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GAUSS–MANIN SYSTEMS OF POLYNOMIALS OF TWO VARIABLES CAN BE MADE FUCHSIAN*

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Abstract. We prove modulo a conjecture due to A. Bolibrukh that every monodromy group in which the operators of local monodromy in their Jordan normal forms have Jordan blocks of size only ≤ 2 can be realized by a fuchsian system of linear differential equations on Riemann's sphere without additional apparent singularities. This implies that the Gauss–Manin system of a polynomial of two variables can always be made fuchsian if a suitable basis in the cohomologies is chosen.

1. Introduction

1.1. Regular and Fuchsian Systems

In the present paper we consider *regular* (resp. *fuchsian*) systems, i. e. linear systems of ordinary differential equations depending meromorphically on complex time (which runs over Riemann's sphere), with *moderate growth rate* of the solutions in neighborhoods of the poles (resp. with logarithmic poles). Fuchsian systems are always regular. By definition, the growth rate is *moderate* at a given pole if any solution to the system when restricted to a sector with vertex at the pole and of arbitrary opening grows no faster than some real power of the distance to the pole. Restricting to a sector is necessary because the poles, in general, are ramification points for the solutions.

When a linear change (meromorphically depending on the time) of the dependent variables is performed, then the system changes and the only object which remains invariant under such changes is its *monodromy group*. A *monodromy*

*Dedicated to Professor O. A. Laudal.

operator corresponding to the class of homotopy equivalence of a given lace on Riemann's sphere (one fixes a base point in advance) is the linear operator mapping a basis of the solution space of the system onto the value of its analytic continuation along the lace. The monodromy operators generate the monodromy group which is defined not in a completely invariant way but only up to conjugacy due to the freedom to choose the base point and the initial value of the solution.

One usually chooses as generators of the monodromy group the ones defined by laces circumventing counterclockwise only one pole of the system (and we call further in the text them and only them monodromy operators). This choice is convenient because the monodromy operators will be conjugate to the operators of *local monodromy* (i. e. obtained when a single pole of the system is circumvented counterclockwise along a small loop around it), the latter being easy to compute algorithmically (see [14] for the case of fuchsian systems; for regular ones one first transforms the system locally into a fuchsian one, see [11], and then uses [14]).

For a suitable choice of the contours defining the monodromy operators M_j one has

$$M_1 \dots M_{p+1} = I \tag{1}$$

where I is the identity matrix.

1.2. The Riemann–Hilbert Problem and Results

The *Riemann–Hilbert problem* is stated like this: *prove that for every set of poles on Riemann's sphere and for every monodromy group there exists a fuchsian linear system with these poles and with no others and with this monodromy group.*

In this formulation the problem admits a negative answer due to A. A. Bolibrukh, see [2]. Therefore it is reasonable to reformulate the problem like this: *give necessary and/or sufficient conditions for the choice of the poles and of the monodromy group so that there exists a fuchsian linear system with these poles and with no others and with this monodromy group.*

In the present article we announce:

Theorem 1. *If every monodromy operator in its Jordan normal form has only Jordan blocks of size* ≤ 2*, then the monodromy group is realizable by a fuchsian system for any prescribed set of poles. Moreover, if the group is block upper-triangular, its diagonal blocks defining irreducible or one-dimensional matrix groups, then the fuchsian system realizing the group can be found of the same block upper-triangular form.*

The theorem is proved modulo a conjecture due to A. A. Bolibrukh formulated in Subsec. 2.2. The importance of the theorem resides in its geometric corollary: *the Gauss-Manin system of a polynomial of two variables can be made fuchsian if one chooses a suitable basis in the cohomologies.* (See Sect. 4 for the details.)

The paper is organized as follows: in Sect. 2 we discuss the question what the local asymptotic behavior of the solutions to a regular system is and how one and the same monodromy group can be realized by fuchsian systems with different asymptotics of the solutions. Theorem 1 is proved in Sect. 3. In Sect. 4 we consider the geometric application of the theorem — polynomials of two complex variables and their Gauss–Manin systems. We also give the precise sense of the above geometric corollary there.

2. Realizing Irreducible Monodromy Groups by Fuchsian Systems with Different Asymptotics of their Solutions

2.1. Levelt's Result

Consider the regular linear system

$$\dot{X} = A(t)X \tag{2}$$

where the $n\times n$-matrix A is meromorphic on $\mathbb{CP}^1$, with poles at $a_1, \dots, a_{p+1}$. The form of its solution X (which is also an $n\times n$-matrix) at its pole a_j is described in [10]. Here we cite the result:

Theorem 2. *In the neighborhood of a pole the solution to the regular linear system* (2) *is representable in the form*

$$X = U_j(t-a_j)(t-a_j)^{D_j}(t-a_j)^{E_j}G_j \tag{3}$$

where the $n \times n$*-matrix* U_j *is holomorphic in a neighborhood of the pole* a_j, $D_j = \operatorname{diag}(\varphi_{1,j}, \dots, \varphi_{n,j})$, $\varphi_{\nu,j} \in \mathbb{Z}$, $\det G_j \neq 0$. *The* $n \times n$*-matrix* E_j *is upper-triangular and for the real parts of its eigenvalues* $\beta_{k,j}$ *one has* $\operatorname{Re}(\beta_{k,j}) \in [0,1)$ *(by definition,* $(t-a_j)^{E_j} = \mathrm{e}^{E_j \ln(t-a_j)}$*). The operator* M_j *of local monodromy (up to conjugacy) equals* $\exp(2\pi \mathrm{i} E_j)$.

The numbers $\varphi_{\nu,j}$ *corresponding to equal eigenvalues* $\beta_{k,j}$ *are valuations in the solution subspace* S *at* a_j *invariant for the operator* M_j *and on which it acts with a single eigenvalue* $\exp(2\pi\mathrm{i}\beta_{k,j})$. *They are defined by the filtration of* S *into subspaces of solutions with different growth rates at* a_j*, see [10], and satisfy the condition (5) formulated below.*

System (2) *is fuchsian at* a_j *if and only if*

$$\det U_j(0) \neq 0\,. \tag{4}$$

We formulate the condition on $\varphi_{\nu,j}$. Let E_j have one and the same eigenvalue in the rows with indices $s_1, s_2, \ldots, s_q$. Then we have

$$\varphi_{s_1,j} \geq \varphi_{s_2,j} \geq \ldots \geq \varphi_{s_q,j} \,. \tag{5}$$

Remark 3.

1) Suppose that system (2) is fuchsian, i. e. of the form

$$\dot{X} = (\sum_{j=1}^{p+1} A_j/(t-a_j))X \,. \tag{6}$$

Then the sums $\beta_{\nu,j} + \varphi_{\nu,j}$ are the eigenvalues $\lambda_{\nu,j}$ of the matrix-residuum A_j at a_j. The sum of the residua of a meromorphic 1-form on Riemann's sphere is 0. Hence, the sum of the traces of the matrices A_j is 0 and one has

$$\sum_{\nu=1}^{n}\sum_{j=1}^{p+1}(\varphi_{\nu,j} + \beta_{\nu,j}) = 0 \,. \tag{7}$$

2) For each subspace K which is invariant for all monodromy operators one has (see [2], Lemma 3.6)

$$0 \geq \sum_{K}(\varphi_{\nu,j} + \beta_{\nu,j}) \in \mathbb{Z} \tag{8}$$

(note that our notation differs from the one in [2]).

3) One can presume (what we do) that equal eigenvalues of the matrices E_j occupy consecutive positions and that the matrices are block-diagonal, the sizes of the diagonal blocks being equal to the multiplicities of the eigenvalues. The blocks themselves are upper-triangular.

4) The local parameter $\tau = t - a_j$ in Levelt's form can be chosen to be a global parameter on $\mathbb{CP}^1$, i. e. a meromorphic function with a single zero (of first order) at a_j and a single pole (of first order) which we choose to be at some other of the points a_i, say, a_q, see [8].

5) In the particular case when the monodromy operator M_j (hence, the matrix E_j as well) has Jordan blocks of sizes only ≤ 2, one can assume that the above-diagonal entries of the matrix E_j are only units and zeros, units occurring only in positions (i,j) such that the i-th and j-th diagonal entries of E_j are equal. Moreover, each row and each column of E_j contains no more than one unit.

Indeed, it suffices to consider the case when E_j has a single eigenvalue; moreover, one can assume that it is equal to 0. Find the lowest row of E_j (say, the i-th one) containing a non-zero entry and choose among these entries the left most one (say, $(E_j)_{i,k}$, $i < k$). Conjugate the matrix E_j with an upper-triangular matrix Q to annihilate all above diagonal entries in the i-th row and in the k-th column except $(E_j)_{i,k}$ and to make $(E_j)_{i,k}$ equal to 1. As $(E_j)^2 = 0$, the matrix $Q^{-1}E_jQ$ has only zero entries in the i-th column. Hence, in the union $U(i,k)$ of entries from the i-th and k-th columns and rows ($4n-4$ entries in all) $(E_j)_{i,k} = 1$ is the only non-zero one. Change in Levelt's form G_j to QG_j and set $(t-a_j)^{D_j}Q^{-1} = H(t-a_j)(t-a_j)^{D_j}$; H is upper-triangular, with constant non-zero diagonal terms and holomorphic ones above the diagonal (this follows from (5)). Change in Levelt's form U_j to U_jH.

Choose then from the non-zero above diagonal entries (if any) not from $U(i,k)$ the one from the lowest row and among these the one from the left most column (say, $(E_j)_{i',k'}$, $i \neq i'$, $k' \neq k$, $i' < k'$) and perform a conjugation with an upper-triangular matrix Q' so that in the set $U(i,k) \cup U(i',k')$ the only non-zero entries of the matrix $(Q')^{-1}Q^{-1}E_jQQ'$ be $((Q')^{-1}Q^{-1}E_jQQ')_{i,k} = ((Q')^{-1}Q^{-1}E_jQQ')_{i'k'} = 1$. Note that the conjugation with Q' does not change the entries from $U(i,k)$ because its non-zero entries from $U(i,k)$ are $Q'_{i,i} = Q'_{k,k} = 1$. Continuing like this we obtain the desired form of the matrix E_j. We call this form *quasi Jordan*. It is Jordan up to conjugacy with a permutation matrix.

6) Hence, in the particular case when the monodromy operator M_j and the matrix E_j have each Jordan blocks of sizes only ≤ 2, E_j is quasi Jordan and Levelt's condition (5) can be simplified – one requires only to have

$$\varphi_{k_1,j} \geq \varphi_{k_2,j} \tag{9}$$

whenever E_j has a unit in position (k_1, k_2) ($k_1 < k_2$). In such a case we say that the numbers $\varphi_{k_1,j}$ and $\varphi_{k_2,j}$ correspond to one and the same Jordan block of M_j (or of E_j).

2.2. Quantum States of an Irreducible Monodromy Group

In this subsection we recall some results proved in [8]. Consider the case of irreducible monodromy groups, i. e. without proper subspace of $\mathbb{C}^n$ invariant for all monodromy operators. All monodromy operators M_j are presumed to have Jordan blocks only of size ≤ 2.

Definition 4. A *local quantum state* at a_j is the couple

$$n\text{-tuple } (\varphi_{1,j}, \dots, \varphi_{n,j}), \quad \varphi_{k,j} \in \mathbb{Z}, \text{ quasi Jordan normal form of } E_j .$$

The integers $\varphi_{k,j}$ are presumed satisfying the inequalities (9). Two $(p+1)$-tuples of local quantum states are *equivalent* if they are obtained from one another by a change

$$X \mapsto (t-a_1)^{l_1} \dots (t-a_{p+1})^{l_{p+1}} X , \qquad l_i \in \mathbb{Z},\ l_1 + \dots + l_{p+1} = 0 .$$

The change results in $\varphi_{k,j} \mapsto \varphi_{k,j} + l_j$. A *quantum state* is a class of equivalent $(p+1)$-tuples of local quantum states. When we change the integers $\varphi_{k,j}$ only for $j = j_0$ and we keep them the same for $j \neq j_0$, then we speak about a_{j_0}-quantum states. Whenever quantum states are changed, we keep all matrices E_j the same. In the definition of a quantum state we presume that there holds equality (7), i. e. that the quantum states are *a priori admissible*.

Definition 5. An a priori admissible quantum state of a given irreducible monodromy group is *admissible* (resp. *forbidden*) if there exists a fuchsian system with the given monodromy group, the given poles and the given local quantum states (resp. if there does not exist such a system). If $n = 1$, then the monodromy group has a single a priori admissible quantum state which is admissible.

Example 6. Let $n = 2$. Assume that $a_1 = 0$. Represent in the neighborhood of 0 a fuchsian system by its Laurent series:

$$\dot{X} = \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} /t + \begin{pmatrix} \phi(t) & \psi(t) \\ c + t\eta(t) & \chi(t) \end{pmatrix} \right) X \, ,$$

where ϕ, ψ, χ, η are holomorphic functions of t. Assume that $\alpha - \beta \notin \mathbb{Z}$ and that $c \neq 0$.

The linear change

$$X \mapsto \begin{pmatrix} 1 & (\alpha - \beta + 1)/(ct) \\ 0 & 1 \end{pmatrix} X$$

changes the eigenvalues α, β of A_1 to $\beta - 1, \alpha + 1$, in this order on the diagonal. If $c = 0$ and $t\eta(t) = c' t^k + o(t^k)$, $c' \neq 0$, then a transformation

$$X \mapsto \begin{pmatrix} 1 & \xi(1/t) \\ 0 & 1 \end{pmatrix} X$$

with ξ being a suitably chosen polynomial of degree $k + 1$ changes them to $\beta - k - 1, \alpha + k + 1$, but one can't change them to $\alpha + l, \beta - l$ for $1 \leq l \leq k$ without changing the eigenvalues of the other residua (we propose to the reader to check this oneself). This means that the k corresponding a_1-quantum states are forbidden.

Definition 7. Call a fuchsian system *of type* $(a_j, m, 0)$, $m \in \mathbb{N}$, $m > 1$ if after a fractionally-linear change of the time t the pole a_j is mapped on 0 and the system is obtained from another fuchsian system (for which $t = 0$ is a pole) after a change

$$\begin{aligned} &\Xi_m : \tau \mapsto \tau^m / (p_m \tau^m + \ldots + p_1 \tau + p_0) \, , \\ &\tau = t - a_{j_0} \, , \; p_i \in \mathbb{C} \, , \; p_0 \neq 0 \end{aligned} \tag{10}$$

Hence, a pole of the initial system will give rise to $\leq m$ poles in the new one, with equal matrices-residua if they are exactly m. If one of the poles of the initial system gives rise to $< m$ poles, then some of them should be regarded as confluences of several poles. Under such a confluence the pole remains of first order and the matrices-residua are added.

Remark 8. Any fractionally-linear transformation maps a fuchsian system into a fuchsian system with the same matrices-residua and the same monodromy group.

Definition 9. A fuchsian system is an (a_j, m) one if it is either of type $(a_j, m, 0)$ or is obtained from such a system by adding a finite number of additional logarithmic poles whose matrices-residua are scalar, with zero sum. Some or all of the new poles can coincide with the already existing poles of the system (in such situations the old and new matrices-residua are added).

Remark 10.
1) An (a_j, m) system can be obtained from an $(a_j, m, 0)$ one by a superposition of transformations

$$X \mapsto ((t-b)/(t-c))^s X\,, \quad b, c, s \in \mathbb{C}\,. \tag{11}$$

2) An $(a_j, m, 0)$ system (and as a result, an (a_j, m) one) has infinitely many forbidden a_j-quantum states because at 0 the map (10) has multiplicity m; one can set $\tau^m = t^m/(p_m t^m + \ldots + p_0)$ (τ is only a local parameter) and thus in Levelt's form (3) every entry of the matrix U_j is a germ at 0 of a function holomorphic in τ^m; hence, the valuations $\varphi_{\nu,j}$ can be changed only by multiples of m.

Theorem 11. *An irreducible monodromy group has infinitely many forbidden a_j-quantum states for a given set of poles if and only if it is realized by an (a_j, m) system with this set of poles.*

Remark 12. When a monodromy group has infinitely many forbidden a_j-quantum states for a given set of poles, then *all* admissible a_j-quantum states are defined by special systems. Indeed, in this case to obtain the fuchsian systems defining all admissible a_j-quantum states (we assume that $a_j = 0$) one finds the admissible quantum states of the fuchsian system which is the preimage of the given one under the map (10) and then applies this map.

The theorem is proved in [8]. It is a particular case of Bolibrukh's conjecture formulated below.

Definition 13. Call a fuchsian system *0-special* if it is obtained from another fuchsian system (S) as a result of a change of time $t \mapsto r(t)$ where $r(t)$ is a rational function one of whose critical points coincides with one of the poles of (S). A fuchsian system is called *special* if it is either 0-special or is obtained from a 0-special system as a result of a finite superposition of changes (11). In the case of $(a_j, m, 0)$-systems which are a particular case of 0-special ones the function $r(t)$ has a critical point of multiplicity m at a pole of (S).

Conjecture 14. (A. A. Bolibrukh) *For given poles a_j an irreducible monodromy group has infinitely many forbidden quantum states if and only if it can be realized by a special fuchsian system with these and with no other poles.*

The conjecture was announced in connection with the results from [3]. We give below a more precise formulation of it.

Definition 15. When we change the integers $\varphi_{k,j}$ only at $a_{i_1}, \ldots, a_{i_s}$ (where $1 \leq i_1 < \ldots < i_s \leq p+1$) by keeping the others fixed we speak about $(a_{i_1}, \ldots, a_{i_s})$-quantum states.

Conjecture 16. *For given poles a_j an irreducible monodromy group has infinitely many forbidden $(a_{i_1}, \ldots, a_{i_s})$-quantum states if and only if it can be realized by a special fuchsian system with poles $a_1, \ldots, a_{p+1}$ and with no other poles; the monodromy group has infinitely many admissible $(a_{i_1}, \ldots, a_{i_s})$-quantum states defined by special systems.*

3. Proof of Theorem 1

3.1. Small Blocks

Although the section is subdivided into subsections, we keep a uniform numeration 1^0, $2^0, \ldots$ throughout it to refer easily to its different parts in the course of the proof. We assume that the monodromy group is reducible (for irreducible monodromy groups the answer to the Riemann–Hilbert problem is positive, see [4] and [9]) and in block upper-triangular form; its restrictions to the diagonal blocks (called further *small blocks*) are presumed to be irreducible or one-dimensional.

1^0. One can realize the given monodromy group by a regular system (RS) of the same block upper-triangular form. After this one constructs the restrictions of the necessary fuchsian system (FS) to the small blocks (they are fuchsian systems of the sizes of these blocks). The numbers $\beta_{k,j}$ and $\varphi_{k,j}$ corresponding to a given small block P satisfy the equation

$$\sum_P (\varphi_{k,j} + \beta_{k,j}) = 0\,. \tag{12}$$

This means that one finds admissible quantum states of the restrictions of the monodromy group to the small blocks. The integers $\varphi_{k,j}$ will be chosen to satisfy conditions (9) (for the system as a whole and for its restrictions to each of the small blocks). Hence, there exists a change of variables $X \mapsto V(t)X$ with matrix V meromorphic on $\mathbb{CP}^1$ and holomorphically invertible outside the poles of system (RS) which transforms it into a regular system (RS1) having the same block upper-triangular form and whose restrictions to the small blocks are the same as the ones of (FS). The matrix V is block-diagonal, with diagonal blocks of the sizes of the respective small blocks.

After this the possible higher order poles in the blocks above the diagonal of system (RS1) are removed by block upper-triangular transformations of the form $X \mapsto W_j(1/(t-a_j))X$ where the entries of the matrix W_j are polynomials of $1/(t-a_j)$ and its diagonal (i. e. small) blocks equal I. Hence, $\det W_j \equiv 1$. To find such matrices W_j is possible because the numbers $\varphi_{k,j}$ satisfy conditions (9). For the reader familiar with the theory of normal forms we'll say only that due to these conditions no resonant monomials appear; the presence of such monomials would imply that the Jordan normal form of at least one operator M_j is not the necessary one. Indeed, to be impossible to eliminate a term $a(t-a_j)^{-g}$, $a \in \mathbb{C}$, $g \in \mathbb{N}^+$, in some position (k_1, k_2) of a block above the diagonal, means that $\beta_{k_1,j} = \beta_{k_2,j}$ and $\varphi_{k_1,j} - \varphi_{k_2,j} = -g$. This implies that $\varphi_{k_1,j}$ and $\varphi_{k_2,j}$ correspond to a Jordan block of size ≥ 2 of M_j (and not to two different Jordan blocks) and that (9) does not hold.

Thus to prove the theorem it suffices to construct system (FS), i. e. to choose the integers $\varphi_{k,j}$ satisfying all conditions (9) and (12).

3.2. Good and Bad Small Blocks. (+) and (−) Bad Blocks

2^0.

Notation: Denote throughout the rest of the proof by $(\varphi_{q,j}, \varphi_{s,j})$, $j = 1, \ldots, p+1$, a couple of integers corresponding to one and the same 2×2-Jordan block of M_j.

Give a "recursive" definition of a *good* small block:

1) if to the small block there belong two integers $\varphi_{k_1,j_1}, \varphi_{k_2,j_2}$ corresponding to Jordan blocks of size 2 (from one and the same or from different matrices M_j and for $j_1 = j_2$ from one and the same or from different Jordan blocks), the first one of which is first and the second one of which is second integer $\varphi_{k,j}$ of their 2×2-Jordan block(s), then the small block is good;

2) if for some j an eigenvalue of a Jordan block of M_j of size 1 corresponds to the small block, then the block is good;

3) if from a couple $(\varphi_{q,j},\ \varphi_{s,j})$ of integers defined as above the two integers correspond to different small blocks and one of them (say, $\varphi_{q,j}$) corresponds to a good block, then the block to which the second number (i. e. $\varphi_{s,j}$) corresponds is also good.

Good blocks of types **1)** and **2)** are the basis of the definition, blocks of type **3)** are the inductive step.

3^0. Small blocks which are not good are called *bad*. A bad block contains no integer $\varphi_{k,j}$ corresponding to a Jordan block of E_j of size 1 (for all j), see **2)**. If a bad block contains one of the two integers of a couple $(\varphi_{q,j}, \varphi_{s,j})$, then the second of these integers also belongs to a bad block, see **3)**. Finally, a bad block never contains a couple of integers $\varphi_{k_1,j_1}, \varphi_{k_2,j_2}$ like in **1)**.

Hence, every bad block contains integers $\varphi_{k,i}$ which are all either first (in this case we say a (+)-block) or second integers $\varphi_{k,i}$ of their 2×2-Jordan blocks (in this case we say a $(-)$-block).

The conditions which the integers $\varphi_{k,j}$ must satisfy are only Levelt's condition (9) and the equalities (12). Hence, the set of these conditions splits into two sets, S_G and S_B, the first of which involving only integers $\varphi_{k,j}$ from good and the second only from bad blocks. This allows one to look for the integers $\varphi_{k,j}$ from the good and from the bad blocks separately.

3.3. How to Find Admissible Quantum States for the Good Blocks?

4^0. Explain how one can find admissible quantum states for the good blocks. This means that one has first to find sets of integers $\varphi_{k,j}$ defining for all good blocks a priori admissible quantum states and satisfying conditions (12) and (9), and then show that among these sets of integers there exists at least one defining admissible quantum states for all good blocks.

Denote by $\mathcal{G}_l$ the set of good blocks obtained by applying l times rule **3)** of the definition of a good block. Define $l_0 \in \mathbb{N}$ by the conditions $\mathcal{G}_{l_0} \neq \emptyset$, $\mathcal{G}_{l_0+1} = \emptyset$. For all good blocks from $\mathcal{G}_{l_0}$ choose admissible quantum states arbitrarily. Note that Levelt's condition (9) imposes no restriction at all on the choice of the numbers $\varphi_{k,j}$ belonging to blocks from $\mathcal{G}_{l_0}$ because from every couple $(\varphi_{q,j},\ \varphi_{s,j})$ one encounters in the block only one of the two integers.

5^0. Suppose that for $l \geq l_1$ $(1 \leq l_1 \leq l_0)$ the integers $\varphi_{k,j}$ from the good blocks from $\mathcal{G}_l$ are chosen such that

a) they define admissible quantum states of these blocks;
b) there hold all Levelt's conditions (9) involving both integers from a couple $(\varphi_{q,j},\ \varphi_{s,j})$ the integers $\varphi_{q,j},\ \varphi_{s,j}$ being among the already defined ones.

Denote by P a good block from $\mathcal{G}_{l-1}$ with $l \geq 2$ (the case $l = 1$ is considered in 9^0). Call an integer $\varphi_{k,j}$ *free w.r.t.* P if it is not an integer from a couple $(\varphi_{q,j},\ \varphi_{s,j})$ of which the other integer is already defined, i. e. belongs to a block from $\mathcal{G}_{l_2}$ with $l_2 > l - 1$. (To be free w.r.t. P means not yet to be obliged to fulfill some of Levelt's conditions (9).) Hence, P always contains a free integer $\varphi_{k,j}$. Denote it (or one of them) by φ^0.

6^0. If P is of size 1, then one can define all integers $\varphi_{k,j}$ from P except φ^0 so that there hold all inequalities (9) encountered when the block P and the blocks from the sets $\mathcal{G}_l$ with $l \geq l_1$ are considered. After this one defines φ^0 so that condition (12) holds for the block P.

7^0. Suppose that P is of size > 1, that $\varphi^0 = \varphi_{k,1}$ and that there are only finitely many forbidden a_1-quantum states of the monodromy group $\mathcal{M}$ defined by the restrictions to the block P of the monodromy operators M_j. Then we define infinitely many a priori admissible a_1-quantum states out of which all but finitely many are admissible; we define them so that there hold all Levelt's conditions (9) mentioned in 6^0.

Namely, we define one such a priori admissible a_1-quantum state like in 6^0. Denote by φ^1 an integer $\varphi_{k',1}$ with $k' \neq k$. Hence, φ^1 is either free w.r.t. P or it must satisfy a single Levelt's inequality (9). In both cases one can perform at least one of the two changes χ_+ or χ_- where

$$\chi_\pm : \varphi^0 \mapsto \varphi^0 \pm g\,, \qquad \varphi^1 \mapsto \varphi^1 \mp g\,, \quad g \in \mathbb{N}\,,$$

(for all $g \in \mathbb{N}$) to obtain infinitely many a priori admissible a_1-quantum states of $\mathcal{M}$. Out of them all but finitely many are admissible.

8^0. Suppose now that P is of size > 1, that $\varphi^0 = \varphi_{k,1}$ and that infinitely many a_1-quantum states of the monodromy group $\mathcal{M}$ are forbidden. Assume that $a_1 = 0$. Hence, $\mathcal{M}$ can be realized by an (a_1, m)-system (T) for some $m \in \mathbb{N}$, $m > 1$. Assume that it is an $(a_1, m, 0)$ one (in the general case one has to take into account the possible scalar changes (11); we leave these details for the reader). System (T) in turn is obtained from some system (T′) as a result of the change (10). System (T′) has a pole $a_1' = 0$.

System (T′) is not an (a_1', m') one, therefore its monodromy group $\mathcal{M}'$ has at most finitely many forbidden and infinitely many admissible a_1'-quantum states. (Its monodromy group must be irreducible, otherwise $\mathcal{M}$ will also be reducible.)

The matrices-residua A_1 and A_1' of (T) and of (T′) at 0 are proportional: $A_1 = mA_1'$. Therefore one can define in a natural way the preimages of φ^0 and φ^1 under (10) (denoted by $\varphi^{0'}$ and $\varphi^{1'}$). One needs not have $\varphi^i = m\varphi^{i'}$ because the condition $\mathrm{Re}(\beta_{k,j}) \in [0,1)$ is not preserved under multiplication by m.

Among the admissible a_1'-quantum states of $\mathcal{M}'$ there are infinitely many in which only $\varphi^{0'}$ and $\varphi^{1'}$ change, the changes being analogous to $\chi_\pm$. Each admissible a_1'-quantum state of $\mathcal{M}'$ defines through the map (10) an admissible a_1-quantum state of $\mathcal{M}$; in these a_1-quantum states only φ^i change, $i = 0, 1$, and the change is of the kind $\chi_\pm$.

9^0. When $l = 1$ (i. e. $P \in \mathcal{G}_0$) we prove the existence of infinitely many admissible a_1-quantum states in almost the same way. If we are in case **1)** (see 2^0), then the roles of φ^i are played by φ_{k_1,j_1}, φ_{k_2,j_2} and either χ_+ or χ_- is possible to be done without violating conditions (9). Even if $j_1 = j_2$ and these two integers correspond to one and the same Jordan block of size 2 of M_{j_1} it is possible to make one of the changes $\chi_\pm$ and one can carry out the proof in the same way.

If we are in case **2)**, then the integer $\varphi_{k,j}$ from the Jordan block of size 1 of M_j is free w.r.t. the block P. Hence, one can always find integers $\varphi_{k,j}$ of the good blocks defining admissible a_1-quantum states of the good blocks and satisfying conditions (12) and (9).

3.4. How to Define the Integers $\varphi_{k,j}$ of the Bad Blocks?

10^0. We first fix the integers $\varphi_{k,j}$, $j > 2$, of the bad blocks and then we choose their integers $\varphi_{k,1}$ and $\varphi_{k,2}$. Recall that the integers $\varphi_{k,j}$ of the bad blocks can be grouped in couples of integers each couple being of the form $(\varphi_{q,j}, \varphi_{s,j})$, see 2^0. We assume that the two integers of every such couple are equal (hence, the corresponding inequalities (9) hold — they become equalities). Thus we consider as unknown integer variables only half of the integers $\varphi_{k,j}$ and we assume that for each j the index k takes only half of the values from 1 to n.

Lemma 17. *The system of equations* (12) *with unknown variables* $\varphi_{k,j}$ *can be decomposed into subsystems such that*

1) *in every subsystem every variable* $\varphi_{k,j}$ *either participates twice (in two equations, one corresponding to a (+) and one corresponding to a (−)-block) or doesn't participate at all;*
2) *the subsystems are the minimal possible;*
3) *every subsystem is of corank 1, i. e. there exists a unique up to multiplication by a constant linear combination (A) of its equations which is an equation of the form* $0 = 0$.
4) *every subsystem admits an integer solution.*

The lemmas from this subsection are proved in Subsect. 3.5.

11^0. Fix an integer solution $\Phi = \{\varphi_{k,j}\}$ to the system of equations (12) (by 4) of the above lemma such a solution exists). However, there might be bad blocks of size > 1 for which the integers $\varphi_{k,j}$ of this solution define forbidden quantum states. Therefore, in general, we change the given solution to another one defining admissible quantum states for all bad blocks of size > 1.

We do this separately for every subsystem (defined by the lemma) like this: for every bad block P of size > 1 define a *cycle* of integers $\varphi_{k,j}$, $j = j'$ or $j = j', j''$, $j' \neq j''$, $1 \leq j^{(i)} \leq p + 1$. As every variable $\varphi_{k,j}$ is encountered exactly twice, we speak about its first and about its second copy (one of them belongs to a $(+)$, the second belongs to a $(-)$-block). The cycle contains either both copies of a given variable $\varphi_{k,j}$ or none of them; it is defined as follows:

A. choose two integers — φ_{k_1,j_1} and φ_{k_2,j_2} — from the block P, with $j_1 = j_2 = j'$;

B. choose the next integers of the cycle also in couples: if an odd number s of couples is chosen and if the last chosen couple is $\varphi_{k_{2s-1},j_{2s-1}}$ and $\varphi_{k_{2s},j_{2s}}$ (each index j_i equals 1 or 2), then the next two chosen integers are the second copies of $\varphi_{k_{2s-1},j_{2s-1}}$ and $\varphi_{k_{2s},j_{2s}}$;

C. if an even number s of couples is chosen, then

   C1. if one of the two last chosen integers $\varphi_{k_{2s-1},j_{2s-1}}$ is from a bad block P' of odd size from which all integers $\varphi_{k,j_{2s-1}}$ are already chosen, then choose as one of the two next integers of the cycle a not yet chosen integer $\varphi_{k_{2s+1},j'+j''-j_{2s-1}}$ from the block P';

   C2. if $\varphi_{k_{2s-1},j_{2s-1}}$ is from a bad block Q containing integers $\varphi_{k,j_{2s-1}}$ which are not chosen yet, then we choose as integer $\varphi_{k_{2s+1},j_{2s+1}}$ one of them;

   C3. choose in the same way $\varphi_{k_{2s+2},j_{2s+2}}$ after $\varphi_{k_{2s},j_{2s}}$ as $\varphi_{k_{2s+1},j_{2s+1}}$ was chosen after $\varphi_{k_{2s-1},j_{2s-1}}$;

D. the construction of the cycle stops if and only if an even number of couples of integers $\varphi_{k,j}$ are chosen (because if the number is odd, then one can apply B) and one of the two things happens:

   - the last chosen couple is the couple of two remaining integers from a bad block of even size (in this case C2 cannot be performed);
   - the last couple is a couple of integers $\varphi_{k_{2s-1},j_{2s-1}}$, $\varphi_{k_{2s},j_{2s}}$ with $j_{2s-1} = j' + j'' - j_{2s}$ from one and the same bad block of odd size and after their choice all integers from this block (both for $j = j'$ and j'') are chosen (in this case C1 cannot be performed).

All cycles constructed in this way (for all possible couples j', j'') are *not* non-intersecting.

12^0. Denote by $c_1, \ldots, c_s$ the different non-intersecting cycles defined for a given subsystem of equations (12). Each cycle contains an even number of couples of integers $\varphi_{k,j}$, see D from 11^0. More exactly, it contains an even number of variables $\varphi_{k,j}$ each encountered twice. For a given cycle assign the number 1 to the variables $\varphi_{k,j}$ with odd and the number -1 to the variables $\varphi_{k,j}$ with even indices. Denote by $\sigma_i c_i$ the same cycle in which these numbers are changed respectively to σ_i and $-\sigma_i$, $\sigma_i \in \mathbb{Z}$.

Fix the integers σ_i, $i = 1, \ldots, s$. Change the solution Φ by adding to each integer $\varphi_{k,j}$ the integers σ_i from the cycles to which $\varphi_{k,j}$ belongs. This is another solution (we denote it by $\Phi(\sigma) = \Phi + \sum_{i=1}^{s} \sigma_i c_i$). Indeed, each bad block contains as many integers $\varphi_{k,j}$ from the given cycle to which σ_i is assigned as it contains ones to which $-\sigma_i$ is assigned. This can easily be deduced from 11^0, see B and C.

Observe also that when thus changing Φ one adds to both integers from one and the same Jordan block of size 2 of a given operator M_j one and the same integer $\pm\sigma_i$, therefore Levelt's conditions (9) are preserved under the change.

13^0. We prove here some statements concerning special and 0-special systems which are necessary to prove the theorem.

(*) A 0-special system (0SS) has ≥ 4 poles with non-scalar matrices-residua.

Indeed, if $A_j = bI$, then $M_j = \exp(2\pi ib)I$. With only two non-scalar operators M_j the monodromy group will be generated by a single monodromy operator, see (1), hence, it will be reducible. System (0SS) is obtained from another fuchsian system (S) via the map

$$\rho : t \mapsto r(t)\,, \qquad r = q_1(t)/q_2(t)\,, \tag{13}$$

the polynomials q_i having no common zero.

This is a rational map $\mathbb{CP}^1 \to \mathbb{CP}^1$ of multiplicity $m = \max(\deg q_1, \deg q_2)$, $m > 1$. When one applies it to a fuchsian system, its polar part $A_j/(t-a_j)$ gives rise to m polar parts $A_j/(t-b_{i,j})$ (counted with the multiplicities) where $b_{i,j}$ are all solutions to the equation $r(t) = a_j$; their set is *the level set* $\rho^{-1}(a_j)$. In the case of a multiple root of this equation (of multiplicity h) the corresponding polar part equals $hA_j/(t-b_{i,j})$. To prove (*) we need another statement:

(**) The map (13) has at most two level sets consisting each of one point of multiplicity m.

Indeed, suppose that there are at least three such level sets corresponding to $a_1 \neq a_2 \neq a_3 \neq a_1$ (we assume that $a_j \neq \infty$ which can be achieved by a fractionally-linear transformation, see Remark 8). For each of them there exists $u_j \in \mathbb{C}$ and $v_j \in \mathbb{C}^*$ such that $q_1(t) - a_j q_2(t) = v_j(t-u_j)^m$. Hence, there exists a non-trivial linear combination of the three polynomials $(t-u_j)^m$ with $m \geq 2$ which is identically 0 — a contradiction.

The monodromy group of system (S) is irreducible, otherwise the one of system (0SS) would be reducible. Hence, at least one of the poles with non-scalar residua of system (S) (which are at least three) gives rise to ≥ 2 poles when the map (13) is applied. This proves statement (*).

(***) A special system has at least 4 poles.

Indeed, a special system is obtained from a 0-special one after a superposition of maps (11) which result in adding scalar matrices-residua. The 0-special system has at least four non-scalar matrices-residua, hence, so does the special one.

(****) The multiplicity m_i of each critical point of the map (13) situated at a pole of system (S) is less than $p+1$.

Indeed, assume that at the given critical point v the solution to system (S) is represented in Levelt's form (3), with $U_j(0) = I$. Then the solution to system (0SS) (at some pole a_{j_0} such that $r(a_{j_0}) = v$) will have the same form, with $U_{j_0} = I + O((t-a_{j_0})^{m_i})$. Hence, $(U_{j_0})^{-1} = I + O((t-a_{j_0})^{m_i})$. Suppose that $m_i \geq p+1$. Then system (0SS) in a neighborhood of a_{j_0} looks like this: $\dot{X} = A(t)X$ with $A(t) = \dot{X}X^{-1}$, i. e.

$$A(t) = \dot{U}U^{-1} + UDU^{-1}/(t-a_{j_0}) + U(t-a_{j_0})^D E(t-a_{j_0})^{-D}U^{-1}/(t-a_{j_0}),$$

see (3); we omit the indices j_0 of U, E and D. Hence,

$$A(t) = (D + (t-a_{j_0})^D E(t-a_{j_0})^{-D})/(t-a_{j_0}) + O((t-a_{j_0})^p). \quad (14)$$

On the other hand, the coefficient before $(t-a_{j_0})^s$ in the local (at a_{j_0}) Laurent series expansion of $A(t)$ equals $B_s := (-1)^s \sum_{j \neq j_0} A_j/(a_{j_0} - a_j)^{s+1}$, (to be checked directly, see (6)). Equality (14) implies that all entries below the diagonal of all matrices B_s, $s = 0, \ldots, p-1$, are 0 (because D and E are upper-triangular); hence, the same is true for the matrices A_j (use Vandermonde's determinant defined by the numbers $(-1)/(a_{j_0} - a_j)$). But then the monodromy group of the system would also be upper-triangular, i. e. reducible — a contradiction.

14^0. Consider the case $p \geq 4$.

Lemma 18. *For $p \geq 4$ one can choose the integers $\varphi_{k,j}$, $j > 2$, of the bad blocks such that for each bad block P the monodromy group defined by the restrictions $M_j|_P$ of the monodromy operators M_j to P has only finitely many forbidden (a_1, a_2)-quantum states.*

When varying the integers σ_i, one obtains infinitely many solutions $\Phi(\sigma)$ defining for each bad block of size > 1 infinitely many a priori admissible quantum states (indeed, each bad block of size > 1 contains integers $\varphi_{k,j}$ from at least one cycle). Among them infinitely many are admissible (this follows from Lemma 18). Hence, the choice of the integers $\varphi_{k,j}$ of the bad blocks satisfying the conditions (12) and (9) and defining admissible quantum states of all bad blocks is possible. This proves Theorem 1 in the case $p \geq 4$.

15^0. In the case $p = 3$ a fuchsian system (FS) is 0-special only if $m = 2$. Indeed, it is obtained from some fuchsian system (S) with three poles via the map (13). Two of these poles (say, at 0 and ∞) must define two level sets consisting each of a single point. The third (say, at 1) must give rise to two poles of (FS). Hence, the map (13) is of the form $r(t) = \alpha t^m$, $\alpha \in \mathbb{C}$. System (FS) will have four poles only if $m = 2$.

Given system (FS) and knowing that it is 0-special, one knows which of its poles (denoted by a_3, a_4) belong to $\rho^{-1}(1)$ (by considering the possible cross-ratios of the four poles — their values are -1, ± 2, $\pm 1/2$).

The same things hold for system (FS) presumed to be special (it can have no scalar matrix-residuum, otherwise system (S) should have one and its monodromy group would be reducible). To prove the theorem in the case $p = 3$ it suffices to construct for each bad block the sets Ψ_4 and Ψ_3 almost non-proportional — hence, no (a_1, a_2)-quantum state at all of $\mathcal{M}|_P$ would be realizable by a special system; by Conjecture 16, only finitely many (a_1, a_2)-quantum states of $\mathcal{M}|_P$ can be forbidden.

16^0. If $p = 2$, then for no bad block P can one have infinitely many forbidden quantum states of $\mathcal{M}|_P$ and the proof is even easier. □

3.5. Proofs of the Lemmas

Proof of Lemma 17:

A^0. Construct the minimal subsystems explicitly. For a variable $\phi = \varphi_{k_1,j}$ from a $(+)$-block denote by $j(\phi)$ the variable $\varphi_{k_2,j}$ (from a $(-)$-block) corresponding to the same Jordan block of size 2 of M_j as ϕ. In the notation from Subsec. 3.2 the couple $(\phi, j(\phi))$ is a couple $(\varphi_{q,j}, \varphi_{s,j})$. Recall that $\varphi_{k_1,j} = \varphi_{k_2,j}$, i. e. $\phi = j(\phi)$; we use the notation ϕ, $j(\phi)$ to show which of the two equal variables is from a $(+)$ and which from a $(-)$-block.

Choose a $(+)$-block P. Denote by $\mathcal{P}$ the set of all its variables ϕ and by (P) the equation (12) corresponding to it. Then the minimal subsystem containing (P) must contain all equations (N_i) which are equations (12) of $(-)$-blocks N_i containing at least one variable $\varphi_{k,j}$ which equals $j(\phi)$, $\phi \in \mathcal{P}$. Indeed, property 1) claimed by the lemma must hold. Hence, the minimal subsystem must contain all equations (P_ν) which are equations (12) of $(+)$-blocks P_ν containing at least one variable $\varphi_{k,j}$ which equals ϕ with $j(\phi) \in \mathcal{N}_i$, etc.

Iterating the procedure "adding the blocks containing variables $j(\phi)$ (resp. ϕ)" (ϕ (resp. $j(\phi)$) being among the variables of the bad blocks whose equations (12) are already known to belong to the subsystem) one obtains the minimal subsystem containing equation (P). The minimality follows from the construction.

B^0. Property 3) also follows from the construction of the minimal subsystems — if the linear combination must be of the form $0 = 0$, then the equations (12) corresponding to the $(+)$ and $(-)$-block containing the variables ϕ and $j(\phi)$ must participate in it with opposite coefficients. Hence, up to a non-zero factor the equations (P) (resp. (N)) of all $(+)$ (resp. $(-)$) blocks of the subsystem must participate in the linear combination with coefficient 1 (resp. -1). The details are left for the reader.

C^0. Prove 4). To this end one can make the subsystem trapezoidal making use of the Gauss method. Recall that by 3) of the lemma, the final system contains exactly one equation of the form $0 = 0$. When one subtracts upper equations (multiplied by suitable coefficients) from lower ones, these coefficients will always equal ± 1 and all non-zero coefficients of all variables in the final system will equal ± 1 (this is proved in D^0). The constant terms of the system will be integers (because they are such at the beginning). The existence of an integer solution to the system obtained after applying the Gauss method is obvious.

D^0. Assume that in the minimal subsystem the equations corresponding to $(-)$-blocks are multiplied by -1. It is obvious that the first subtraction of the Gauss method must be done with coefficient -1 (one tries to eliminate from an equation $(L-)$ corresponding to a $(-)$-block a variable $j(\phi)$, hence, one finds the unique equation $(L+)$ of a $(+)$-block where ϕ participates and one replaces in the system the equation $(L-)$ by $(L-) + (L+)$). All coefficients of unknown variables in equation $(L-) + (L+)$ equal 0, 1 or -1.

Suppose that at some stage of applying the Gauss method all non-zero coefficients of all equations of the system equal ± 1, all signs of the variables ϕ (resp. $j(\phi)$) being positive (resp. negative; this is the case of equation $(L-)+(L+)$). Hence, when one carries out the next subtraction, it will be carried out again with coefficient -1.

Suppose that one replaces equation (H_2) of the system by $(H_2) + (H_1)$. Then whenever two variables ϕ and $j(\phi)$ participate respectively in (H_1) and (H_2), then their coefficients will be opposite and they will not participate in $(H_2) + (H_1)$. If only one of them participates (in one of the two equations), then in $(H_2) + (H_1)$ its coefficient will equal 1 if it is a variable ϕ and to -1 if it is a variable $j(\phi)$. $\square$

Proof of Lemma 18:

A^0. We choose the integers $\varphi_{k,j}$ for $j > 2$ such that only finitely many a priori admissible (a_1, a_2)-quantum states would be possible to realize by special systems. By Conjecture 16, the monodromy group can have only finitely many forbidden (a_1, a_2)-quantum states.

For each bad block P of size $l > 1$ define the numbers $\psi_{k,j}$ as follows:

$$\psi_{k,j} = \varphi_{k,j} + \beta_{k,j} - \frac{\sum_k (\varphi_{k,j} + \beta_{k,j})}{l}$$

(the sum is taken over the integers $\varphi_{k,j}$ from the block P, j is fixed). Recall that the eigenvalues of A_j equal $\beta_{k,j} + \varphi_{k,j}$; the numbers $\beta_{k,j}$ are known and the numbers $\psi_{k,j}$ define the integers $\varphi_{k,j}$ up to a simultaneous shift of the latter.

Definition 19. Set $\Lambda = \{(s_1/s_2);\ s_i \in \{0, 1, \ldots, p\}, s_2 \neq 0\}$. Call two sets Ψ_j $(j = j_1, j_2)$ of numbers $\psi_{k,j}$ (for one and the same bad block, for two different values of j) *almost non-proportional* if either there exists no permutation of the elements of Ψ_{j_1} after which they become proportional or such a permutation exists but the ratio of the sets and its inverse do not belong to the set Λ.

If the two sets Ψ_{p+1}, Ψ_p are almost non-proportional, then the matrices $A_{p+1}|_P$ and $A_p|_P$ with eigenvalues defined after the sets Ψ_{p+1}, Ψ_p, cannot be matrices-residua of a special system such that the poles a_{p+1}, a_p belong to one and the same level set of the map (13). Indeed, for a 0-special system with a_{p+1}, a_p from the same level set the ratio $A_{p+1}|_P/A_p|_P$ must equal the ratio of the multiplicities of two points of the map (13) (a non-critical point has multiplicity 1). The ratio of these multiplicities belongs to Λ (statement (****) from 13^0 of the proof of Theorem 1). When constructing a special system after a 0-special one by means of maps (11) the sets Ψ_j do not change.

B^0. Choose as integers $\varphi_{k,p+1}$ the ones from Φ (see 11^0 from the proof of Theorem 1). We assume that the numbers $\varphi_{k,p+1} + \beta_{k,p+1}$ are not all equal (i. e. A_{p+1} is not scalar); if this is not the case, then we can change Φ to $\Phi(\sigma)$ with σ_i from some cycle with $j' = p + 1$, $j'' = 1$. Consider all cycles c_i (see 11^0 from the proof of Theorem 1) with $j' = p$, $j'' = 1$. It is possible to find integers σ_i such that the sets Ψ_{p+1} and Ψ_p computed after the integers $\varphi_{k,j}$ from $\Phi(\sigma)$ be almost non-proportional.

Indeed, for each bad block of size ≥ 2 there exists such a cycle containing two integers $\varphi_{k,p}$ from P, to which one assigns the integers $\pm\sigma_i$. If σ_i is big enough, then Ψ_{p+1} and Ψ_p are almost non-proportional (Ψ_{p+1} does not depend on σ_i). Moreover, one can choose the integers σ_i so that the sets Ψ_{p+1} and Ψ_p be almost non-proportional for all bad blocks simultaneously.

C^0. After this one considers the cycles with $j' = p - 1$, $j'' = 1$ and chooses their integers σ_i such that every two of the sets Ψ_{p+1}, Ψ_p and Ψ_{p-1} be almost non-proportional, for all bad blocks simultaneously (in this case Ψ_p and Ψ_{p-1} do not depend on the choice of the integers σ_i of the cycles with $j' = p - 1$, $j'' = 1$). Continuing in the same way, one makes every two of the sets Ψ_3, $\ldots$, Ψ_{p+1} almost non-proportional. Hence, no two of the matrices $A_3|_P$, $\ldots$, $A_{p+1}|_P$ with eigenvalues defined after the respective sets Ψ_j can be matrices-residua of a special system at two poles belonging to one and the same level set of the map (13).

D^0. If a fuchsian system with ≥ 5 non-scalar matrices-residua is special, then for everyone except at most two of its matrices-residua A_j there exists another matrix-residuum A_μ such that the poles a_j and a_μ belong to one and the same level set of the map (13) (see 13^0 from the proof of Theorem 1, statement (**)). Hence, the sets Ψ_j and Ψ_μ will not be almost non-proportional.

E^0. Fix the integers $\varphi_{k,j}$ for $j \geq 3$ like they were defined in A^0 – C^0 and vary $\varphi_{k,1}$, $\varphi_{k,2}$ (by considering the cycles with $j' = 2$, $j'' = 1$ like this was done for $j' > 2$). Their integers σ_i define (for each bad block P) infinitely many a priori admissible (a_1, a_2)-quantum states of the restriction $\mathcal{M}|_P$ of the monodromy group $\mathcal{M}$ to the block P. Only for finitely many of them each of the sets Ψ_1 and Ψ_2 is not almost non-proportional to everyone of the sets Ψ_j for $j \geq 3$, i. e. only for finitely many of them each of the matrices $A_1|_P$, $A_2|_P$ with eigenvalues defined after Ψ_1, Ψ_2 can be a matrix-residuum resp. at a_1, a_2 of a special system.

Indeed, by D^0, each of the points a_1, a_2 must belong to one and the same level set of the map (13) as some of the points $a_3, \ldots, a_{p+1}$ — each two of the sets Ψ_j being almost non-proportional for $j \geq 3$, if Ψ_1 or Ψ_2 is also almost non-proportional with Ψ_j, $j \geq 3$, then the map (13) will have more than one level set consisting each of a single point which contradicts statement (**) from 13^0 of the proof of Theorem 1.

Hence, for all but finitely many choices of the integers σ_i the corresponding (a_1, a_2)-quantum states will not be realizable by special systems. Conjecture 16 implies that only finitely many of them can be forbidden. □

4. Gauss–Manin Systems of Polynomials of Two Variables

4.1. Definition of the Gauss–Manin System

In this section we recall some facts from singularity theory. We recall them only in the case of two variables, for the general case good references are [1], [6], [12] and [13] (and the list is very far from being exhaustive).

Denote by $f(x, y) \in \mathbb{C}[x, y]$ a polynomial of two complex variables, with only isolated singularities if any; a *singularity* or a *singular point* in $\mathbb{C}^2$ is a point (x_0, y_0) at which $\mathrm{d}f$ vanishes. Denote by (x_i, y_i) the singular points of f and by $t_i = f(x_i, y_i)$ its corresponding *critical values*. The level sets $f^{-1}(t)$, $t \in \mathbb{C}$, are algebraic curves in $\mathbb{C}^2$ all but finitely many of which (namely, except for $t = t_i$) are non-singular. The projectivizations of these curves might have singularities at infinity.

In the neighborhood of all but finitely many values of t the fibration of $\mathbb{C}^2$ over $\mathbb{C}$ with fibre $f^{-1}(t)$ is locally trivial. The exceptional values of t include all critical values t_i of f and there might be a finite number of values s_j for which the fibre $f^{-1}(s_j)$ is non-singular but the fibration in the neighborhood of s_j is not trivial. Denote by Σ the set of all values t_i and s_j.

Given a value $t' \notin \Sigma$, one can fix a basis of $H_1(f^{-1}(t'), \mathbb{C})$, i. e. cycles δ_k on $f^{-1}(t')$ of real dimension 1. The local triviality of the fibration allows to define families (continuously depending on t) of such cycles for all t close to t'.

Denote by $\psi = p(x, y)\,\mathrm{d}x \wedge \mathrm{d}y$ a holomorphic differential 1-form where p is a polynomial of (x, y). Then the *relative* form $\psi/\,\mathrm{d}f$ is well-defined on every non-singular level set; it is closed on it. The integrals $\int_{\delta_k} \psi/\,\mathrm{d}f$ are well-defined functions of t. For t close to t' these integrals depend holomorphically on t.

Set $l = \dim H_1(f^{-1}(t'), \mathbb{C})$. Choose l forms ψ_l such that the corresponding relative forms are a basis of $H^1(f^{-1}(t'), \mathbb{C})$ dual to $\{\delta_k\}$. Hence, for the *period matrix* $\mathcal{I} = \left\{ \int_{\delta_k} \psi_\nu/\,\mathrm{d}f \right\}$ one has $\det \mathcal{I}|_{t=t'} = 1$. (The k-th column of the period matrix consists of the integrals of the forms $\psi_\nu/\,\mathrm{d}f$ over δ_k.)

The matrix $\mathcal{I}$ can be defined for nearby values of t and its dependence on t will be holomorphic. It is a fundamental solution to a system of linear differential equations $\mathrm{d}\mathcal{I}/\,\mathrm{d}t = A(t)\mathcal{I}$ called *the Gauss–Manin system*. This system has finitely many poles among which the points from Σ. There might be some more poles at the points where $\det \mathcal{I}$ vanishes (indeed, one has $A(t) = (\,\mathrm{d}\mathcal{I}/\,\mathrm{d}t)\mathcal{I}^{-1}$); these singularities are *apparent*, i. e. with trivial local monodromy; denote their set by Θ.

One can continue the solution $\mathcal{I}$ to the Gauss–Manin system along any path in $\mathbb{C}$ not passing through any point from $\Sigma \cup \Theta$. Thus globally on $\mathbb{C}$ the matrix $\mathcal{I}$ will be a multivalued matrix-function with branching points at t_i and s_j and with poles on Θ. All these points turn out to be regular singularities of the Gauss–Manin system (in general, ∞ is also a singular point of the system; it is also regular).

The monodromy operators corresponding to these singular points have (in their Jordan normal forms) Jordan blocks of size only ≤ 2 (in the case of n variables of size only $\leq n$). This fact is known to specialists but a complete proof (including all cases) is hard to find. Good references are [7] and [5].

4.2. A Geometric Corollary from Theorem 1

Theorem 1 implies that the monodromy group of the Gauss–Manin system (GM) of any polynomial of two variables with only isolated singularities if any can be realized by some fuchsian system (F); if the monodromy group is block upper-triangular, its diagonal blocks defining irreducible or one-dimensional matrix groups, then system (F) can be presumed to be of the same block upper-triangular form. System (F) has poles only on Σ.

Hence, there exists a change of variables

$$\mathcal{I} \mapsto W(t)\mathcal{I} \tag{15}$$

transforming (GM) into (F). The matrix W has, in general, poles on $\Sigma \cup \Theta$. It is holomorphic and holomorphically invertible outside this set because both (GM) and (F) are holomorphic outside it.

This means that one can choose l differential 1-forms ψ_i^0 such that the Gauss-Manin system of f constructed after them will be fuchsian, with poles only from Σ (this system is in fact (F)). However, these forms will not be polynomials and not even holomorphic in x and y. They will be finite combinations of terms of the form $p(x,y)/(f-a_j)^g$ where $p \in \mathbb{C}[x,y]$, $a_j \in (\Sigma \cup \Theta)$, $g \in \mathbb{N}$. Indeed, the change (15) transforms every integral $\int_{\delta_k} \psi_\nu / \,\mathrm{d}f$ into a linear combination of such integrals with coefficients of the form $d/(t-a_j)^g$, $d \in \mathbb{C}$ (it transforms it into one and the same combination for every k fixed). Note that one has

$$(t-a_j)^{-g} \int_{\delta_k} \psi_\nu / \,\mathrm{d}f = \int_{\delta_k} (f-a_j)^{-g} \psi_\nu / \,\mathrm{d}f$$

because the integration is performed in the fibre $f^{-1}(t)$. Hence, the new basis of $H^1(f^{-1}(t), \mathbb{C})$ will be of forms $\psi_j \,\mathrm{d}x \wedge \mathrm{d}y$ with $\psi_j \in \mathbb{C}[x,y][(f-a_1)^{-1}, \ldots, (f-a_r)^{-1}]$ where $\{a_1, \ldots, a_r\} = (\Sigma \cup \Theta)$.

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