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A Note on the Upper and Lower Henstock Integrals

Abstract

It is shown that the upper and lower Henstock integrals coincide with upper and lower Perron integrals, when the former exist.

1 Introduction

In [2], the notions of upper and lower Henstock integrals are defined using a set-valued map of an interval variable, called a contraction. In this paper, we show that when defined, these agree with the usual upper and lower Perron integrals.

The usual definition of the upper Perron integral is given in terms of major functions, see [5]. Pfeffer has shown that it can also be defined in terms of Riemann-Stieltjes sums, see [3] and [4]. See also [1].

2 Notation and Terminology

Throughout the paper, f will refer to a measurable function $f : [a, b] \to \mathbf{R}$, and δ will refer to a *gauge* $\delta : [a, b] \to (0, +\infty)$.

A *Perron partition* is a set $P := \{(t_i , [x_{i-1}, x_i]) : i = 1, \dots n \}$, where $a = x_0 \leq x_1 \leq ... \leq x_n = b$ and $x_{i-1} \leq t_i \leq x_i$ for $i = 1, \dots n$.

We let $I[a, b]$ denote the set of closed subintervals of $[a, b]$. We include singletons in this set. Let $2^{[a,b]}$ denote the set of all subsets of $[a, b]$. A *contraction* is a map $\nu : I[a, b] \to 2^{[a,b]}$ having the following three properties:

C1. For each $J \in I[a, b]$, $v(J) \subseteq J$.

C2. For each $J_1, J_2, \in I[a, b], J_1 \subseteq J_2$ implies that $\nu(J_2) \cap J_1 \subseteq \nu(J_1)$.

C3. $\cup\{\nu(J)\cap J^\circ\} = [a,b]$, where J° is the interior of J in the relative topology of $[a,b]$.

For any gauge $\delta[a,b] \to (0,\infty)$, define a mapping $\nu_\delta : I[a,b] \to 2^{[a,b]}$, called the *gauge contraction corresponding to* δ by $\nu_\delta : J \mapsto \{x \in J : J \subset (x-\delta(x), x+\delta(x))\}$. It is an easy exercise to verify that ν_δ is a contraction.

Let $\nu : I[a,b] \to 2^{[a,b]}$ be a contraction and let $P = \{(t_i \ , [x_{i-1}, x_i]) : i = 1, \ldots n\ \}$ be a Perron partition. We say that P is $\nu - fine$ if $\nu([x_{i-1}, x_i]) \neq \emptyset$ for $i = 1, \ldots, n$. We note in passing that the tags of P are irrelevant to this definition. However, given a gauge δ , it is easy to see that if P is δ- fine, then P is ν_δ- fine.

If ν_1 and ν_2 are contractions, we say that $\nu_1 \prec \nu_2$ if for every closed interval I, $\nu_1(I) \supseteq \nu_2(I)$. It is easy to see that if $\nu_1 \prec \nu_2$, and P is ν_1-fine, then P is ν_2-fine.

Let $f : [a,b] \to \mathbf{R}$ be a function. Let $\nu : I[a,b] \to 2^{[a,b]}$ be a contraction and $P = \{(t_i \ , [x_{i-1}, x_i]) : i = 1, \ldots n\ \}$ be a ν-fine Perron partition. As in [2], we define the following analogues of lower Riemann sums for f bounded on $\nu([x_{i-1}, x_i]),\ i = 1, \ldots, n$.

$$S^l(P,f) = \sum_{i=1}^{n} \inf\{f(t) : t \in \nu([x_{i-1}, x_i])\}\,(x_i - x_{i-1})$$

$$S^l(\nu, f) = \inf_P S^l(P,f),$$

where the infimum is taken over the set of all ν-fine partitions $P = \{(t_i \ , [x_{i-1}, x_i]) : i = 1, \ldots, n\ \}$.

The analogs of upper Riemann sums, $S^u(P,f)$ and $S^u(\nu,f)$ are defined similarly using supremums in place of infimums. As is noted in [2] , $S^l(\nu,f)$ and $S^u(\nu,f)$ can be written in terms of a single infimum or supremum. In fact,

$$S^l(\nu, f) = \inf_P \ \sigma(P,f),$$

where the infimum is taken over the set of all ν-fine Perron partitions $P = \{(t_i \ , [x_{i-1}, x_i]) : i = 1, \ldots n\ \}$. $S^u(\nu,f)$ is given by the same formula with inf replaced by sup. We remark that $S^l(P,f)$, $S^l(\nu,f)$, $S^u(P,f)$, and $S^u(\nu,f)$ are defined in [2] only when they are finite. It is shown in [2] that $S^l(\nu,f)$ is monotone increasing and $S^u(\nu,f)$ is monotone decreasing in ν.

The *upper Henstock integral* of f is defined to be:

$$(\overline{H})\int_a^b f(x)dx := \inf_{\nu} S^u(\nu, f)$$

where the infimum is taken over the set of all contractions ν for which $S^u(\nu, f)$ is defined and $\nu_0 \prec \nu$, where ν_0 is a fixed contraction for which $S^u(\nu_0, f)$ is defined. We follow [2] in declining to define this upper integral if the appropriate set of contractions is empty. The *lower Henstock integral* is defined analogously

$$(\underline{H})\int_a^b f(x)dx := \sup_{\nu} S^l(\nu, f).$$

Major and minor functions are defined as in [5]. The *upper Perron integral* of f is defined to be

$$(\overline{P})\int_a^b f(x)dx := \inf\{U_\delta(b) : U \text{ is a major function for } f(x)\text{on } [a, b]\}$$

The *lower Perron integral* of f is defined to be

$$(\underline{P})\int_a^b f(x)dx := \sup\{U_\delta(b) : U \text{ is a major function for } f(x)\text{on } [a, b]\}$$

We adopt the conventions that $\sup\emptyset = -\infty$ and $\inf\emptyset = +\infty$. The *Perron integral* exists if and only if the upper and lower integrals are finite and equal.

Let $f : [a, b] \to \mathbf{R}$ be a measurable function, and P a Perron partition. We use the following notation for the Riemann sum of f with respect to P $\sigma(P, f) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$. Let $\delta : [a, b] \to (0, +\infty)$ be a gauge and P a Perron partition. P is $\delta - fine$ if $[x_{i-1}, x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i)), i = 1, ..., n$.

Let $f : [a, b] \to \mathbf{R}$ be a measurable function, and $\delta : [a, b] \to (0, +\infty)$ be a gauge. We define the upper interval function $U_\delta : I[a, b] \to \mathbf{R}_e$ as follows. For each $c \in [a, b]$, we define $U_\delta[c, c] = 0$ and for $a \le c < d \le b$, we define

$$U_\delta[c, d] = \sup\{\sigma(P, f) : P\text{is a } \delta\text{-fine partition of } [c, d]\}.$$

We note that $U_\delta[c, d] > -\infty$. We allow the possibility that $U_\delta[c, d] = +\infty$.

$L_\delta[c,d]$ is defined analogously, with the supremum replaced by infimum. We note that $L_\delta[c,d] < +\infty$. We allow the possibility that $L_\delta[c,d] = -\infty$.

Let δ_1 and δ_2 be gauges. We say that δ_2 is a *refinement* of δ_1 and write $\delta_1 \prec \delta_2$ if for each $x \in [a,b]$ we have $\delta_1(x) \geq \delta_2(x)$.

The following result was proven in [3].

Theorem 2.1 *The upper and lower Perron integrals may be defined as:*

2.1.1. $(\overline{P}) \int\limits_a^b f(x)dx := \inf\{U_\delta[a,b] : \delta \text{ is a gauge}\}$.

2.1.2. $(\underline{P}) \int\limits_a^b f(x)dx := \sup\{L_\delta[a,b] : \delta \text{ is a gauge}\}$.

3 Relation of Upper and Lower Perron Integrals to Upper and Lower Henstock Integrals

Lemma 3.1 *For each contraction ν, there is a gauge δ such that $\nu \prec \nu_\delta$.*

Proof. For each $x \in [a,b]$, define

$$\delta(x) := \min\{2, \inf\{\max\{x-c, d-x\} : c \leq x \leq d \text{ and } x \notin \nu([c,d])\}\}.$$

For clarity, we point out that x is being considered as fixed, and that the infimum is taken over all *intervals* $[c,d]$ for which $x \in [c,d]\backslash\nu([c,d])$.

Clearly, $0 \leq \delta(x) \leq 2 < +\infty$. In order to establish that δ is a gauge, we need only show that $\delta(x) > 0$. Consider x to be fixed. By the third of the defining properties of a contraction, [C3], there is a subinterval, J, of $[a,b]$ such that $x \in \nu(J) \cap J^\circ$. Call $J = [g,h]$.

We proceed by examining three cases. First, suppose that $g \neq a$ and $h \neq b$. Then $g < x < h$. Assume without loss of generality that $[c,d]$ is an interval for which $x \in [c,d]\backslash\nu([c,d])$. By way of contradiction, suppose $[c,d] \subseteq [g,h]$, then by defining property [C2] of a contraction, $\nu[g,h] \cap [c,d] \subseteq \nu([c,d])$. But $x \notin \nu([c,d])$ and $x \in \nu([g,h]) \cap [c,d]$. This contradiction shows that $[c,d]$ is not a subset of $[g,h]$. Therefore, either $h < d$ or $g > c$. Hence, either $d-x > h-x$ or $x-c > x-g$. Therefore, $\delta(x) \geq \max(h-x, x-g) > 0$.

The cases where $g = a$ or $h = b$ are handled similarly. Thus, δ is a gauge.

In order to verify $\nu \prec \nu_\delta$, it suffices to show that $\nu_\delta([g,h)] \subseteq \nu([g,h])$ for each subinterval $[g,h]$ of $[a,b]$. Let $x \in \nu_\delta([g,h])$. By definition of ν_δ, $x \in [g,h]$ and $[g,h] \subseteq (x-\delta(x), x+\delta(x))$, or equivalently, $\delta(x) > \max(h-x, x-g)$. Therefore, $x \in \nu([g,h])$, and hence $\nu_\delta([g,h]) \subseteq \nu([g,h])$. $\square$

Theorem 3.2 *The upper Perron integral of a function is finite if and only if the upper Henstock integral exists. It this case, the two upper integrals have the same value. A similar assertion holds for the lower Perron and Henstock integrals.*

Proof. We give the proof only for the upper integrals. The result for the lower integrals follows upon replacing the integrand by its negative. Assume that $(\overline{P})\int_a^b f(x)dx < +\infty$. Pick $\epsilon > 0$. By Theorem 2.1, there is a gauge δ such that

$$(\overline{P})\int_a^b f(x)dx \leq U_\delta[a,b] < (\overline{P})\int_a^b f(x)dx + \epsilon.$$

It follows readily from the definitions that $U_\delta[a,b] = S^u(\nu_\delta, f)$. We may take the contraction, ν_0, in the definition of the Henstock integral to be ν_δ. Thus, the upper Henstock integral is defined, and moreover

$$(\overline{H})\int_a^b f(x)dx \leq U_\delta[a,b] < (\overline{P})\int_a^b f(x)dx + \epsilon.$$

Letting ϵ go to zero, we obtain

$$(\overline{H})\int_a^b f(x)dx \leq (\overline{P})\int_a^b f(x)dx.$$

Now assume that the upper Henstock integral of f is defined. Pick $\epsilon > 0$. There is a contraction ν such that

$$(\overline{H})\int_a^b f(x)dx \leq S^u(\nu, f) < (\overline{H})\int_a^b f(x)dx + \epsilon.$$

By Lemma 3.1, there is gauge δ such that $\nu \prec \nu_\delta$. Hence,

$$(\overline{P})\int_a^b f(x)dx \leq S^u(\nu_\delta, f) \leq S^u(\nu, f) < (\overline{H})\int_a^b f(x)dx + \epsilon.$$

Since ϵ is arbitrary, the result follows. $\square$

References

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