$ we have the existence of local solutions for any initial condition. This means that in the neighborhood of any point $q\in\widehat{(HE)}_C$ we can built an integral quantum B-quaternionic manifold of dimension 2 over C, $V\subset\widehat{(HE)}_C$, such that $V\cong\pi_2(V)\subset M$, where π_2 is the canonical projection $\pi_2:\hat{J}\mathcal{D}^2(W)\to M$. Then by using a Theorem 5.6 given in ref.[12] we have that the first integral bordism groups of $\widehat{(HE)}_C$ is: $\Omega_1^{\widehat{(HE)}_C}\cong H_1(W;\mathbf{K})\otimes_{\mathbf{K}}C\cong 0$. Hence we get that any admissible closed integral 1-dimensional quantum B-quaternionic manifold, $N\subset\widehat{(HE)}_C$ is the boundary of an integral 2-dimensional quantum B-quaternionic manifold V, $\partial V\subset N$, $V\subset\widehat{(HE)}_C$, such that V is diffeomorphic to its projection into W by means of the canonical projection $\pi_{2,0}:\hat{J}^2_2(W)\to W$. ∎

EXAMPLE 2.8 - **Quantum quaternionic heat equation**. As a particular case of above equation one can take $B\equiv\mathbf{R}$. Then one has:

$$\left\{\begin{array}{l}\pi: W\equiv\mathbf{H}^4\to\mathbf{H}^2\equiv M;\quad (t,x,u)\mapsto(t,x)\\ \widehat{(HE)}_{\mathbf{H}}\subset J\mathcal{D}^2(W)\subset J^2_2(W):\quad u_{xx}-u_t=0\end{array}\right\},\quad \Omega_1^{\widehat{(HE)}_{\mathbf{H}}}\cong H_1(W;\mathbf{R})\otimes_{\mathbf{R}}\mathbf{H}\cong 0.$$

We can see that the set $\mathcal{S}ol(\widehat{(HE)}_{\mathbf{H}})$ of solutions of $\widehat{(HE)}_{\mathbf{H}}$ contains also quaternionic manifolds, i.e., affine quantum quaternionic solutions. For example a torus [3] $X\subset\mathbf{H}^2\equiv M$ can be embedded into

[2] An **almost complex structure** on a C^∞ manifold M is a fiberwise endomorphism J of the tangent bundle TM such that $J^2=-1$. A **complex analytic map** between almost complex manifolds (X,J_1) and (Y,J_2) is a C^∞ map $\phi:X\to Y$, such that $T(\phi)\circ J_1=J_2\circ T(\phi)$. An **almost quaternionic structure** on a C^∞ manifold M is a pair of two almost complex structures J and K such that $JK+KJ=0$. A **quaternionic map** ϕ between two almost quaternionic manifolds (X,J_1,K_1) and (Y,J_2,K_2) is a map $\phi:X\to Y$ that is complex analytic from (X,J_1) to (Y,J_2) and from (X,K_1) to (Y,K_2). A **quaternionic manifold** is a C^∞ manifold M endowed with an atlas $\{\phi_i:U_i\to\mathbf{R}^{4n}\}$, for some n, such that $\phi_j\circ\phi_i^{-1}:\phi_i(U_i\cap U_j)\to\phi_j(U_i\cap U_j)$ is a quaternionic function with respect to the standard structure on $\mathbf{R}^{4n}$. (See also ref.[1].)

[3] Recall [15] that if (X,J,K) is a quaternionic manifold, then X with the complex structure $aJ+bK+c(JK)$, $a,b,c\in\mathbf{R}$, is an affine complex manifold, hence has zero rational Pontryagin classes. Furthermore, if X is compact has zero index and Euler characteristic. Moreover, if $\dim_{\mathbf{H}}X=1$ and, for some a,b,c, X is Kähler, then it is a torus.

$(\widehat{HE})_{\mathbf{H}}$ by means of the second holonomic prolongation of the zero section $u \equiv 0 : M \to W$. In fact, $(\widehat{HE})_{\mathbf{H}}$ is a linear equation. Therefore, $X^{(2)} \equiv D^2u(X)$ is a 1-dimensional smooth closed compact admissible integral manifold contained into $(\widehat{HE})_{\mathbf{H}}$, that is the boundary of a 2-dimensional integral admissible manifold contained into $(\widehat{HE})_{\mathbf{H}}$ too. This last is also a quaternionic manifold. Moreover, all the regular solutions of $(\widehat{HE})_{\mathbf{H}} \subset J\mathcal{D}^2(W)$ are quaternionic manifolds, as they are diffeomorphic to $\mathbf{H}^2$. However, no all the regular solutions of $(\widehat{HE})_{\mathbf{H}} \subset J^2_2(W)$ are necessarily quaternionic manifolds too. ■

We are ready now to state the main results of this paper.

THEOREM 2.3 - *Let B be a quantum algebra such that its centre B_0 is a Noetherian $\mathbf{R}$-algebra. Let $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ be a quantum regular QPDE in the category $\mathcal{C}^B_{\mathbf{H}}$, where $\pi : W \to M$ is a fibre bundle with $dim_C M = m$, $C \equiv B \otimes_{\mathbf{R}} \mathbf{H}$. If $\dot{g}_{k+r+1}$ is a bundle of C_0-modules over $\hat{E}_k$, and $\hat{E}_{k+r+1} \to \hat{E}_{k+r}$ is surjective for $0 \leq r \leq m$, then $\hat{E}_k$ is formally quantumintegrable. In such a case, and further assuming that W is p-connected, $p \in \{0, \dots, m-1\}$, then the integral bordism groups of $\hat{E}_k \subset \hat{J}^k_m(W)$ are given by:*

$$\Omega_p^{\hat{E}_k} \cong H_p(W; \mathbf{R}) \otimes_{\mathbf{R}} C, \quad 0 \leq p \leq m-1.$$

All the regular solutions of $\hat{E}_k \subset \hat{J}^k_m(W)$ are quantum B-quaternionic submanifolds of $\hat{E}_k$ of dimension m, over C, identified with m-dimensional quantum B-quaternionic submanifolds of W.

PROOF. It follows directly from above definitions and remarks by specializing Theorem 1.1, Theorem 1.2 and our results in ref.[12], about integral bordism groups in QPDEs, to the category $\mathcal{C}^B_{\mathbf{H}}$. □

COROLLARY 2.1 - *Let $\hat{E}_k \subset J\mathcal{D}^k(W)$ be a quantum regular QPDE in the category $\mathcal{C}^{\mathbf{R}}_{\mathbf{H}}$, (resp. $\widetilde{\mathcal{C}}_{\mathbf{H}}$), where $\pi : W \to M$ is a fibre bundle with $dim_{\mathbf{H}} M = m$. If $\dot{g}_{k+r+1}$ is a bundle of $\mathbf{R}$-modules over $\hat{E}_k$, and $\hat{E}_{k+r+1} \to \hat{E}_{k+r}$ is surjective for $0 \leq r \leq m$, then $\hat{E}_k$ is formally quantumintegrable. In such a case, further assuming that W is p-connected, $p \in \{0, \dots, m-1\}$, then the integral bordism groups of $\hat{E}_k \subset J^k_m(W)$ are given by:*

$$\Omega_p^{\hat{E}_k} \cong H_p(W; \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{H}, \quad 0 \leq p \leq m-1.$$

All the regular solutions of $\hat{E}_k \subset J^k_m(W)$ are quantum quaternionic, (resp. almost quaternionic), submanifolds of $\hat{E}_k$ of dimension m, identified with m-dimensional quantum quaternionic, (resp. almost quaternionic), submanifolds of W.

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[14] Regular solutions of noncommutative B-quaternionic PDEs are noncommutative B-quaternionic manifolds. So our approach fits also the well known question to find quaternionic submanifolds of a given one. (See e.g. the following reference: Marchiafava, S., Boll. Un. Mat. Ital. **7(5-B)**(1991), 417–447.)

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Acknowledgment. Work partially supported by grants MURST "Geometry of PDEs and Applications", MURST "Geometric Properties of Real and Complex Manifolds" and GNFM/INDAM.

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